PERSONAL NOTES

Computational Fluid Dynamics

Hanfeng Zhai

Disclaimer: These notes are intended solely for personal reference and study purposes. They represent my own understanding of the course material and may contain errors or inaccuracies. The content presented here should not be considered as an authoritative source, and reliance solely on these materials is not recommended. If you notice any materials that potentially infringe upon the copyright of others, please contact me at hz253@cornell.edu so that appropriate action can be taken. Your feedback is greatly appreciated.

2020

CHAPTER 1. Introduction to Computational Fluid Dynamics

© Hanfeng Zhai

School of Mechanics and Engineering Science, Shanghai University

Shanghai 200444, China

Abstract

Introduce the concept of computational fluid dynamics (CFD) continued from fluid mechanics. Provide the governing equations, conceptions, and related parameters in CFD. Explained the basic ideology and goal for CFD.

Computational fluid dynamics (CFD), is a method to solve fluid mechanics problems through numerical methods. Navier-Stokes equation is the governing equation for fluid phenomena and physics. Due to the nonlinear nature of the Navier-Stokes equation, analytical solution for fluidic problems are limited to specific situations. Hence, to obtain solutions for more generalized fluidic problems, we introduce numerical methods like CFD to discretize the equations for results.

In fluid mechanics (or fluid dynamics), the governing equation for fluid motion and physics is the Navier-Stokes equation, which takes the form

$$\rho \frac{d\boldsymbol{V}}{dt} = \rho g - \nabla P + \mu \nabla^2 \boldsymbol{V} \tag{1}$$

Which can be written in the coordinate system for 3D situation:

$$\begin{aligned}
\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) &= f_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\
\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) &= f_y - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \\
\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) &= f_z - \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)
\end{aligned} \tag{2}$$

Where ρ is the fluidic density, $\boldsymbol{V}$ is the speed vector; P is the pressure on fluid, u, v, w are the velocity component in the x, y, z directions, respectively. $\boldsymbol{f}$ is the force acting on the unit volume. The constant μ is the dynamics viscosity.

The Navier-Stokes equation taking the form in Eq. 2 elucidate the physics of fluidic nature. Yet for computation, we write the Navier-Stokes equation in the following terms for discretization.

$$\frac{\partial U}{\partial t} + \frac{\partial E_i}{\partial x_i} - \frac{\partial E_{vi}}{\partial x_i} - P_i = 0 \tag{3}$$

Where P_i, E_{vi} marks the external forces and viscosity terms, respectively.

In 3D situation, consider the coordinate system as compared with Eq. 2, Eq. 3 takes the form:

$$\frac{\partial U}{\partial t}+\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}+\frac{\partial G}{\partial z}-\frac{\partial E_v}{\partial x}-\frac{\partial F_v}{\partial y}-\frac{\partial G_v}{\partial z}-P=0 \tag{4}$$

Where

$$U=\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ \rho E\end{pmatrix},E=\begin{pmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ \rho uw\\ \rho uH\end{pmatrix},F=\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ \rho vw\\ \rho vH\end{pmatrix},G=\begin{pmatrix}\rho w\\ \rho uw\\ \rho vw\\ \rho w^2+p\\ \rho wH\end{pmatrix} \tag{5}$$

$$P=\begin{pmatrix}0\\ \rho f_x\\ \rho f_y\\ \rho f_z\\ \rho\left(uf_x+vf_y+wf_z\right)\end{pmatrix},E_v=\begin{pmatrix}0\\ \sigma_{xx}\\ \sigma_{xy}\\ \sigma_{xz}\\ u\sigma_{xx}+v\sigma_{xy}+w\sigma_{xz}+\kappa\frac{\partial T}{\partial x}\end{pmatrix}, \tag{6}$$

$$F_v=\begin{pmatrix}0\\ \sigma_{yx}\\ \sigma_{yy}\\ \sigma_{yz}\\ u\sigma_{yx}+v\sigma_{yy}+w\sigma_{yz}+\kappa\frac{\partial T}{\partial x}\end{pmatrix},G_v=\begin{pmatrix}0\\ \sigma_{zx}\\ \sigma_{zy}\\ \sigma_{zz}\\ u\sigma_{zx}+v\sigma_{zy}+w\sigma_{zz}+\kappa\frac{\partial T}{\partial x}\end{pmatrix}$$

Where $p=\rho RT=\rho(1-\gamma)\left(\rho E-\frac{1}{2}(u^2+v^2+w^2)\right)$.

Here, for classic aerodynamics problems, we neglect the external force as in specific flow fields (Lagrangian viewpoint). we take the equation in 2D situation for example:

$$\frac{\partial U}{\partial t}+\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}-\frac{\partial E_v}{\partial x}-\frac{\partial F_v}{\partial y}=0 \tag{7}$$

Where

$$U=\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ \rho E\end{pmatrix},E=\begin{pmatrix}\rho\\ \rho u^2+p\\ \rho uv\\ \rho uH\end{pmatrix},F=\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ \rho vH\end{pmatrix} \tag{8}$$

$$E_v=\begin{pmatrix}0\\ \sigma_{xx}\\ \sigma_{xy}\\ u\sigma_{xx}+v\sigma_{xy}-\kappa\frac{\partial T}{\partial x}\end{pmatrix},F_v=\begin{pmatrix}0\\ \sigma_{xy}\\ \sigma_{yy}\\ u\sigma_{xy}+v\sigma_{yy}-\kappa\frac{\partial T}{\partial y}\end{pmatrix} \tag{9}$$

In the given terms, p is the pressure, $\sigma_{x_ix_i}$ are the viscous terms. The given variables have the following relations:

The pressure p, temperature T, and terms H, E follows:

$$p=\rho RT \tag{10}$$

$$\rho H=\rho E+p \tag{11}$$

$$E=\frac{R}{\gamma-1} \tag{12}$$

Where $\gamma = \rho g$ and R is the thermal constant.

The viscous term $\sigma_{x_i x_i}$ takes the form:

$$\sigma_{x_i x_i} = \mu\left(\frac{\partial u_i}{\partial x_i}\right) \tag{13}$$

Where

$$\begin{cases} \sigma_{xx} = \mu\left(\dfrac{4}{3}\dfrac{\partial u}{\partial x} - \dfrac{2}{3}\dfrac{\partial u}{\partial y}\right) \\ \sigma_{yy} = \mu\left(\dfrac{4}{3}\dfrac{\partial v}{\partial x} - \dfrac{2}{3}\dfrac{\partial v}{\partial y}\right) \\ \sigma_{xy} = \sigma_{yx} = \mu\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) \end{cases} \tag{13}$$

In fluid mechanics, the constants have the following relations:

The sound velocity:

$$c = \sqrt{\frac{\gamma P}{\rho}} \tag{14}$$

The Reynold's number:

$$Re = \frac{\rho u l}{\mu} \tag{15}$$

The thermal constant:

$$\kappa = \frac{\gamma R}{Pr(\gamma - 1)}\mu \tag{16}$$

Where

$$\mu = \frac{M}{Re}\sqrt{\gamma} \tag{17}$$

In which Re, Pr, M is the Reynold's number, Prandtl number and Mach number, respectively.

