Problem Session #6 2/15/2025

D 2D !!!

Recall Problem Session #4, we solved 1D diffusion-advection equation using FEA. Today we are going to solve it in 2D.

2D diffusion-advection problem, $k<0$, f, v_1 & v_2 are constants.

find T smooth enough such that.

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+v_1\frac{\partial T}{\partial x}+v_2\frac{\partial T}{\partial y}=f$$

in $\Omega=[-1,1]\times[0.1]$.

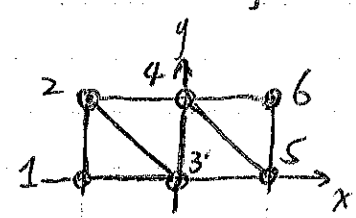
$$T(-1,y)=\tilde{T}_1$$

$$T(1,y)=\tilde{T}_2$$

$T_{,i}\,n_i=0$ on $y=-1$ and $y=1$.

Consider a simple mesh → 6 nodes using linear triangles.

node	coordinate
# 1	(-1, 0)
# 2	(-1, 1)
# 3	(0, 0)
# 4	(0, 1)
# 5	(1, 0)
# 6	(1, 1)



State the Galerkin formulation :

$\rightarrow T_i w_i = T_1 w_1 + T_2 w_2$

$$\int_\Omega (kT_{,ii} + v_i T_{,i}) w \, d\Omega = \int_\Omega f w \, d\Omega$$

$$\int_\Omega (-kT_{,i} w_{,i} + v_i T_{,i} w) \, d\Omega + \int_{\partial\Omega} k w T_{,i} n_i \, d\Gamma$$

$$= \int_\Omega f w \, d\Omega .$$

Since $w = 0$ on Γ_g, therefore $\int_{\Gamma_g} k w T_{,i} n_i \, d\Gamma = 0.$

$$\int_\Omega (-kT_{,i} w_{,i} + v_i T_{,i} w) \, d\Omega + \int_{\Gamma_h} k w T_{,i} n_i \, d\Gamma$$

$$= \int_\Omega f w \, d\Omega$$

$\because$ $T_{,i} n_i = 0$ on Γ_h, we then have

$$\int_\Omega (kT_{,i} w_{,i} - v_i T_{,i} w) \, d\Omega = -\int_\Omega f w \, d\Omega$$

$$a(T, w) = \int_\Omega (kT_{,i} w_{,i} - v_i T_{,i} w) \, d\Omega$$

$$l(w) = -\int_\Omega f w \, d\Omega$$

The Galerkin formulation is stated as:

Find $T_h \in \mathcal{T}_h = \text{span}\{N_3, N_4\}$ s.t.

$$a(T_h, w_h) = l(w_h) - a(w_h, T_h^g),$$

$$\forall w_h \in \mathcal{W}_h = \mathcal{T}_h$$

* LG matrix:

$$LG = \begin{bmatrix} 1 & 2 & 4 & 4 \\ 3 & 3 & 3 & 5 \\ 2 & 4 & 5 & 6 \end{bmatrix}$$

Global version of finite element.

$$T_h^g = T_1 N_1 + T_2 N_2 + T_5 N_5 + T_6 N_6.$$

$$\left.\begin{aligned} T_h^a &= \sum_{b=3}^{4} T_b N_b \\ w_h &= \sum_{a=3}^{4} w_a N_a. \end{aligned}\right\} T_h = T_h^a + T_h^g$$

Substitute into the weak form.

$$a\left(\sum_{b=3}^{4} T_b N_b, \sum_{a=3}^{4} T_a N_b\right) = l\left(\sum_{a=3}^{4} w_a N_a\right) - a\left(T_h^g, \sum_{a=3}^{4} w_a N_a\right).$$

$$\forall w_h = \mathcal{W}_h = \mathcal{T}_h.$$

reorganize the sign of summation.

$$\sum_{a=3}^{4}\sum_{b=3}^{4} w_a\, a(T_b N_b, N_a) = \sum_{a=3}^{4} w_a\, l(N_a) - \sum_{a=3}^{4} w_a\, a(T_h^g, N_a),$$

$$\forall w_h \in \mathcal{V}_h = \mathcal{T}_h$$

We can reformulate the equation as.

$$\sum_{b=3}^{4} a(N_b, N_a) T_b = l(N_a) - a(T_h^g, N_a).$$

$$K_{LG(b),\, LG(a)} = a(N_b, N_a)$$

$$F_{LG(a)} = l(N_a) - a(T_h^g, N_a)$$

$K \rightarrow$ not symmetric because $a(T, w)$ is not symmetric.

