Problem Session #1

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Overview

▷ Problem 1.1. recall strong form.

$$\underbrace{-\left(k(x)\,u'(x)\right)' + b(x)\,u'(x) + c(x)\,u(x)}_{\mathcal{L}(u,\,x)} = f(x)$$

$\forall x \in \Omega$ ← domain.

(1) $\mathcal{L}(u, x) - f = \phi$

(2) $\hat{\mathcal{L}}(\hat{u}, x) - f \neq \phi$ ← approximated

Construct residual: $R_\Omega = (2) - \cancel{(1)}^{\,0} \neq \phi$.

Variational formulation → Integrate the residual:

$$\int_\Omega R_\Omega \, v \, d\Omega = \phi$$

v ↳ test functions (weighting func.)

Weak form can be constructed as.

$$\int_{\Omega} R_{\Omega}\, v \, d\Omega + \int_{\Gamma} R_{\Gamma}\, d\Gamma = 0$$

$\downarrow$ Residual over domain

$\downarrow$ Residual over boundaries.

R_{Ω}: linear combinations of basis functions
$\sum_{m=1}^{M} a_m N_m$ } Galerkin method.

$v(x)$: "can be any function of x that is sufficiently well behaved for the integrals to exist."

You may put on constraints on $v(x)$ based on your problems.

$\Downarrow$

You will explore this in your HW 1.

m LaForce
NERGY 281
006.

▸ Boundary Conditions.

- Dirichlet B.C.s. $u(x=a) = g_0$

- Neumann B.C.s $u'(x=b) = d_L$

- Robin B.C.s $u'(x=c) + u(x=c) = \alpha$

Trial space. $\mathcal{S} = \{ w : \Omega \to \mathbb{R} \text{ smooth} \}$.

Test space: $\mathcal{V} = \{ w : \Omega \to \mathbb{R} \text{ smooth} \}$

- trial functions → approximation of the solution.

... represents the sol'n to the problem,

examples. polynomials: $u(x) = a_0 + bx + cx^2 + \cdots$

- test functions → test how well trial function satisfies the governing equations

↳ PDE.

... used to evaluate the error.

<u>Example</u> (1.10)

$f : [a,b] \to \mathbb{R}$. find $u : [a,b] \to \mathbb{R}$ s.t.

$$u''' = f, \qquad x \in (a,b)$$

$$u(a) = 1$$

$$u'(b) = 2$$

$$u''(a) = 3.$$

<u>Solution</u> (Exact)

$$\int_a^x f(y)\,dy = \int_a^x u'''(y)\,dy$$

$$= u''(x) - u''(a) = u''(x) - 3.$$

$$\int_b^x \int_a^z f(y)\,dy\,dz = \int_b^x u''(z) - 3\,dz.$$

$$= u'(x) - u'(b) - 3(x-b)$$

$$= u'(x) - 2 - 3(x-b)$$

$$\int_a^x \int_b^w \int_a^z f(y)\,dy\,dz\,dw = \int_a^x u'(w) - 2 - 3(w-b)\,dw$$

$$= u(x) - \overset{1}{\cancel{u(a)}} - 2(x-a) - \tfrac{3}{2}(x^2 - a^2)$$
$$+ 3b(x-a)$$

Exact solution writes.

$$u(x) = 1 + (2-3b)(x-a) + \frac{3}{2}(x^2 - a^2)$$
$$+ \int_a^x \int_b^w \int_a^z f(y)\, dy\, dz\, dw.$$

Solving it w/ variational method.

(a) form the residual. $r = u''' - f$.

(b) multiply by test function and integrate.

$$\int_a^b (u''' - f)\, v \, dx = 0$$

Smooth

(c). Integration by parts.

$$u''(b) v(b) - u''(a) v(a) - \int_a^b u'' v' + f v \, dx = 0$$

for all v smooth.

(d). Subs. B.C.s. we know $u''(a) = 3$.

... needs to request $v(b) = 0$

$$\rightarrow \ -3v(a) - \int_a^b u''v' + fv \, dx = 0.$$

Hence, $u''(a) = 3$ is a natural B.C.s

(e). formulate the weak form.

essential B.C.s: $u(a) = 1$ & $u'(b) = 2$.

let: $\mathcal{S} = \{u : [a, b] \to \mathbb{R} \text{ smooth} \mid u(a) = 1,\ u'(b) = 2\}$.

$\mathcal{V}^g = \{v : [a, b] \to \mathbb{R} \text{ smooth} \mid v(b) = 0\}$

* Weak form of the problem.

find $u \in \mathcal{S}$ s.t. for all $v \in \mathcal{V}^g$.

$$-\int_a^b u''v' \, dx = \int_a^b fv \, dx - 3v(a)$$