The constants relations showed from Eq. 14 to 17 will be further applied on the nondimensionalization for the equations to be given in Chap. 2.

References

[1] Yuxin Ren. Computational Fluid Dynamics (Basic). Department of Engineering Mechanics, Tsinghua University. 2003.

[2] Eleuterio F. Toro. Riemann Solvers and Numerical Methods for Fluid Dynamics. Springer-Verlag Berlin Heidelberg 2009. ISBN 978-3-540-25202-3. DOI 10.1007/978-3-540-49834-6.

[3] Antony Jameson. Advanced Computational Fluid Dynamics AA215A Lecture 4. Winter Quarter, 2012, Stanford, CA.

X. -Q. Yang. **Computational Fluid Dynamics** (2020).

Full notes series could be found at http://hanfengzhai.net/categories/note

CHAPTER 2. Nondimensionalization for Variables

© Hanfeng Zhai

School of Mechanics and Engineering Science, Shanghai University

Shanghai 200444, China

Abstract

Introduce the nondimensionalization methods for the equations involved in computational fluid dynamics. Provide an example with different reference variables on Euler equation to show basic strategies of nondimensionalization.

To solve the governing equations as given in Chap. 1, magnitude of variables needed to be considered. In storing the variables for calculations, the last few decimal points are neglected when magnitude doesn't match. Hence, nondimensionalization of variables in equations is important for computations for accurate results.

For nondimensionalizations, here we adopt the Navier-Stokes equation in 1D situation neglecting the spatial term as an example:

$$\frac{\partial U}{\partial t} - \frac{\partial E_v}{\partial x} = 0 \tag{1}$$

Where

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix}, E = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u H \end{pmatrix} \tag{2}$$

In coordinate system, Eq. 1 can be expanded to the form:

$$\frac{\partial \rho u}{\partial t} - \mu \frac{\partial^2 u}{\partial x^2} = 0 \tag{3}$$

Recalled from Chap. 1, The pressure p, temperature T, and terms H, E follows:

$$\begin{cases} p = \rho R T \\ \rho H = \rho E + p \\ E = \dfrac{R}{\gamma - 1} \end{cases} \tag{4}$$

Hence, we obtains:

$$\rho E = \frac{pR}{\gamma - 1} + \frac{\rho}{2}(u^2 + v^2) \tag{5}$$

$$\Rightarrow \rho = (\gamma - 1)\left[\rho E - \frac{1}{2}\rho(u^2 + v^2)\right] \tag{6}$$

Here, we chose the reference variables for nondimensionalization:

$$\begin{cases} [\rho] = \rho_\infty = 1 \\ [p] = p_\infty = 1 \\ [T] = T_\infty = 1 \\ [l] = l_\infty = 1 \end{cases} \tag{7}$$

Therefore, the time and velocity can be obtained through calculation as:

$$\begin{cases} [t] = \dfrac{[x]}{[u]} = \dfrac{l}{\sqrt{\dfrac{p_\infty}{\rho_\infty}}} \\ u_\infty = c = \sqrt{\dfrac{\gamma p}{\rho}} \end{cases} \tag{8}$$

Hence, all the terms involved in the equation can be nondimensionalized to the following forms:

$$\begin{cases} \rho = \rho'[\rho] = \rho'\rho_\infty \\ p = p'[p] = p'p_\infty \\ T = T'[T] = T'T_\infty \\ x = x'[l] = x'l_\infty \\ \mu = \mu'[\mu] = \mu'\mu_\infty \\ u = u'[u] = u'[c] = u'\sqrt{\dfrac{p_\infty}{\rho_\infty}} \\ t = t'[t] = t'\dfrac{l_\infty}{u} = t'\dfrac{l_\infty}{\sqrt{\dfrac{p_\infty}{\rho_\infty}}} \end{cases} \tag{9}$$

Substituting Eq. 9 into Eq. 3, we obtains:

$$\frac{\partial \rho' u'}{\partial t'} + \frac{\mu_\infty}{\rho_\infty l_\infty \sqrt{\dfrac{p_\infty}{\rho_\infty}}} \frac{\partial^2 u'}{\partial x'^2} = 0 \tag{10}$$

Due to $Re = \frac{\rho_\infty u_\infty l_\infty}{\mu_\infty}$ and $M = \frac{u_\infty}{c} = u_\infty \sqrt{\frac{\rho_\infty}{\gamma p_\infty}}$, the nondimensionalized equation can be written in the form:

$$\frac{\partial \rho' u'}{\partial t'} + \frac{M\sqrt{\gamma}}{Re} \frac{\partial^2 u'}{\partial x'^2} = 0 \tag{11}$$

References

[1] Yuxin Ren. Computational Fluid Dynamics (Basic). Department of Engineering Mechanics, Tsinghua University. 2003.

[2] Eleuterio F. Toro. Riemann Solvers and Numerical Methods for Fluid Dynamics. Springer-Verlag Berlin Heidelberg 2009. ISBN 978-3-540-25202-3. DOI 10.1007/978-3-540-49834-6.

[3] Antony Jameson. Advanced Computational Fluid Dynamics AA215A Lecture 4. Winter Quarter, 2012, Stanford, CA.

X. -Q. Yang. **Computational Fluid Dynamics** (2020).

Full notes series could be found at http://hanfengzhai.net/categories/note

CHAPTER 3. Finite Volume Method

© Hanfeng Zhai

School of Mechanics and Engineering Science, Shanghai University

Shanghai 200444, China

Abstract

Provide the reasoning of finite volume method based on the integration to discretize the Euler equation. We first integrate the Euler equation and hence simplify each term of U and H separately. We therefore give the full term to discretize the Euler equation.

We first give the Euler equation as the controlling equation:

$$\frac{\partial U}{\partial t}+\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}=0 \tag{1}$$

Where

$$U=\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ \rho E\end{pmatrix}, E=\begin{pmatrix}\rho u\\ \rho u^2+P\\ \rho uv\\ \rho vu\end{pmatrix}, F=\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+P\\ \rho vu\end{pmatrix} \tag{2}$$

We hence integral the Euler equation, as visualized in Fig. 1:

$$\int\left(\frac{\partial U}{\partial t}+\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}\right)dS=0 \tag{3}$$

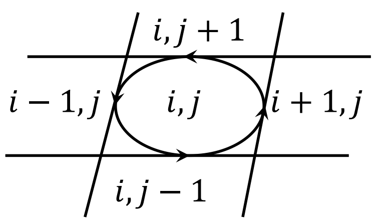

Fig. 1 The integral area of the finite volume method based on single mesh element.

