Problem Session 2 1/16/2025

▷ Vector space.

▷ Euler-Lagrange

▷ (possibly) test functions, shape functions & basis functions ...

<u>Vector Space</u>

~ Renzo Cavalieri, CSO

"Nerdy definition" 

"A vector space is a set that is closed under addition and scalar multiplication".

**basis** for a vector space

↳ sets w/ simple structure

⇝ they can be **added** together & **multiplied** by Scalars.

Definition.

Additive Multiplicative

Additive Closure. $u+v \in V$

Additive Commutativity $u+v = v+u$

Additive Associativity $(u+v)+w = u+(v+w)$.

Zero. $u + 0_V = u \quad \forall u \in V$

Additive Inverse For every $u$, exists $w$ $u+w = 0_V$

Multiplicative Closure $c \cdot v \in V$

Distributivity $(c+d)\cdot v = c\cdot v + d\cdot v$.

Distributivity $c\cdot(u+v) = c\cdot u + c\cdot v$

Associativity $(cd)\cdot v = c\cdot(d\cdot v)$

Unity $1\cdot v = v \quad \forall v \in V$

Examples Credit : Sebastian Tomaskovic-Moore, UPenn.

① $\{(a,b) \in \mathbb{R}^2 : b = 3a+1\}$.

counter-example :
- No zero vector.
- Not closed addition & multiplication.

② $\{(a,b) \in \mathbb{R}^2\}$ w/ scalar mul. $k(a,b) = (ka, b)$

$$(r+s)(a,b) = ((r+s)a, b) = (ra+sa, b)$$

$$r(a,b) + s(a,b) = (ra, b) + (sa, b) = (ra+sa, 2b).$$

violates the distributivity !!!!

③ $\{(a,b) \in \mathbb{R}^2\}$ w/ Scalar mul. $k(a,b) = (ka, 0)$.

$$1(a,b) = (1a, 0) = (a, 0) \neq (a,b)$$

violates both Mul. closure & Unity mul.

## Euler Lagrange Equation

credit: Norbert Stoop, MIT

Let us define an "Energy functional".

$$P(u) = \int_0^1 F(u, u')\,dx \quad \text{w/} \quad \begin{cases} u(0) = a \\ u(1) = b \end{cases}$$

Recall functional derivative:

$$J[f] = \int_a^b L(x, f(x), f'(x))\,dx.$$

$$\frac{\delta J}{\delta f} = \frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'} \longleftrightarrow \delta J = \int_a^b \left( \frac{\partial L}{\partial f}\,\delta f(x) + \frac{\partial L}{\partial f'}\frac{d}{dx}\delta f(x) \right) dx$$

Giaquinta & Hildebrandt, 1996

First variation (not required for this course)

$$\frac{\delta P}{\delta u} = \int_0^1 \left( v\frac{\partial F}{\partial u} + v'\frac{\partial F}{\partial u'} \right) dx \quad \text{for every } v$$

our old friend, test function !!

Weak form: $\underbrace{\int_0^1 v(x)\left(\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right)dx}_{\text{integral}} + \underbrace{\left[v\frac{\partial F}{\partial u'}\right]_0^1}_{\text{boundary terms}} = 0$

Note that this is satisfied for **ALL** test functions

Euler-Lagrange equation for $u$:

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0.$$

<u>Example</u> (Ex. 2.20; P. 36)

function $u$ satisfies

$$\int_0^L u'v'\,dx + u'(0)\,v(0) + u(0)\,v'(0) + \cancel{\mu\, u(0)\, v(0)}$$
$$- \int_0^L f(x)\,v(x)\,dx - d_L\, v(L) - g_0\, v'(0)$$
$$- \cancel{\mu g_0\, v(0)} = 0.$$

for all $v \in \mathcal{V} = \{v : [0, L] \to \mathbb{R} \text{ smooth}\}$

For general procedure, See P. 35 ~36

Step 1: eliminate the derivative on $v$

$$u'(L)\,v(L) - \underline{u'(0)\,v(0)} - \int_0^L u''\,v\,dx + \underline{u'(0)\,v(0)}$$

$$+ u(0)\,v'(0) + \mu\,u(0)\,v(0) = \int_0^L f\,v\,dx + d_L\,v(L)$$

$$+ g_0\,v'(0) + \mu\,g_0\,v(0)$$

Step 2: Collect $v$ terms

$$\int_0^L (u'' + f)\,v\,dx = \left(u'(L) - d_L\right) v(L)$$

$$+ \left(u(0) - g_0\right) v'(0) + \mu\left(u(0) - g_0\right) v(0)$$

For $v \in \mathcal{V}$: $\rightarrow$ $v(0) = v(L) = v'(0) = 0$

implying RHS $=0$

We conclude: $\int_0^L (u'' + f)\,v\,dx = 0$

Step 3: Obtain PDE & B.C.s

$$\rightarrow \quad u''(x) + f(x) = 0 \qquad x \in (0, L).$$

u needs to satisfy this PDE

For such $u$, the previous RHS should be satisfied for all $v$, not just $v(0) = v(L) = v'(0) = 0$ $\sim v \in \mathcal{V}$.

