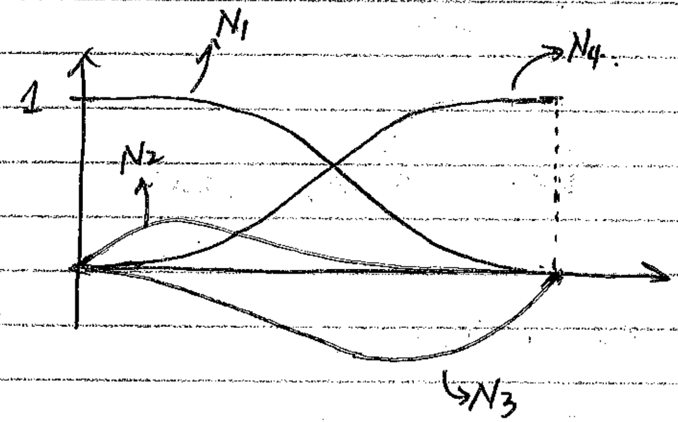
Problem Session #5. 2/10/2025.

... LG matrix.

Definitions of Hermite element



Eqns. (4.29)

$$N_1^e(x) = \left(\frac{x_2^e - x_1}{x_2^e - x_1^e}\right)^2 \left(1 + 2\,\frac{x - x_1^e}{x_2^e - x_1^e}\right)$$

$$N_2^e(x) = \left(\frac{x_2^e - x}{x_2^e - x_1^e}\right)^2 (x - x_1^e)$$

$$N_3^e(x) = \left(\frac{x_1^e - x}{x_1^e - x_2^e}\right)^2 \left(1 + 2\,\frac{x - x_2^e}{x_1^e - x_2^e}\right)$$

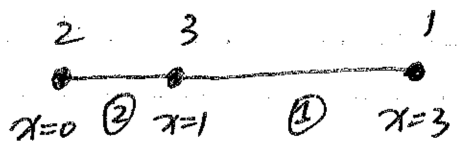
$$N_4^e(x) = \left(\frac{x_1^e - x}{x_1^e - x_2^e}\right)^2 (x - x_2^e)$$

Cubic polynomial in e :

$$f^e(x) = \phi_1^e N_1^e(x) + \phi_2^e N_2^e(x) + \phi_3^e N_3^e(x) + \phi_4^e N_4^e(x)$$

Example 4.8

Consider a two-element mesh.



nodal coordinates: $x_1 = 3$, $x_2 = 0$, $x_3 = 1$

Local-to-global map writes:

$$LG = \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 1 & 5 \\ 2 & 6 \end{bmatrix}$$

Using the definition of LG matrix:

$$N_A = \sum_{\{(a,e) \mid LG(a,e) = A\}} N_a^e$$

One writes for $A = 1, \ldots, 6$

$$N_1 = N_3^1$$

$$N_2 = N_4^1$$

$$N_3 = N_1^2$$

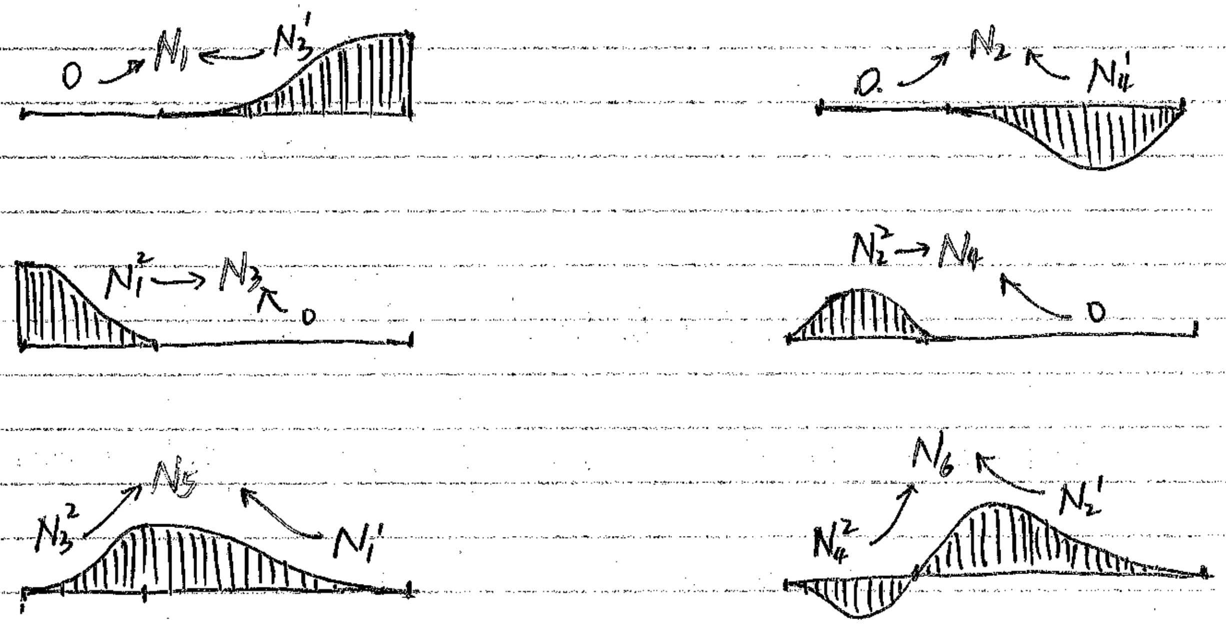
$$N_4 = N_2^2$$

$$N_5 = N_1^1 + N_3^2$$

$$N_6 = N_2^1 + N_4^2$$

Global shape functions

N_1, N_3, N_5 $\qquad\qquad$ N_2, N_4, N_6



... just a review

Recall the standard form for a 2^{nd} order diff. eqn.