Eq. 3 can be written as:

$$\int\left(\frac{\partial U}{\partial t}\right)dS+\int\left(\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}\right)dS=0 \tag{4}$$

Now we define the term *H*:

$$\vec{H}=E\vec{\imath}+F\vec{\jmath} \tag{5}$$

With the Green-Gauss integration, the right term in Eq. 4 can be written as:

$$\int\vec{v}\cdot\vec{H}d\Omega=\oint\vec{H}d\vec{S} \tag{6}$$

$$=\sum_{i=1}^{\Delta}H_i n_i\,\Delta S_i \tag{7}$$

Based on the Taylor expansion, the term *U* can be written as:

$$U = U_c + \frac{\partial U}{\partial x}(x - x_c) + \frac{\partial U}{\partial y}(y - y_c) + O(\Delta^2) \tag{8}$$

Hence, the mean of the term *U* can be obtained through integration:

$$\int \frac{U d\Omega}{d\Omega} = \int U_c d\Omega \tag{9}$$

$$= \frac{U d\Omega}{d\Omega} + \int \frac{\partial U}{\partial x}(x - x_c) d\Omega \tag{10}$$

Here we define the mean of *U* as:

$$\int \frac{U_c d\Omega}{d\Omega} = \bar{U} \tag{11}$$

The term $U = U(x, t)$ can be decomposed as the separation of variation:

$$U = \sum_{i=1}^{N} U_i(t) b(x_i) \tag{12}$$

Thence, the term *U* can be considered:

$$U(x, t) \cong \bar{U}(t) \tag{13}$$

The integration of the first term is written as:

$$\int \frac{\partial U}{\partial t} d\Omega = \int \frac{dU}{dt} d\Omega = \frac{dU}{dt} \Omega \tag{14}$$

Hence, Euler equation can be written as:

$$\frac{dU}{dt} \Omega + \oint \vec{H} \cdot \vec{n} \cdot dS = 0 \tag{15}$$

The right term can be discretized as the following form.

$$\oint \vec{H} \cdot \vec{n} \cdot dS = \bar{H}$$

$$= H_{i+\frac{1}{2},j} - H_{i-\frac{1}{2},j} + H_{i,j+\frac{1}{2}} - H_{i,j-\frac{1}{2}} \tag{16}$$

We therefore define the term in Eq. 16 as *RHS*.

$$RHS = H_{i+\frac{1}{2},j} - H_{i-\frac{1}{2},j} + H_{i,j+\frac{1}{2}} - H_{i,j-\frac{1}{2}} \tag{17}$$

The discretized form of the Euler equation is given as the form.

$$\frac{dU}{dt} \Omega + RHS = 0 \tag{18}$$

APPENDIX. Jacobian Matrix

The Jacobian matrix A as formerly introduced in Chap. 1 can be further diagonalized for obtaining the eigenvalue λ. Here we show how the A matrix and the eigen value is derived.

We first give the term H based on Eq. 5:

$$H = \vec{H} \cdot \vec{n}$$

$$= \begin{pmatrix} \rho g \\ \rho u g + P n_x \\ \rho v g + P n_y \\ \rho H g \end{pmatrix} \tag{19}$$

Where

$$\rho H = \rho E + P \tag{20}$$

Hence, we deduce that the Jacobian matrix A can be written as:

$$A = \frac{\partial H}{\partial U}$$

$$= \begin{pmatrix} g & 0 & 0 & 0 \\ \frac{P n_x + \rho g u}{r} & 0 & 0 & 0 \\ \frac{P n_y + \rho g v}{r} & 0 & 0 & 0 \\ \frac{P g + \rho E g}{r} & 0 & 0 & 0 \end{pmatrix} \tag{21}$$

Therefore, matrix A is written as:

$$A = \frac{\partial E}{\partial u} n_x + \frac{\partial E}{\partial v} n_y \tag{22}$$

The eigenvalues of matrix A are

$$\begin{cases} \lambda_1 = q = u n_x + v n_y \\ \lambda_2 = q = u n_x + v n_y \\ \lambda_3 = q + c = u n_x + v n_y + c \\ \lambda_4 = q - c = u n_x + v n_y + c \end{cases} \tag{23}$$

REFERENCES

[1] Yuxin Ren. Computational Fluid Dynamics (Basic). Department of Engineering Mechanics, Tsinghua University. 2003.

[2] Eleuterio F. Toro. Riemann Solvers and Numerical Methods for Fluid Dynamics. Springer-Verlag Berlin Heidelberg 2009. ISBN 978-3-540-25202-3. DOI 10.1007/978-3-540-49834-6.

[3] Antony Jameson. Advanced Computational Fluid Dynamics AA215A Lecture 4. Winter Quarter, 2012, Stanford, CA.

X. -Q. Yang. **Computational Fluid Dynamics** (2020).

Full notes series could be found at http://hanfengzhai.net/categories/note

CHAPTER 4. Finite Difference Method

© Hanfeng Zhai

School of Mechanics and Engineering Science, Shanghai University

Shanghai 200444, China

Abstract

Provide the reasoning of finite difference method (FDM) from discretize the derivative of the flux. A continued method for discretization of unequal distance meshing is provided based on FDM from the meshing transformation as to be introduced in Chap. 7.

We first give the Euler equation as in 1D situation:

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} = 0 \tag{1}$$

For the 1D situation, the boundary conditions and the governing equation can be written in combined as in Eq. 2, as visualized in 0:

$$\frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} = 0 \tag{2}$$

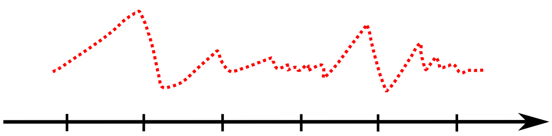

Fig. 1 Schematic for a 1D shock wave equation.

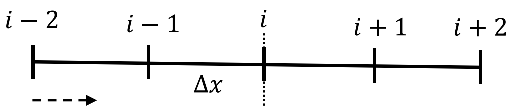

Fig. 2 Schematic for spatial discretization in a row.

As shown in Fig. 2, the spatial term in x direction can be expanded as Taylor series:

$$U_{i-1} = U_i - \frac{\partial U}{\partial x_i}\Delta x + O(\Delta^2) + \cdots \tag{3}$$

$$U_{i+1} = U_i + \frac{\partial U}{\partial x_i}\Delta x + O(\Delta^2) + \cdots \tag{4}$$

Subtracting Eq. 4 to Eq. 3, one obtains:

$$U_{i+1} - U_{i-1} = 2\frac{\partial U}{\partial x_i}\Delta x \tag{5}$$

Hence, Eq. 5 can be reduced to:

$$\left.\frac{\partial U}{\partial x}\right|_i = \frac{U_{i+1} - U_{i-1}}{2\Delta x} \tag{6}$$

Which the discretization method is nominated as the central difference method (CDS).

Based on Eq. 3, we can write:

$$U_{i+1} \approx U_i + \frac{\partial U}{\partial x_i}\Delta x \tag{7}$$

Subtracting Eq. 7 to U_i one obtains:

$$U_{i+1} - U_i = \frac{\partial U}{\partial x_i}\Delta x \tag{8}$$

Hence, Eq. 7 can be reduced to:

$$\left.\frac{\partial U}{\partial x}\right|_i = \frac{U_{i+1} - U_i}{\Delta x} \tag{9}$$

Which the discretization is called the front difference scheme (FDS).

Referred from what is shown form Eq. 7 to 8, we obtains:

$$U_{i-1} = U_i - \frac{\partial U}{\partial x_i}\Delta x \tag{10}$$

Subtracting Eq. 10 to U_i one obtains:

$$U_i - U_{i-1} = \frac{U_i - U_{i-1}}{\Delta x} \tag{11}$$

Therefore, the derivative takes the form

$$\left.\frac{\partial U}{\partial x}\right|_i = \frac{U_i - U_{i-1}}{\Delta x} \tag{12}$$

Which is called the back-difference scheme (BDS).