∴ local version of finite element.

$$a(T_h, w_h)_{\Omega} = \sum_{LG}^{e} a(T_h, w_h)_{\Omega^e}$$

a simplified symbol for global assembly

$$l(w_h)_{\Omega} = \sum_{LG}^{e} l(w_h)_{\Omega^e}$$

on Ω^e $\quad T_h = \sum_{b=1}^{3} T_b^e N_b^e$

$$w^h = \sum_{a=1}^{3} w_a^e N_a^e$$

We have $K^e_{ab} = a(N^e_b, N^e_a)_{\Omega^e}$

$$l^e_a = l(N^e_a)_{\Omega^e} - a(T^g_h, N^e_a)_{\Omega^e}$$

$\rightarrow a(T^g_h, N^e_a)_{\Omega^e} = K^e_{ab}\, g^e$

$$IG^e = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

on Ω^1:

$$T^g_h = \tilde{T}_1 N_1' + \tilde{T}_1 N_3'$$

$$a(T^g_h, N_a')_{\Omega'} = a(\tilde{T}_1 N_1' + \tilde{T}_1 N_3', N_a')_{\Omega'}$$

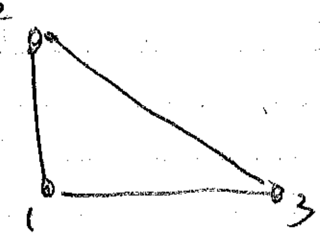
$$= a(\tilde{T}_1 N_1' + 0 N_2' + \tilde{T}_1 N_3', N_a')_{\Omega'}$$

$$a(T^g_h, N_a')_{\Omega'} = a(N_1', N_a')_{\Omega'} \tilde{T}_1 + a(N_2', N_a')_{\Omega'} \cdot 0 + a(N_3', N_a')_{\Omega'} \tilde{T}_1$$

Therefore

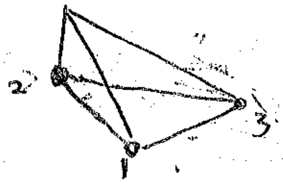
$$\begin{bmatrix} a(T^g_h, N_1')_{\Omega'} \\ a(T^g_h, N_2')_{\Omega'} \\ a(T^g_h, N_3')_{\Omega'} \end{bmatrix} = \begin{bmatrix} a(N_1', N_1') & a(N_2', N_1') & a(N_3', N_1') \\ a(N_1', N_2') & a(N_2', N_2') & a(N_3', N_2') \\ a(N_1', N_3') & a(N_2', N_3') & a(N_3', N_3') \end{bmatrix} \begin{bmatrix} \tilde{T}_1 \\ 0 \\ \tilde{T}_1 \end{bmatrix}$$

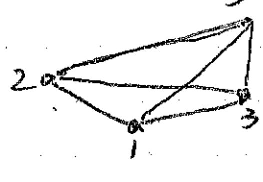
Compute $a(N_2', N_3')|_{\Omega'}$



$$N_3'(x, y) = x + 1$$

$$N_2'(x, y) = y$$





$$a(N_2^1, N_3^1)\big|_{\Omega^1} = \int_{\Omega^1} \left(k N^1_{3,x} N^1_{2,x} + k N^1_{3,y} N^1_{2,y} - v_1 N^1_{2,x} N^1_3 - v_2 N^1_{2,y} N^1_3\right) d\Omega .$$

$$= \int_{\Omega^1} \left(-v_1 N^1_{2,x} N^1_3\right) d\Omega$$

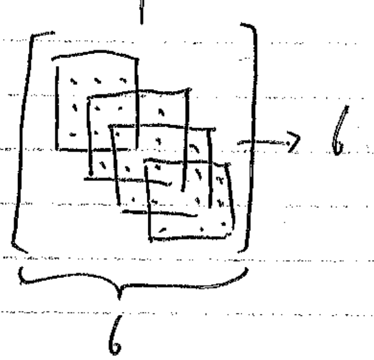
$$= -v_1 \int_{\Omega^1} y \, d\Omega = -\frac{v_1}{6}.$$

We can then do the assembly of K and F

$$K = \begin{bmatrix} K^1_{22} + K^2_{22} & K^2_{23} \\ & \\ K^2_{32} & K^2_{33} \end{bmatrix} \overset{\circ}{+} \cdots$$

the nest steps should be the same

... Dimension of overall K ?



$$\begin{bmatrix} \vec{F} \end{bmatrix} \to 6$$