$$-\left(k(x)\,u'(x)\right)' + b(x)\,u'(x) + c(x)\,u(x) = f(x) \qquad \cdots (2.1)$$

After variational formulation, some algebra, defining the finite element, ..., we have:

$$a_h(u_h, v_h) = \sum_{e=1}^{n_{el}} \underbrace{\int_{K_e} \left[k(x)\,u_h'(x)\,v_h'(x) + b(x)\,u_h'(x)\,v_h(x) + c(x)\,u_h(x)\,v_h(x) \right] dx}_{= a_h^e(u_h, v_h)}$$

... assembly step

$$= \sum_{e=1}^{n_{el}} a_h^e(u_h, v_h)$$

$$l_h(v_h) = k(L)\, d_L v_h(L) + \sum_{e=1}^{n_{el}} \underbrace{\int_{K_e} f(x)\, v_h(x)\, dx}_{l_h^e(v_h)}$$

$$= k(L)\, d_L v_h(L) + \sum_{e=1}^{n_{el}} l_h^e(v_h)$$

Consider BVP: constant f, EI, find smooth u s.t.

$$(EI\, u_{,xx})_{,xx} = f \qquad x \in (0,1)$$

$$u(0) = 0$$
$$u'(0) = 0$$
$$u(1) = 0$$
$$u'(1) = 0$$

Galerkin form:

Find $u^h \in S_h \subset S = \{u : [0,1] \to \mathbb{R} \text{ smooth} \mid u(0)=0,\ u'(0)=0,\ u(1)=0,\ u'(1)=0\}$

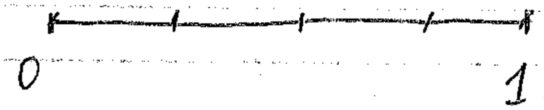
$$a(u_h, v_h) = l(v_h)$$

$$a(u_h, v_h) = \int_0^1 u_h''\, EI\, v_h''\, dx$$

$$l(v_h) = \int_0^1 f v_h\, dx.$$

for all $v_h \in V_h \subset V = \{v : [0,1] \to \mathbb{R} \text{ smooth} \mid v(1)=0,\ v'(1)=0,\ v(0)=0,\ v'(0)=0\}$

consider a mesh of 4 elements l_1, l_2, l_3, l_4



$$l_e = x_2 - x_1 > 0$$

$$\downarrow$$

$$N_1^e = \frac{-(x - x_2)^2\left[-l_e + 2(x_1 - x)\right]}{l_e^3} \quad \Omega^e = [x_1, x_2]$$

$$N_2^e = \frac{(x - x_1)(x - x_2)^2}{l_e^2}$$

$$N_3^e = \frac{(x - x_1)^2\left[l_e + 2(x_2 - x)\right]}{l_e^3}$$

$$N_4^e = \frac{(x - x_1)^2(x - x_2)}{l_e^2}$$

for general element w/ length l_e,

entries $k_{ab}^e = a(N_b^e, N_a^e)$

$\rightarrow$ take second derivative of N_1^e:

$$N_{1,xx}^e = \frac{2(l_e + 6x - 2x_1 - 4x_2)}{l_e^3} = \frac{2(-3l_e + 6x - 6x_1)}{l_e^3}$$

using change of variable: $x = x_1 + \xi(x_2 - x_1)$, $\xi \in [0, 1]$

we have: $dx = l_e \, d\xi$

$$\frac{d^2x}{d\xi^2} = 0$$

$$\frac{dN}{dx} = \frac{dN}{d\xi}\frac{d\xi}{dx}.$$

$$\frac{d^2N}{dx^2} = \frac{d^2N}{d\xi^2}\left(\frac{d\xi}{dx}\right)^2 + \frac{dN}{d\xi}\frac{d^2\xi}{dx^2}$$

because $\frac{d^2\xi}{dx^2} = 0.$

$$\frac{d^2N}{dx^2} = \frac{d^2N}{d\xi^2}\left(\frac{d\xi}{dx}\right)^2$$

We have $\quad N_1^e = (1-\xi)^2(1+2\xi)$

$$\frac{d^2N_1^e}{d\xi^2} = -6 + 12\xi$$

$$\frac{d^2N_1^e}{dx^2} = \frac{d^2N_1^e}{d\xi}\left(\frac{d\xi}{dx}\right)^2 = \frac{-6+12\xi}{l_e^2}$$