For spatial discretization, we have

$$\left.\frac{\partial U}{\partial x}\right|_i = \begin{cases} \dfrac{U_{i+1} - U_i}{\Delta x} \rightarrow \text{FDS} \\ \dfrac{U_i - U_{i-1}}{\Delta x} \rightarrow \text{BDS} \\ \dfrac{U_{i+1} - U_{i-1}}{2\Delta x} \rightarrow \text{CDS} \end{cases} \tag{13}$$

For discretization of time derivatives, we apply similar strategies:

$$\left.\frac{dU}{dt}\right|_i = \begin{cases} \dfrac{U^{n+1} - U^n}{\Delta x} \rightarrow \text{FDS} \\ \dfrac{U^n - U^{n-1}}{\Delta x} \rightarrow \text{BDS} \\ \dfrac{U^{n+1} - U^{n-1}}{2\Delta x} \rightarrow \text{CDS} \end{cases} \tag{14}$$

Here we give the artificial dissipative term as to be discussed in Chap. 6 in spatial discretization based on CDS:

$$\frac{U_{i+1} - U_{i-1}}{2\Delta x} + \kappa_1 \frac{\partial^2 u}{\partial x^2} - \kappa_2 \frac{\partial^4 u}{\partial t^4} = 0 \tag{15}$$

Also, for unequal distance meshing as to be introduced in Chap. 7 as visualized in Fig. 3, we can discretize the term $\frac{\partial U}{\partial x}$ in the forms:

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial \zeta}\frac{\partial \zeta}{\partial x} \tag{16}$$

$$= \frac{U^{\zeta+1} - U^{\zeta}}{\Delta \zeta}\frac{\partial \zeta}{\partial x} \tag{17}$$

$$= \frac{U^{i+1} - U^{i}}{\Delta x} \tag{18}$$

Where $\zeta = \frac{1}{\Delta x}$.

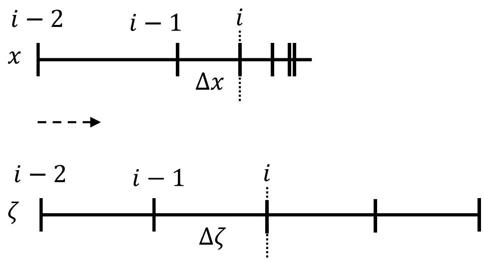

Fig. 3 Schematic for the unequal distance meshing in a row.

Here, we expand the Euler equation as in 2D situation:

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = 0 \tag{19}$$

Where

$$\begin{cases} \frac{\partial E}{\partial x} = \frac{\partial E}{\partial \zeta}\zeta_x + \frac{\partial E}{\partial \eta}\eta_x \\ \frac{\partial F}{\partial y} = \frac{\partial F}{\partial \zeta}\zeta_y + \frac{\partial F}{\partial \eta}\eta_y \end{cases} \tag{20}$$

Hence, the 2D Euler equation takes the form:

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial \zeta}\zeta_x + \frac{\partial E}{\partial \eta}\eta_x + \frac{\partial F}{\partial \zeta}\zeta_y + \frac{\partial F}{\partial \eta}\eta_y = 0 \tag{21}$$

REFERENCES

[1] Yuxin Ren. Computational Fluid Dynamics (Basic). Department of Engineering Mechanics, Tsinghua University. 2003.

[2] Eleuterio F. Toro. Riemann Solvers and Numerical Methods for Fluid Dynamics. Springer-Verlag Berlin Heidelberg 2009. ISBN 978-3-540-25202-3. DOI 10.1007/978-3-540-49834-6.

[3] Antony Jameson. Advanced Computational Fluid Dynamics AA215A Lecture 4. Winter Quarter, 2012, Stanford, CA.

X. -Q. Yang. **Computational Fluid Dynamics** (2020).

Full notes series could be found at http://hanfengzhai.net/categories/note

CHAPTER 5. Boundary Conditions

© Hanfeng Zhai

School of Mechanics and Engineering Science, Shanghai University

Shanghai 200444, China

Abstract

Introduce the basic strategy to analyze and give the boundary conditions in different scenarios based on an 1D non-viscous situation (Euler equation). Introduce the boundary conditions in three different scenarios.

In solving mechanics problems, whether involved in fluid or solid, boundary conditions (BCs) are always considered one of the most important factors. In computational fluid dynamics (CFD), slightly difference in BCs may strongly variate the calculation results. Therefore, choosing the right BCs for whether specific engineering problems or research works are critical and must be scrutinized. Here, we introduce three basic BCs model commonly encountered in CFD.

We first give the Euler equation as in 1D situation:

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} = 0 \tag{1}$$

For the 1D situation, the boundary conditions and the governing equation can be written in combined as in Eq. 2, as recalled from Chap. 4, can be visualized in Fig. 1:

$$\begin{cases} \frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} = 0 \\ U = U_0 \\ U_{BC} = C \end{cases} \tag{2}$$

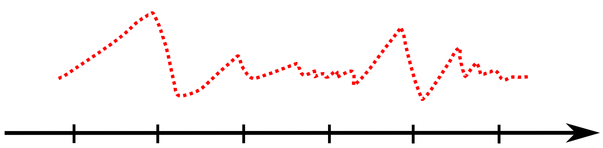

Fig. 1 Schematic for a 1D shock wave equation.

Recall the flux H as referred form Eq. 7 and Eq. 11 in Chap. 1, we have:

$$H = \begin{pmatrix} \rho q \\ \rho u q + p n_x \\ \rho v q + p n_y \\ \rho H q \end{pmatrix} \tag{3}$$

Where $q = u n_x + v n_y$.

Here, we introduce the three basic scenarios for different boundary conditions:

I. *No-slip wall*

The no-slip wall BCs is the most commonly encountered BCs when the fluid-solid interactions are involved. Here, Fig. 2 visualize a typical model with meshing on of the no-slip BCs, which shows how the continuum pressure values is discretized nearing the wall boundary when the normal vector is parallel to the pressure variation.

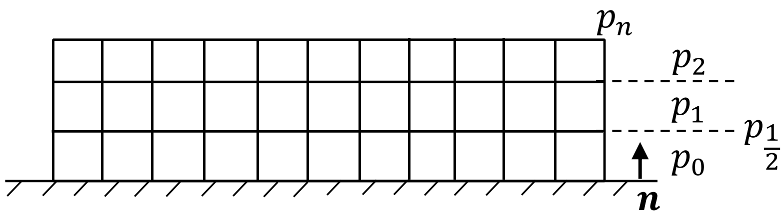

Fig. 2 Schematic for the no-slip boundary conditions.

