(from Yi Shu)

1/27/2025 Problem Session #3.

2 point BVP. $\underbrace{f(x)=1,\ g=0,\ h=0}$

constant D & μ parameters.

find smooth u s.t.

$$-Du'' + \mu u' = f \qquad x \in (0,1)$$
$$u(0) = g$$
$$u(1) = h.$$

$\Rightarrow$ exact solution: $u(x) = \frac{1}{\mu}\left(x - \frac{1 - e^{\frac{\mu}{D}x}}{1 - e^{\frac{\mu}{D}}} \right)$

Standard procedure

(a) form residual: $r = -Du'' + \upsilon u' - f$.

exact sol'n should satisfy $r = 0$ $x \in (0,1)$

(b). for $v \in \mathcal{V}$ integrated over $(0,1)$,

$$\int_0^1 r(x)\, v(x)\, dx = 0$$

(c) Integration by part. for any $v \in \mathcal{V}$.

$$\mu \int_0^1 u' v \, dx - D u' v \Big|_0^1 + D \int_0^1 u' v' \, dx$$

$$= \int_0^1 v f \, dx, \quad x \in (0, 1). \quad \cdots (*)$$

(d). Use B.C.s & I.C.s for $\mathcal{V}$, → we do not have requirement for u' @ $x=0$ & $x=1$.

Eq. (*) holds $\begin{cases} v(0) = 0 \\ v(1) = 0 \end{cases}$

Eq. (*) becomes

$$\mu \int_0^1 u' v \, dx + D \int_0^1 u' v' \, dx = \int_0^1 v f \, dx \quad x \in (0, 1).$$

Formulate weak form. Find $u \in \mathcal{S}$ s.t.

$a(u, v) = l(v)$ for all $v \in \mathcal{V}$.

$$a(u, v) = \mu \int_0^1 u' v \, dx + D \int_0^1 u' v' \, dx.$$

$$l(v) = \int_0^1 f v \, dx$$

$\mathcal{S} = \{ u : [a, b] \to \mathbb{R},$ smooth $| \; u(0) = g,$
$u(1) = h \}.$

$\mathcal{V} = \{ v : [a, b] \to \mathbb{R},$ smooth $| \; v(0) = 0,$
$v(1) = 0 \}.$

Note that $a(u, v) \neq a(v, u).$

→ State Galerkin formulation.

Let $\mathcal{S}_h \subset \mathcal{S}, \; \mathcal{V}_h \subset \mathcal{V}.$

Find $u_h \in \mathcal{S}_h$ s.t. ... (**)

$$a(u_h, v_h) = l(v_h) \quad \text{for all } v_h \in \mathcal{V}_h$$

because $g = 0, \; h = 0.$ $\mathcal{S}$ & $\mathcal{V}$ are the same

Consider equidistant mesh w/ nodes $x_a = \frac{a}{N} = a\Delta x$ in $[0, 1]$. $a = 0, 1, 2, \dots, N$. piecewise linear shape functions $\{N_a\}$ defined to span $\mathcal{S}_h$ & $\mathcal{V}_h$.

$$N_a = \begin{cases} 0, & x < x_{a-1} \\ \dfrac{x - x_{a-1}}{x_a - x_{a-1}}, & x_{a-1} \le x < x_a. \\ 1, & \text{if } x = x_a \\ \dfrac{x_{a+1} - x}{x_{a+1} - x_a} & x_a < x \le x_{a+1} \\ 0 & x_{a+1} < x \end{cases}$$

approximation writes.

$$v_h = \sum_{a=1}^{N-1} N_a v_a .$$

$$u_h = \sum_{b=1}^{N-1} N_b u_b$$

proceed to compute $a(u_h, v_h)$ & $l(v_h)$.

$$a(v_h, u_h) = a\left(\sum_{a=1}^{N-1} N_a v_a, \sum_{b=1}^{N-1} N_b u_b\right).$$

$$= \sum_{a=1}^{N-1} \sum_{b=1}^{N-1} v_a u_b \, a(N_a, N_b)$$

$$l(v_h) = l\left(\sum_{a=1}^{N-1} N_a v_a\right)$$

$$= \sum_{a=1}^{N-1} v_a \, l(N_a)$$

rewriting Eq. (**).

$$\sum_{a=1}^{N-1} \sum_{b=1}^{N-1} v_a u_b \, a(N_b, N_a) = \sum_{a=1}^{N-1} v_a \, l(N_a) \quad \text{for all } v_a$$

$$\sum_{b=1}^{N-1} a(N_b, N_a) = l(N_a)$$

entries of load vector $\underline{F}$. $f_a = l(N_a)$.

stiffness matrix $\underline{\underline{K}}$, $K_{ab} = a(N_b, N_a)$

one can solve for $\underline{U}$

$$\underline{\underline{K}} \, \underline{U} = \underline{F}$$

$$a(N_b, N_a) = \mu \int_0^1 N_a' N_b \, dx + D \int_0^1 N_b' N_a' \, dx$$

$$l(N_a) = \int_0^1 N_a \, dx$$

Case study $N=3$. nodes in the mesh

$0, \frac{1}{3}, \frac{2}{3}, 1.$

$\underline{\underline{K}} \rightarrow 2\times 2$

$\Downarrow$

$\Delta x = \frac{1}{3}.$

$\underline{U} \rightarrow 2\times 1$

$\underline{F} \rightarrow 2\times 1$

$$K_{11} = a(N_1, N_1) = \mu \int_0^1 N_1 N_1' dx + D \int_0^1 N_1' N_1' dx$$

$$= \frac{2D}{\Delta x}$$

$$K_{22} = a(N_2, N_2) = \mu \int_0^1 N_2 N_2' dx + D \int_0^1 N_2' N_2' dx$$

$$= \frac{2D}{\Delta x}.$$

$$K_{12} = a(N_2, N_1) = \mu \int_0^1 N_2' N_1 dx + D \int_0^1 N_2' N_1' dx$$

$$= \mu/2 - D/\Delta x$$

$$K_{21} = a(N_1, N_2) = \mu \int_0^1 N_1' N_2 dx + D \int_0^1 N_1' N_2' dx$$

$$= -\mu/2 - D/\Delta x$$

Solving for load vector.

$$\underline{F} \rightarrow \begin{cases} F_1 = 1 \\ F_2 = 1. \end{cases}$$

linear system.

$$\begin{bmatrix} 2D/\Delta x & \mu/2 - D/\Delta x \\ -\mu/2 - D/\Delta x & 2D/\Delta x \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\det(\underline{\underline{K}}) = 3D^2/\Delta x^2 + \mu^2/4 \neq 0$$

$$\Downarrow$$

the system is invertible