$$\rightsquigarrow \quad 0 = \left(u'(L) - d_L\right) v(L) + \left(u(0) - g_0\right) v'(0)$$
$$+ \mu \left(u(0) - g_0\right) v(0)$$

if $v(L) \neq 0 \Rightarrow u'(L) - d_L = 0.$

(Neumann B.C.s)

if $\left.\begin{matrix} v'(0) \neq 0 \\ v(0) \neq 0 \end{matrix}\right\} \quad u(0) - g_0 = 0$

(Dirchilet B.C.s)

Euler-Lagrange equation:

$$u''(x) + f(x) = 0, \qquad x \in (0, L)$$
$$u(0) = g_0,$$
$$u'(L) = d_L$$

# Conceptual Clarifications

- test functions

~ test how well trial functions satisfy sol'n (Recall last P.S.)

how do we use

$\Downarrow$

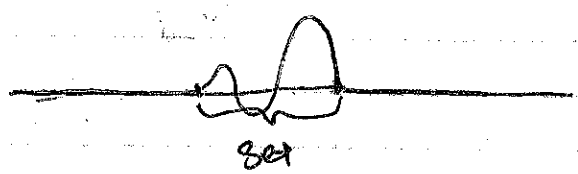
"A test function is an infinitely differentiable function of compact support." (NIST. Math. Func)

how does it look like

$\Downarrow$

"A function has compact support if it is zero outside of a compact set." (Wolfram)

(topological space)

1D example 



example $\int_0^1 u'' v \, dx < \infty$ ... test & trial func. not necessarily same.

$\Downarrow$

$u$ must be twice differentiable

doesn't even have to be continuous

- basis functions

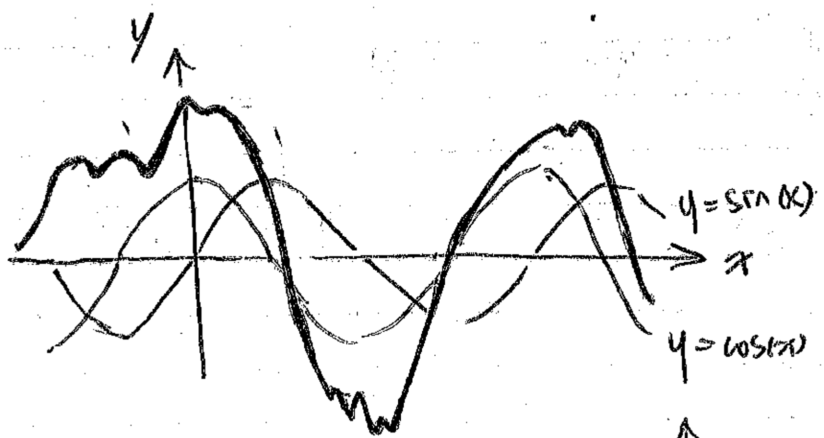
↳ usually referred in the context of **approximation** in FEA

$$u(x) = \sum_{i=1}^{n} c_i \varphi_i$$

"an element of a particular basis for a function space."

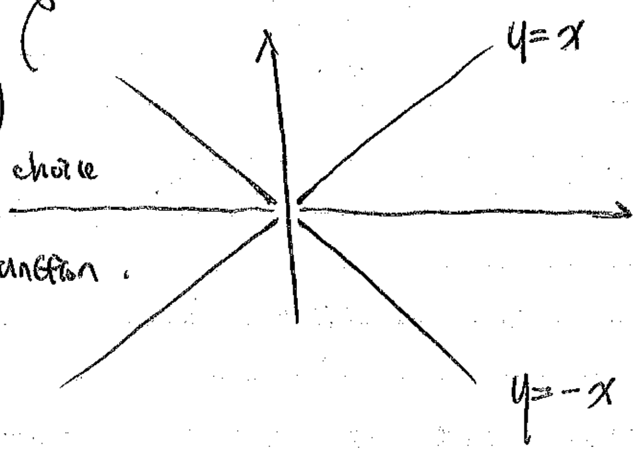
↳ every function in the function space can be represented as a linear combination of basis functions.

Example




use basis function trying to interpolate a weird-shaped function

probably a good choice of basis function.