For no-slip BCs, we have $q = 0$, thence the flux takes the form:

$$H = \begin{pmatrix} 0 \\ pn_x \\ pn_y \\ 0 \end{pmatrix} \tag{4}$$

The velocities on the wall obeys:

$$\begin{cases} u_{wall} = 0 \\ v_{wall} = 0 \end{cases} \tag{5}$$

The pressure on the wall obeys:

$$P_{wall} = P_L \tag{6}$$

Here we provide a coding example (FORTRAN) on giving the no-slip wall boundary conditions. Where the mesh generation of a 2D situation applied on a no-slip BCs is shown.

```
FOR J = JL, i = 1, IL
    FOR I = 1, j = 1, JL
        FOR = 1L, j = 1, JL, J = JL
              CUT J = 1, I = 1, IOI
              CUT J = 1, I = IOI, LL
        END
    END
END
```

II. *Symmetry wall*

Symmetry wall BCs are widely encountered for symmetric geometries including the NACA "00 series" airfoils, bullets, ships, etc. Here in Fig. 3 we show a schematic of the symmetry BCs with $\boldsymbol{V} = \binom{u_1}{v_1}$ as the velocity vector and $\boldsymbol{n}$ as the position vector.

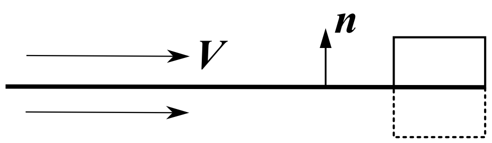

Fig. 3 Schematic for the symmetry boundary conditions.

For symmetry situation, the velocities follows:

$$u_{wall}n_x + v_{wall}n_y = 0 \tag{7}$$

Where the velocities in the two directions obeys

$$u_{wall} = u_1 - (u_1 n_x + v_1 n_y) \cdot n_x \tag{8}$$

$$v_{wall} = v_1 - (u_1 n_x + v_1 n_y) \cdot n_y \tag{9}$$

The flux on the boundary takes the form

$$H = \begin{pmatrix} 0 \\ pn_x \\ pn_y \\ 0 \end{pmatrix} \tag{10}$$

The pressure obeys:

$$P_{wall} = P_L \tag{11}$$

III. *Far field*

Far field BCs are widely applied on noise calculation, thermal estimation and other problems common in aerodynamics. Here in Fig. 4 we show how a typical far field BCs is adopted based on an airfoil meshing, where mostly applied on noise calculations. Usually, for aerodynamic and fluid mechanics, the flow field is chosen to be much larger than the targeted structure to obey the far field BCs.

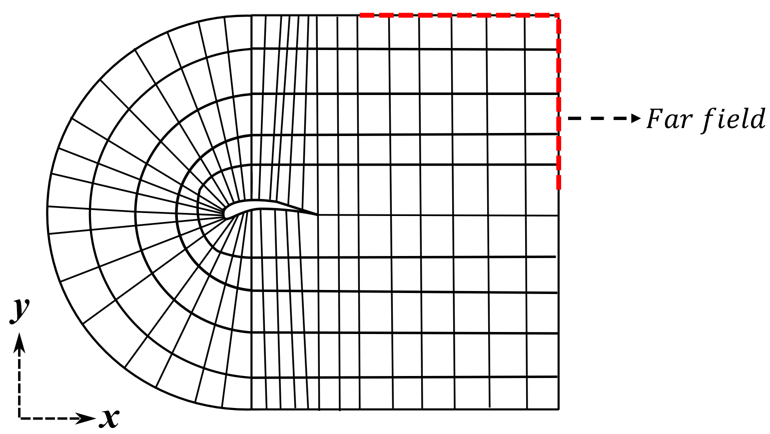

Fig. 4 Schematic for the far field boundary conditions, with a structural meshing of an airfoil.

The far field BCs involves the following situations:

i. For $Ma \leq -1$:

When the Mach number obeys $Ma \leq -1$, we refer the situation as "supersonic outlet", where the flux U can be written as:

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}_1 = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}_2 = \begin{pmatrix} \rho_\infty \\ \rho u_\infty \\ \rho v_\infty \\ \rho E_\infty \end{pmatrix} \tag{12}$$

Hence the flux H can be reasoned:

$$\Rightarrow H = \begin{pmatrix} \rho_\infty q_\infty \\ \rho u_\infty q_\infty + p_\infty n_x \\ \rho v_\infty q_\infty + p_\infty n_y \\ \rho H_\infty q_\infty \end{pmatrix} \tag{13}$$

ii. For $-1 < Ma < 0$:

The BCs obeys $-1 < Ma < 0$ are considered as "subsonic inlet", where the flux H can also be reasoned through the flux U:

$$\begin{pmatrix} \rho \\ \rho u_1 \\ \rho v_1 \\ \rho E_1 \end{pmatrix} = \begin{pmatrix} \rho_\infty \\ \rho u_\infty \\ \rho v_\infty \\ \rho E_\infty \end{pmatrix} \Rightarrow H = F(u_1) + F(u_2) + D_1 + \cdots \tag{14}$$

iii. For $0 < Ma < 1$:

The BCs is considered "subsonic outlet", where the flux takes the form:

$$\begin{pmatrix} \rho \\ \rho u_1 \\ \rho v_1 \end{pmatrix} = \begin{pmatrix} \rho_\infty \\ \rho u_\infty \\ \rho v_\infty \end{pmatrix} \Rightarrow \rho E_1 = \frac{p_\infty}{\gamma - 1} + \frac{\rho_2}{2}({u_2}^2 + {v_2}^2) \tag{15}$$

Hence the flux H is reasoned:

$$\Rightarrow H = \begin{pmatrix} \rho_2 q_2 \\ \rho u_2 q_2 + p_2 n_x \\ \rho v_2 q_2 + p_2 n_y \\ \rho E_2 + p_\infty \end{pmatrix} \tag{16}$$

iv. For $Ma > 1$:

The BCs $Ma > 1$ is the "supersonic outlet", where the flux H can be reasoned:

$$\begin{pmatrix} \rho \\ \rho u_1 \\ \rho v_1 \\ \rho E_1 \end{pmatrix} = \begin{pmatrix} \rho \\ \rho u_2 \\ \rho v_2 \\ \rho E_2 \end{pmatrix} \Rightarrow H = \begin{pmatrix} \rho_2 q_2 \\ \rho u_2 q_2 + p_2 n_x \\ \rho v_2 q_2 + p_2 n_y \\ \rho E_2 \end{pmatrix} \tag{17}$$

Where

$$\rho E = \frac{p_\infty}{\gamma - 1} + \frac{\rho}{2} V^2 \tag{18}$$

and

$$\rho H = \rho E + p \tag{19}$$

References

[1] Yuxin Ren. Computational Fluid Dynamics (Basic). Department of Engineering Mechanics, Tsinghua University. 2003.

[2] Eleuterio F. Toro. Riemann Solvers and Numerical Methods for Fluid Dynamics. Springer-Verlag Berlin Heidelberg 2009. ISBN 978-3-540-25202-3. DOI 10.1007/978-3-540-49834-6.

[3] Antony Jameson. Advanced Computational Fluid Dynamics AA215A Lecture 4. Winter Quarter, 2012, Stanford, CA.

X. -Q. Yang. **Computational Fluid Dynamics** (2020).

Full notes series could be found at http://hanfengzhai.net/categories/note

CHAPTER 6. Artificial Dissipative Terms

© Hanfeng Zhai

School of Mechanics and Engineering Science, Shanghai University

Shanghai 200444, China

Abstract

The artificial dissipative term was raised by A. Jameson in applying Runge-Kutta method solving Euler equations by finite volume method [Jameson et al., 1981]. Here we show how the artificial dissipative term is derived from the flux H.