We can calculate k_1^e as an example:

$$a(N_1^e, N_1^e) = EI\int_{x_1}^{x_2} N_{1,xx}^e N_{1,xx}^e \, dx$$

$$= \frac{36EI}{l_e^3}\int_0^1 (-1+2\xi)^2 \, d\xi.$$

$$= \cdots = \frac{12EI}{l_e^3}$$

Using this transformation, we derive the other shape functions $\quad N_1^e = (1-\xi)^2(1+2\xi)$

$$N_2^e = l_e \xi (\xi - 1)^2$$

$$N_3^e = \xi^2 (3 - 2\xi)$$

$$N_4^e = l_e \xi^2 (\xi - 1)$$

2nd order derivative:

$$N_{1,xx}^e = -6 + 12\xi$$

$$N_{2,xx}^e = l_e (6\xi - 4)$$

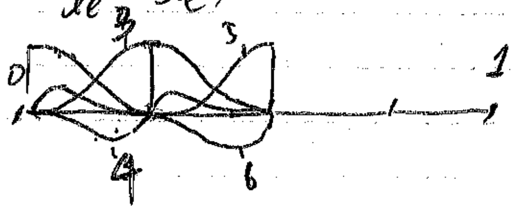
$$N_{3,xx}^e = 6 - 12\xi$$

$$N_{4,xx}^e = l_e (6\xi - 2)$$

elemental stiffness matrix.

$$\underline{\underline{k}}^e = EI \begin{bmatrix} \frac{12}{l_e^3} & \frac{6}{l_e^2} & -\frac{12}{l_e^3} & \frac{6}{l_e^2} \\ \frac{6}{l_e^2} & \frac{4}{l_e} & -\frac{6}{l_e^2} & \frac{2}{l_e} \\ -\frac{12}{l_e^3} & -\frac{6}{l_e^2} & \frac{12}{l_e^3} & -\frac{6}{l_e^2} \\ \frac{6}{l_e^2} & \frac{2}{l_e} & -\frac{6}{l_e^2} & \frac{4}{l_e} \end{bmatrix}$$

Write LG matrix



$$LG = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 3 & 5 & 7 & 9 \\ 4 & 6 & 8 & 10 \end{bmatrix}$$

Assemble the global stiffness matrix

$$K = EI\begin{bmatrix} \frac{12}{l_1^3} & -\frac{6}{l_1^2} & & & & \\ -\frac{6}{l_1^2} & \frac{4}{l_1} & & & & \\ & & & & & \\ & & & & & \\ & & & & \frac{12}{l_4^3} & \frac{6}{l_4^2} \\ & & & & \frac{6}{l_4^2} & \frac{4}{l_4} \end{bmatrix} +$$

$$EI\begin{bmatrix} \frac{12}{l_2^3} & \frac{6}{l_2^2} & -\frac{12}{l_2^3} & \frac{6}{l_2^2} & 0 & 0 \\ \frac{6}{l_2^2} & \frac{4}{l_2} & -\frac{6}{l_2^2} & \frac{2}{l_2} & 0 & 0 \\ -\frac{12}{l_2^3} & -\frac{6}{l_2^2} & \frac{12}{l_2^3}+\frac{12}{l_3^3} & -\frac{6}{l_2^2}+\frac{6}{l_3^2} & -\frac{12}{l_3^3} & \frac{6}{l_3^2} \\ \frac{6}{l_2^2} & \frac{2}{l_2} & -\frac{6}{l_2^2}+\frac{6}{l_3^2} & \frac{4}{l_2}+\frac{4}{l_3} & -\frac{6}{l_3^2} & \frac{2}{l_3} \\ 0 & 0 & -\frac{12}{l_3^3} & -\frac{6}{l_3^2} & \frac{12}{l_3^2} & -\frac{6}{l_3^2} \\ 0 & 0 & \frac{6}{l_3^2} & \frac{2}{l_3} & -\frac{6}{l_3^2} & \frac{4}{l_3} \end{bmatrix}$$

Assemble force vector $f_a^e = f\int_{x_1}^{x_2} N_a^e\,dx$

$f_1^e = l_e f\int_0^1 (1-\xi)^2(1+2\xi)\,d\xi = \frac{1}{2} f l_e$

$f_2^e = l_e^2 f\int_0^1 \xi(\xi-1)^2\,d\xi = \frac{1}{12} f l_e^2$

$f_3^e = l_e f\int_0^1 \xi(3-2\xi)\,d\xi = \frac{1}{2} f l_e$

$f_4^e = l_e^2 f\int_0^1 \xi^2(\xi-1)\,d\xi = -\frac{1}{12} f l_e^2$

Assemble global $\underline{F}$

$$\underline{F} = f \begin{bmatrix} \frac{1}{2}l_1 + \frac{1}{2}l_2 \\ -\frac{1}{12}l_1^2 + \frac{1}{12}l_2^2 \\ \frac{1}{2}l_2 + \frac{1}{2}l_3 \\ -\frac{1}{12}l_2^2 + \frac{1}{12}l_3^2 \\ \frac{1}{2}l_3 + \frac{1}{2}l_4 \\ -\frac{1}{12}l_3^2 + \frac{1}{12}l_4^2 \end{bmatrix}$$