Problem Session 4 2/3/2025

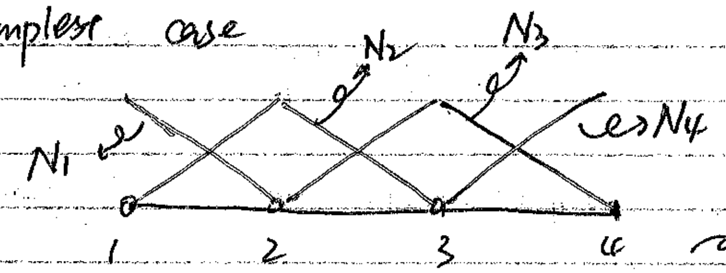
▷ Local - to - Global Map

▷ Examples on FEM implementation

▷ Q & A.

Local to Global Map

Simplest case



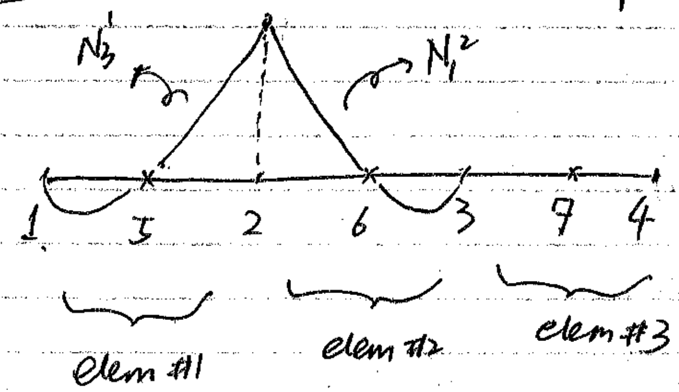
node number

$$LG = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

element #1, elem. #2, elem. #3.

$$LG = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 2 & 2 & 4 \end{bmatrix}$$

Example 3.11 continuous piecewise quadratic functions.



$N_1 = N_1^1$

$N_2 = N_3^1 \,\&\, N_1^2$

$N_3 = N_3^2 \,\&\, N_1^3$

$N_4 = N_3^3$

$N_5 = N_2^1$

$N_6 = N_2^2$, $N_7 = N_2^3$

N_a^b → b: element number
→ a: shape functions (local)

Modified example from Philip DePond

▷ Consider a 1D diffusion-advection equation.

given constant $k < 0$, f, v.

... find T smooth enough s.t.

$$k \frac{d^2 T}{dx^2} + v \frac{dT}{dx} = f \quad : \quad \text{in } \Omega \in [-1, 1].$$

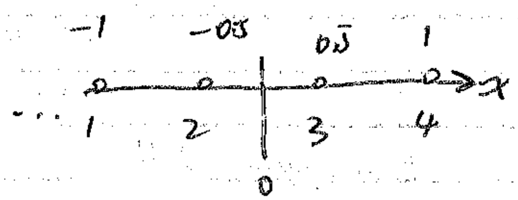
$$T(x=-1) = T_1$$

$$T(x=1) = T_2 \qquad \text{B.C.s.}$$

Consider a simple mesh w/ 4 nodes

using linear elements

Node	coordinate
1	-1
2	-0.5
3	0.5
4	1



State Galerkin form:

Starting from strong form:

$$\int_\Omega \left(k \frac{d^2 T}{dx^2} + v \frac{dT}{dx} \right) w \, dx = \int_\Omega f w \, dx$$

$$\int T'' w\,dx = \int w\,dT'$$

$$\hookrightarrow T'w\Big|_{\partial\Omega} - \int T'w'\,dx$$

$$-k\int_\Omega \frac{dT}{dx}\frac{dw}{dx}\,dx + v\int_\Omega \frac{dT}{dx} w\,dx = \int_\Omega f w\,dx$$

the Galerkin form is stated:

$$a(w, T) = \int_\Omega \left(k\frac{dw}{dx}\frac{dT}{dx} - v\frac{dT}{dx}w\right)dx$$

$$\ell(w) = -\int_\Omega f w\,dx$$

Find $T_h \in \mathcal{T}_h = \text{span}\{N_1, N_2, N_3, N_4\}$ s.t.

$$a(w_h, T_h) = \ell(w_h) - a(w_h, T_h^g),$$

$$\forall w_h \in \mathcal{W}_h = \mathcal{T}_h$$

~ Determine the LG matrix

Impose B.C.s

#1 #2 #3

$$LG = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

Global statement for the finite element problem

Dirichlet B.C.s: $T_h^g = T_1 N_1 + T_4 N_4$.

unknown part: $T_h = T_2 N_2 + T_3 N_3$

Full sol'n: $T_h^{total} = T_h^g + T_h = \sum_{i=1}^{4} T_i N_i$

test function: $w_h = \sum_{i=1}^{4} w_i N_i$

$w_1 = w_4 = 0$ (due to Dirichlet B.Cs)

Recall def'n of bilinear func.