We first start with the Euler equation:

$$\frac{\partial U}{\partial t}+\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}=0 \tag{1}$$

We apply the finite volume method as given in Chap. 3 on the equation and obtains

$$\int\left(\frac{\partial U}{\partial t}\right)dS+\int\left(\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}\right)dS=0 \tag{2}$$

The second term can be transformed through the Green-Gauss transformation:

$$\int\left(\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}\right)dS=\oint_{\partial\Omega}\left(En_x+Fn_y\right)dS \tag{3}$$

Referred from Chap. 3, the term in Eq. 3 is considered as RHS, taking the form:

$$RHS=H_{i+\frac{1}{2},j}-H_{i-\frac{1}{2},j}+H_{i,j+\frac{1}{2}}-H_{i,j-\frac{1}{2}} \tag{4}$$

Referred from Eq. 18 which is given in Chap. 3, the form of Euler equation that discretized by the finite volume method can be written as:

$$\frac{dU}{dt}\Omega+RHS=0 \tag{5}$$

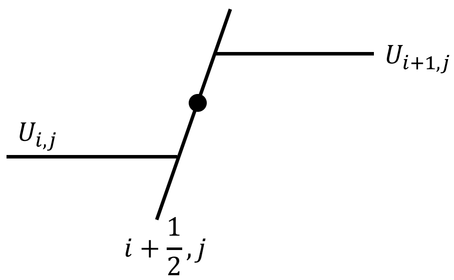

Fig. 1 Discretization on the boundary.

The timing step can be discretized as:

$$U_{i+\frac{1}{2},j}=\frac{1}{2}\left(U_{i+1,j}+U_{i,j}\right) \tag{6}$$

Based on the discretization on the boundary surface, and as referred from Eq. 4, the discretization of the flux H on the boundary surface can be written as:

$$H_{i+\frac{1}{2},j} = \frac{1}{2}\left(H_{i+1,j} + H_{i,j}\right) - D \tag{7}$$

Where D is the artificial dissipative term.

The artificial dissipative term $D = D_{i,j}$ is derived from the discretization of the boundary surface flux $H_{i+\frac{1}{2},j}$. Here, the term $D_{i,j}$ takes the form:

$$D_{i,j} = D_{i+\frac{1}{2},j} - D_{i-\frac{1}{2},j} + D_{i,j+\frac{1}{2}} - D_{i,j-\frac{1}{2}} \tag{8}$$

Where the artificial dissipative term on the boundary surface can be discretized as:

$$D_{i+\frac{1}{2},j} = \frac{H_{i+\frac{1}{2},j}}{\Delta t}\left(\varepsilon^{(2)}_{i+\frac{1}{2},j}\left(U_{i+1,j} - U_{i,j}\right) - \varepsilon^{(4)}_{i+\frac{1}{2},j}\left(U_{i+2,j} - 3U_{i+1,j} + 3U_{i,j} - U_{i-1,j}\right)\right) \tag{9}$$

Where

$$\varepsilon^{(2)}_{i+\frac{1}{2},j} = \kappa^{(2)} max\left(v^{(2)}_{i+1}, v^{(2)}_{j}\right) \tag{10}$$

$$\varepsilon^{(4)}_{i+\frac{1}{2},j} = max\left(0, \left(\kappa^{(4)} - \varepsilon^{(2)}_{i+\frac{1}{2},j}\right)\right) \tag{11}$$

Hence, based on Eq. 7, and the discretization of the artificial dissipative term on the boundary surface shown from Eq. 8 and 9, we deduce that

$$H_{i+\frac{1}{2},j} = \frac{1}{2}\left(H_{i,j} - H_{i+1,j}\right) - \frac{1}{2} D_{i+\frac{1}{2},j} \tag{9}$$

References

[1] Antony Jameson. Numerical Solution of the Euler Equations by Finite Volume Methods Using Runge-Kutta Time-Stepping Schemes. AIAA 14th Fluid and Plasma Dynamic Conference. June 23–25, 1981. Palo Alto, California.

[2] Yuxin Ren. Computational Fluid Dynamics (Basic). Department of Engineering Mechanics, Tsinghua University. 2003.

[3] Eleuterio F. Toro. Riemann Solvers and Numerical Methods for Fluid Dynamics. Springer-Verlag Berlin Heidelberg 2009. ISBN 978-3-540-25202-3. DOI 10.1007/978-3-540-49834-6.

[4] Antony Jameson. Advanced Computational Fluid Dynamics AA215A Lecture 4. Winter Quarter, 2012, Stanford, CA.

X. -Q. Yang. **Computational Fluid Dynamics** (2020).

Full notes series could be found at http://hanfengzhai.net/categories/note

CHAPTER 7. Meshing Transformation & Viscosity Discretization

© Hanfeng Zhai

School of Mechanics and Engineering Science, Shanghai University

Shanghai 200444, China

Abstract

In meshing objects with structural meshes such as meshing of airfoils, we need to transform the meshing to the coordinate system. The transformation process of the meshing coordinates is provided. The methods for discretization of the viscosity is hence given.

We first recall the viscous term as gives in the Navier-Stokes equation in Chap. 1:

$$\sigma_{x_i x_i} = \mu \left(\frac{\partial u_i}{\partial x_i} \right) \tag{1}$$

Here, the derivative term takes the form based on the Green-Gauss integration:

$$\frac{\partial u}{\partial x} = \frac{\int \frac{\partial u}{\partial x} d\Omega}{\int d\Omega} = \frac{1}{\Omega} \oint \vec{u} \cdot \vec{n} dS \tag{2}$$

$$= \frac{1}{\Omega} u_{i+\frac{1}{2},j} n_{i+\frac{1}{2},j} \Delta S \tag{3}$$

Hence, the partial derivatives of u can be discretized taking the forms:

$$\left. \frac{\partial u}{\partial x} \right|_{i,j} = \frac{1}{\Omega_{i,j}} \left(u_{s_{i+\frac{1}{2},j}} - u_{s_{i-\frac{1}{2},j}} + u_{s_{i,j+\frac{1}{2}}} - u_{s_{i,j-\frac{1}{2}}} \right) \tag{4}$$

Substituting the relation as given in Eq. 3, we deduce

$$\left. \frac{\partial u}{\partial x} \right|_{i,j} = \frac{1}{2\Omega_{i,j}} \left(u_{i+1,j} + u_{i,j} \right) n_{i+\frac{1}{2},j} S_{i+\frac{1}{2},j} + \left(u_{i,j+1} + u_{i,j} \right) n_{i,j+\frac{1}{2}} S_{i,j+\frac{1}{2}} \tag{5}$$
$$- \left(u_{i-1,j} + u_{i,j} \right) n_{i-\frac{1}{2},j} S_{i-\frac{1}{2},j} - \left(u_{i,j} + u_{i,j-1} \right) n_{i,j-\frac{1}{2}} S_{i,j-\frac{1}{2}}$$

Here, the meshing elements in a row can be visualized as shown in Fig. 1. The structural mesh are shown in the *x*-*y* coordinates (adapted mesh). The transformed mesh in the coordinate system is the *ζ*-*η* system as shown in the down view.