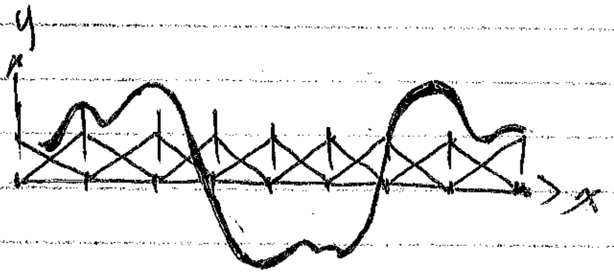



Shape functions ----- (usually referred specifically in FEA) ———

The shape function is the function which interpolates the solution between the discrete values obtained at the mesh nodes

TU Wien

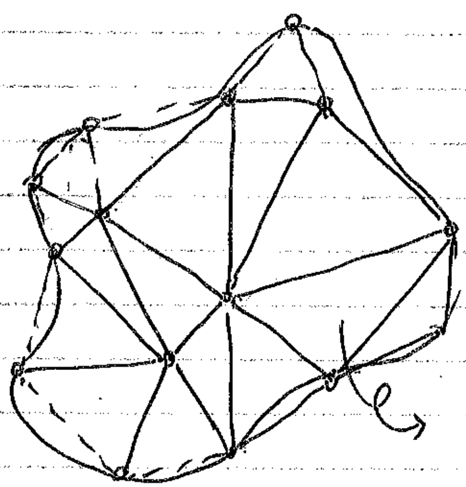
(Credit: Roberto Lacerda de Orio)



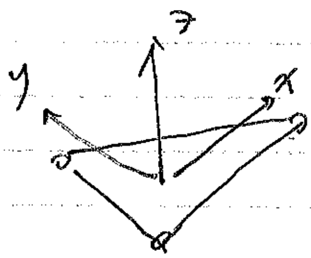

So, in FEA, sometimes they (the 3 functions) are talking about the same thing!

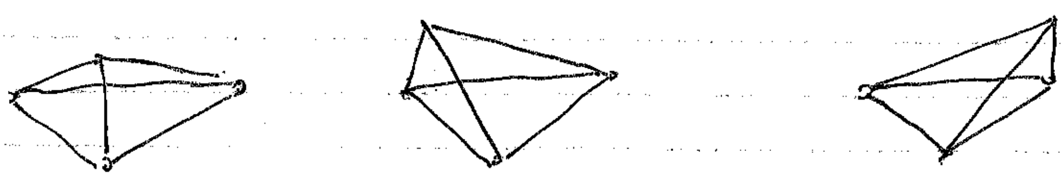
Much easier to approximate :)) !!!

... in higher dimensions ⊙⊙



pick this element



three basis functions!

Example 2.40

base space $\mathcal{V}_h = \text{span}(\{1, x, x^2, x^3\})$

test space $\mathcal{V}_h^0 = \text{span}(\{x, x^2, x^3\})$

trial space $\mathcal{S}_h = \{3 + v_h \mid v_h \in \mathcal{V}_h^0\}$

$$\begin{cases} m = 4 \\ n = 3 \end{cases} \quad \underbrace{\underbrace{\overset{x}{N_1(x)} \quad \overset{x^2}{N_2(x)} \quad \overset{x^3}{N_3(x)}}_{n} \quad , \overset{1}{N_4(x)}}_{m}$$

$\bar{u}_h(x) = 3N_4(x) = 3.$

Recall definition of consistency

$R_h(u, v_h) = 0 \quad a_h(u_h, v_h) = l_h(v_h)$

$\Downarrow$ solves

$$a_h(u, v_h) = l_h(v_h)$$

$$\begin{cases} a_h(u_h, N_1) = l_h(N_1) \\ a_h(u_h, N_2) = l_h(N_2) \\ a_h(u_h, N_3) = l_h(N_3) \end{cases}$$

$\because u_h \in \mathcal{S}_h$ , $\therefore \bar{u}_4 = 3$.

▷ load vector.

$$\underline{F} = \begin{bmatrix} l_h(N_1) \\ l_h(N_2) \\ l_h(N_3) \\ \bar{u}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

▷ Stiffness matrix

$$\underline{\underline{K}} = \begin{bmatrix} a_h(N_1, N_1) & a_h(N_2, N_1) & a_h(N_3, N_1) & a_h(N_4, N_1) \\ a_h(N_1, N_2) & a_h(N_2, N_2) & a_h(N_3, N_2) & a_h(N_4, N_2) \\ a_h(N_1, N_3) & a_h(N_2, N_3) & a_h(N_3, N_3) & a_h(N_4, N_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$g$

Write some codes (MATLAB & Python) to solve for the numerical values in $K_{ij}$.

Solving for $\underline{\underline{K}}\,\underline{U} = \underline{F} \rightarrow \underline{U} = \underline{\underline{K}}^{-1}\underline{F}$

$$\underline{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \dashrightarrow u_h(x) = u_1 N_1(x) + u_2 N_2(x) + u_3 N_3(x) + u_4 \underbrace{N_4(x)}_{1} \quad (-\bar{u}_4)$$

some values