$a(u, w+v) = a(u,w) + a(u,v)$

...

Substitute into bilinear form:

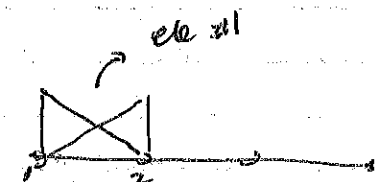
$$\sum_{i=1}^{4}\sum_{j=1}^{4} w_i\, a(N_i, N_j)\, T_j = \sum_{i=1}^{4} w_i\, l(N_i) - \sum_{i=1}^{4} w_i\, a(N_i, T_h^g)$$

Since $w_1 = w_4 = 0$, system reduces to

$$\sum_{j=2}^{3} a(N_i, N_j)\, T_j = l(N_i) - a(N_i, T_h^g)$$

$i = 2, 3$.

Local version of Finite Element.



Element #1

$$K_{ab}^1 = \int_{\Omega^1} \left(k \frac{dN_a^1}{dx}\frac{dN_b^1}{dx} - v\frac{dN_b^1}{dx} N_a^1 \right) dx, \quad a,b = 1,2$$

$$F_a^1 = -\int_{\Omega^1} f N_a^1\, dx - a(N_a^1, T_h^g), \quad a = 1,2.$$

K^1, F^1 corresponding to $LG(a,1)$ & $LG(b,1)$

Element #2 (nodes #2 & #3)

$$K^2_{ab} = \int_{\Omega^2} \left(k \frac{dN_a^2}{dx} \frac{dN_b^2}{dx} - v \frac{dN_b^2}{dx} N_a^2 \right) dx, \quad a,b = 1,2 .$$

$$F_a^2 = -\int_{\Omega^2} f N_a^2 \, dx - a(N_a^2, T_h^g), \quad a = 1,2 .$$

K^2, F^2 correspond to $LG(a,2)$ & $LG(b,2)$

... same procedure with elements #3 & #4.

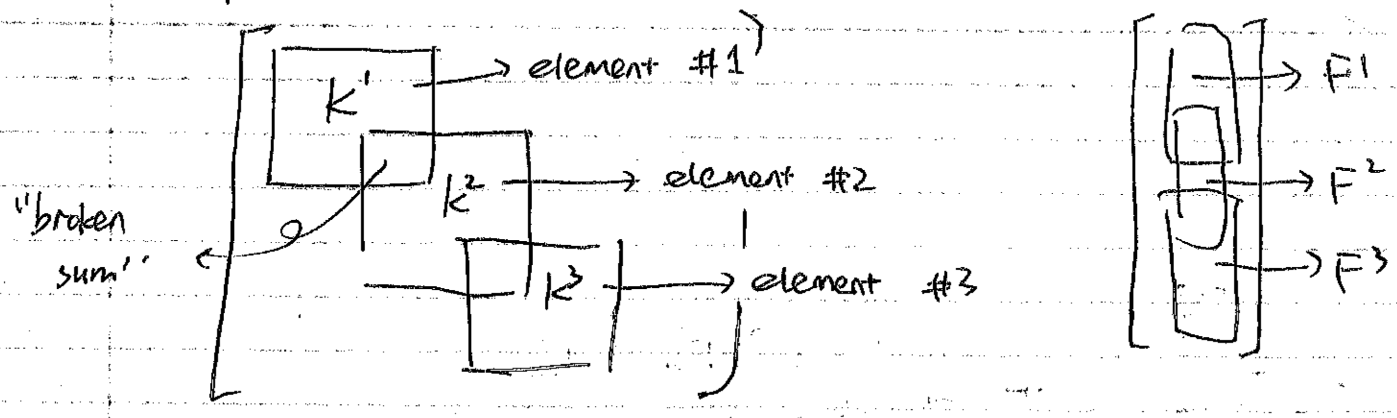
↳ corresponding to $LG(a,3)$, $LG(b,3)$ ...?
$LG(a,4)$, $LG(b,4)$

(Important !!) Assemble the Global System.

$$K_{LG(a,e),\,LG(b,e)} \leftarrow K_{LG(a,e),\,LG(b,e)} + K^e_{ab} \quad \text{for all } e, a, b$$

$$F_{LG(a,e)} \leftarrow F_{LG(a,e)} + F^e_a$$

Stiffness matrix



Final step: Solve the global system

$$KT = F$$

$$T = K^{-1}F$$

after solving for T, the sol'n:

$$T_h = T_1 N_1 + T_2 N_2 + T_3 N_3 + T_4 N_4$$

... implement these in

Python / MATLAB

~ your first FEM code !