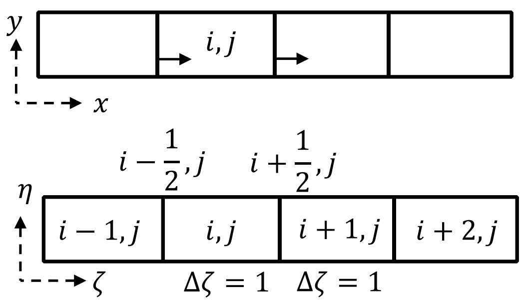

Fig. 1 Schematic for meshing elements in a row.

The derivatives on the top side can be transformed to the coordinate system as written:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \zeta}\zeta_x + \frac{\partial u}{\partial \eta}\eta_x \tag{6}$$

In which the term $\frac{\partial u}{\partial \zeta}$ can be written as

$$\frac{\partial u}{\partial \zeta} = \left(u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}\right) \tag{7}$$

Where the two terms on the right side can be derived from the down view in Fig. 1:

$$u_{i+\frac{1}{2},j} = \frac{u_{i+1,j} - u_{i,j}}{2} \tag{8}$$

$$u_{i-\frac{1}{2},j} = \frac{u_{i,j} - u_{i-1,j}}{2}$$

Similarly, the component in the η can also be decomposed as:

$$\frac{\partial u}{\partial \eta} = \left(u_{i,j+\frac{1}{2}} - u_{i,j-\frac{1}{2}}\right) \tag{9}$$

Where the two terms follows

$$u_{i,j+\frac{1}{2}} = \frac{u_{i,j+1} - u_{i,j}}{2} \tag{10}$$

$$u_{i,j-\frac{1}{2}} = \frac{u_{i,j} - u_{i,j-1}}{2}$$

Thence, the term $\frac{\partial u}{\partial x}$ can be discretized as:

$$\left.\frac{\partial u}{\partial x}\right|_{i,j} = \left.\frac{\partial u}{\partial \zeta}\zeta_x\right|_{i+\frac{1}{2},j} - \left.\frac{\partial u}{\partial \eta}\eta_x\right|_{i+\frac{1}{2},j} \tag{11}$$

$$= \frac{1}{4}\left(\left.\frac{\partial u}{\partial x}\right|_{i+\frac{1}{2},j} + \left.\frac{\partial u}{\partial x}\right|_{i-\frac{1}{2},j} + \left.\frac{\partial u}{\partial x}\right|_{i,j+\frac{1}{2}} + \left.\frac{\partial u}{\partial x}\right|_{i,j-\frac{1}{2}}\right) \tag{12}$$

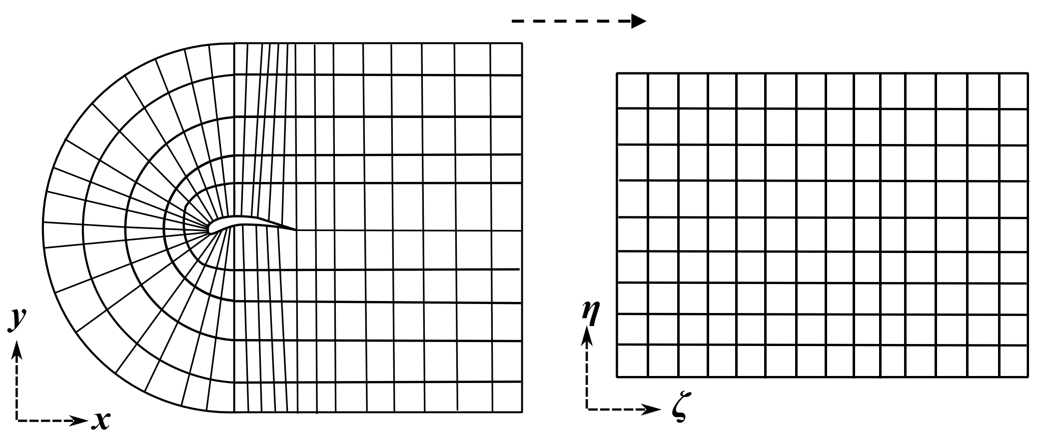

Fig. 2 Schematic for the meshing transformation of the two coordinates with an airfoil.

Here in Fig. 2 we present an example of the meshing transformation based on an airfoil. The

x-y coordinate indicate the original structural meshing corresponds to the coordinate system ζ-η.

With the presented transformation, we can discretize the terms $\vec{H}\cdot\vec{n}$ of the viscosity term as given in Chap. 3 considering a 2D situation:

$$\vec{H}\cdot\vec{n}=\begin{pmatrix} 0 \\ \sigma_{11}n_x+\sigma_{12}n_y \\ \sigma_{12}n_x+\sigma_{22}n_y \\ (u_1\sigma_{11}+u_2\sigma_{12})n_x+(u_1\sigma_{12}+u_2\sigma_{22})n_x+\kappa\left(\frac{\partial T}{\partial x}n_x+\frac{\partial T}{\partial y}n_y\right) \end{pmatrix} \tag{13}$$

Where the derivatives can be transformed through the Green-Gauss integration:

$$\int\frac{\partial T}{\partial x}d\Omega=\oint T\cdot n_x dS \tag{14}$$

$$\int\frac{\partial T}{\partial y}d\Omega=\oint T\cdot n_y dS \tag{15}$$

Referring from Eq. 7 to 8, we can write derivatives

$$\begin{cases} \dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial \zeta}\zeta_x+\dfrac{\partial u}{\partial \eta}\eta_x \\ \dfrac{\partial u}{\partial y}=\dfrac{\partial u}{\partial \zeta}\zeta_y+\dfrac{\partial u}{\partial \eta}\eta_y \end{cases} \tag{16}$$

Here, based on Eq. 13, let us consider the derivative terms on the boundary as shown in Fig. 3:

$$\left.\frac{\partial u}{\partial x}\right|_{i+\frac{1}{2}}=\frac{u_{i+1}-u_i}{\Delta\zeta} \tag{17}$$

$$=(u_{i+1}-u_i)\zeta_{x_{i+\frac{1}{2}}}+(u_{j+1}-u_j)\eta_{x_{j+\frac{1}{2}}} \tag{18}$$

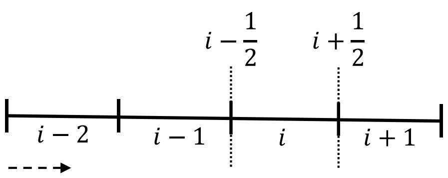

Fig. 3 The meshing points in the row in i-direction.

References

[1] Yuxin Ren. Computational Fluid Dynamics (Basic). Department of Engineering Mechanics, Tsinghua University. 2003.

[2] Eleuterio F. Toro. Riemann Solvers and Numerical Methods for Fluid Dynamics. Springer-Verlag Berlin Heidelberg 2009. ISBN 978-3-540-25202-3. DOI 10.1007/978-3-540-49834-6.

[3] Antony Jameson. Advanced Computational Fluid Dynamics AA215A Lecture 4. Winter Quarter, 2012, Stanford, CA.

X. -Q. Yang. **Computational Fluid Dynamics** (2020).

Full notes series could be found at http://hanfengzhai.net/categories/note

CHAPTER 8. Implicit Difference Scheme

© Hanfeng Zhai

School of Mechanics and Engineering Science, Shanghai University

Shanghai 200444, China

Abstract

Provide an implicit scheme method with a given half differential equation. We first discretize the equation with finite difference; hence elicit the terms D, L, U to linearize the given equation and update the solution w.

Let us consider a simple differential equation:

$$\frac{dw}{dt} = RHS \tag{1}$$

The equation can be discretized with the finite difference method; in the *n+1* moment:

$$\frac{w^{n+1} - w^n}{\Delta t} = RHS^{n+1} \tag{2}$$

In which

$$RHS^{n+1} = RHS^n + \frac{\partial RHS}{\partial w}(w^{n+1} - w^n) + O(\Delta w^2) \tag{3}$$

Let us consider $w^{n+1} - w^n = \Delta w$. Hence, Eq. 3 can be reduced to

$$RHS^{n+1} \approx RHS^n + \left(\frac{\partial RHS}{\partial w}\Delta w\right)^n \tag{4}$$

Eq. 2 can be further written as

$$\frac{\Delta w}{\Delta t} - \frac{\partial RHS}{\partial w}\Delta w = RHS^n \tag{5}$$

Which can be reduced to

$$\left(\frac{I}{\Delta t} - \frac{\partial RHS}{\partial w}\right)\Delta w = RHS^n \tag{6}$$

Now, we can consider Eq. 6 as a simple linear equation form, which is written as:

$$A\Delta w = RHS^n \tag{7}$$

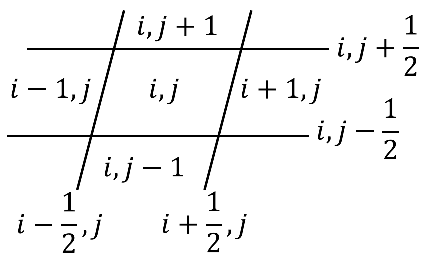

Fig. 1 Schematic for the single mesh element.

From Fig. 1, the term *RHS* can be decomposed as:

$$(RHS)_{ij} = H_{i+\frac{1}{2},j} - H_{i-\frac{1}{2},j} + H_{i,j+\frac{1}{2}} - H_{i,j-\frac{1}{2}} \tag{8}$$

In which

$$H_{i+\frac{1}{2},j} = H\left(w_{i,j}, w_{i+1,j}\right) \tag{9}$$

$$H_{i-\frac{1}{2},j} = H\left(w_{i,j}, w_{i-1,j}\right) \tag{10}$$

$$H_{i,j+\frac{1}{2}} = H\left(w_{i,j}, w_{i,j+1}\right) \tag{11}$$

$$H_{i,j-\frac{1}{2}} = H\left(w_{i,j}, w_{i,j-1}\right) \tag{12}$$

Here we define three terms from term *RHS* and *w*:

$$D = \frac{\partial RHS_{ij}}{\partial w_{ij}} \tag{13}$$

$$L = \frac{\partial RHS_{ij}}{\partial w_{i-1,j}} + \frac{\partial RHS_{ij}}{\partial w_{i,j-1}} \tag{14}$$

$$U = \frac{\partial RHS_{ij}}{\partial w_{i+1,j}} + \frac{\partial RHS_{ij}}{\partial w_{i,j+1}} \tag{15}$$

Hence the linear equation Eq. 7 could be written as:

$$(L + D + U)\Delta w = \frac{\partial RHS}{\partial w} \tag{16}$$

From Eq. 14 to 15, we obtain:

$$L\Delta w = \frac{\partial RHS_{ij}}{\partial w_{i-1,j}}\Delta w_{i-1,j} + \frac{\partial RHS_{ij}}{\partial w_{i,j-1}}\Delta w_{i,j-1} \tag{17}$$

$$U\Delta w = \frac{\partial RHS_{ij}}{\partial w_{i+1,j}}\Delta w_{i+1,j} + \frac{\partial RHS_{ij}}{\partial w_{i,j+1}}\,\Delta w_{i,j+1} \tag{18}$$

Therefore, we could simplify Eq. 16 as

$$(L + D + U)\Delta w \approx (L + D)D^{-1}(U + D)\,\Delta w \tag{19}$$

The right term in Eq. 19 could be further simplified as:

$$\begin{aligned}(L + D)D^{-1}(U + D)\Delta w &= (L \cdot D^{-1} + I)(U + D)\,\Delta w \\ &= L \cdot D^{-1}U + (L + U + D)\,\Delta w^{-1}\end{aligned} \tag{20}$$

Where $L \cdot D^{-1}U$ could be considered as an infinitesimal of high order.

Substituting Eq. 19 into Eq. 16:

$$(L + D)D^{-1}(U + D)\Delta w = RHS \tag{21}$$

Eq. 21 can be further linearized as:

$$(L + D)\Delta Q = RHS \tag{22}$$

$$D\Delta Q = RHS - L\Delta Q \tag{23}$$

$$\Delta Q = D^{-1}(RHS - L\Delta Q) \tag{24}$$

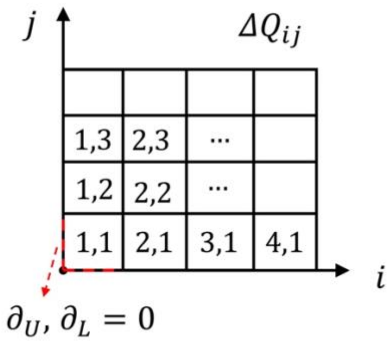

Fig. 2 Schematic for the meshing.

From Fig. 2 we could write the term ΔQ_{ij} as the following terms:

$$\Delta Q_{11} = {D_{11}}^{-1} RHS_{11} \tag{25}$$

$$\Delta Q_{21} = {D_{11}}^{-1}(RHS_{21} - L_{21}\Delta Q_{11}) \tag{26}$$

$$\Delta Q_{12} = {D_{11}}^{-1}(RHS_{12} - L_{12}\Delta Q_{11}) \tag{27}$$

$$\cdots$$

$$\Delta Q_{nm} = \cdots \tag{28}$$

Hence, the full sets of the implicit scheme algorithm could be summarized as:

Step 1:	$\Delta Q = D^{-1}(RHS - L\Delta Q)$
Step 2:	$\Delta Q = (D^{-1}U + I)\Delta w$
	$\Delta w = \Delta Q - D^{-1}U\,\Delta w$
	The solution can be updated as
	↓
Step 3:	$w^{n+1} = w^n + \Delta w$

References

[1] Yuxin Ren. Computational Fluid Dynamics (Basic). Department of Engineering Mechanics, Tsinghua University. 2003.

[2] Eleuterio F. Toro. Riemann Solvers and Numerical Methods for Fluid Dynamics. Springer-Verlag Berlin Heidelberg 2009. ISBN 978-3-540-25202-3. DOI 10.1007/978-3-540-49834-6.

[3] Antony Jameson. Advanced Computational Fluid Dynamics AA215A Lecture 4. Winter Quarter, 2012, Stanford, CA.