Problem Session #8 3/3/2025

Definition B.1. V: vector space.

a norm is a function: $\|\cdot\|: V \to \mathbb{R}$

such that for $v, u \in V$ & $\alpha \in \mathbb{R}$

1) $\|v\| \geq 0$ & $\|v\| = 0$ IFF $v = 0$

2) $\|\alpha v\| = |\alpha| \, \|v\|$

3) $\|v + u\| \leq \|v\| + \|u\|$

B.5. For $v \in V_1$, H^1-norm

$$\|v\|_{1,2} = \left[\int_a^b v(x)^2 dx + \int_a^b v'(x)^2 dx \right]^{1/2}$$

$$= \Big[\underbrace{\|v\|_{0,2}^2}_{L^2\text{-norm}} + \underbrace{|v|_{1,2}^2}_{H^1\text{-Seminorm}} \Big]^{1/2}$$

Definition B.2 A vector space V with a norm defined over $\|\cdot\|: V \to \mathbb{R}$ is called a normed space, denoted as $(V, \|\cdot\|)$

B.10. Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$. For such domain Ω, the norm $\|v\|_{0,2}$ of $v : \Omega \to \mathbb{R}$ is defined as:

$$\|v\|_{0,2} = \left[\int_\Omega v(x)^2 \, d\Omega \right]^{1/2}$$

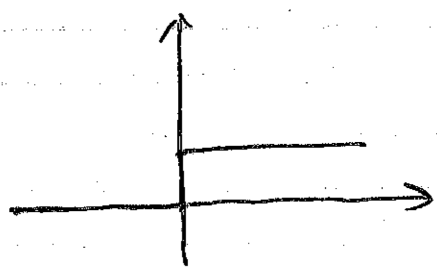
The set

$$L^2(\Omega) = \{ v : \Omega \to \mathbb{R} \mid \|v\|_{0,2} < \infty \}$$

is called the $L^2(\Omega)$ space, and $(L^2(\Omega), \|\cdot\|_{0,2})$ is a normed space. The space $L^2(\Omega)$ is said to contain all square-integrable functions.

→ does not need to be smooth, e.g.,

$\Omega = [-1, 1]$ contains $H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$,



integral of the square of the abs. value is finite.

However, $H(x) \notin L^2(\mathbb{R})$.

↳ why? $\int_{-\infty}^{+\infty} (H(x))^2 \, dx = \int_0^{+\infty} 1 \, dx = +\infty$

B.11 Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$. For such domain Ω, we define H^1-norm :

$$\|v\|_{1,2} = \left[\|v\|_{0,2}^2 + \sum_{i=1}^{n} \left\| \frac{\partial v}{\partial x_i} \right\|_{0,2}^2 \right]^{1/2}$$

→ we define $H^1(\Omega)$-space

$$H^1(\Omega) = \{ v : \Omega \to \mathbb{R} \mid \|v\|_{1,2} < \infty \}.$$

normed space : $(H^1(\Omega), \|\cdot\|_{1,2})$

Functions in $H^1(\Omega)$ contain "both function & each one of its partial derivatives is square integrable". Alternatively, the function & each of its partial derivatives is in $L^2(\Omega)$.

if a function $v \in H^1(\Omega)$, then $v \in L^2(\Omega)$.

e.g., Let $\Omega = [-1, 1] \times [-1, 1]$.

① function $v(x_1, x_2) = x_1^2 + x_2^3 \in H^1(\Omega)$,

$$\|v\|_{1,2}^2 = \int_{-1}^{1}\int_{-1}^{1} (x_1^2 + x_2^3)^2 \, dx_1 dx_2 + \int_{-1}^{1}\int_{-1}^{1} (2x_1)^2 dx_1 dx_2$$

$$+\int_{-1}^{1}\int_{-1}^{1}(3x_2^2)^2\,dx_1dx_2=\frac{292}{21}<\infty$$

② the function $v(x_1,x_2)=\ln(1+x_1)+\ln(1+x_2)$

$\notin H^1(\Omega)$, but $v\in L^2(\Omega)$, since

$$\|v\|_{0,2}^2=\int_{-1}^{1}\int_{-1}^{1}\left(\ln(1+x_1)+\ln(1+x_2)\right)^2dx_1dx_2$$

$$=24+8\ln(4)(\ln(2)-2)<\infty$$

$$\|v\|_{1,2}^2=\|v\|_{0,2}^2$$

$$+\int_{-1}^{1}\frac{1}{(1+x_1)^2}\,dx_1dx_2+\int_{-1}^{1}\frac{1}{(1+x_2)^2}\,dx_1dx_2$$

$$=\infty$$

~ A Simple Example

$-u''(x)=\overset{=x}{f(x)}$ on $[0,1]$ ... 1D.

$$\begin{cases} u''(x)=x, & x\in(0,1).\\ u(0)=0, & u(1)=0\end{cases}$$

Weak form.

$$\int_0^1 (-u''(x))v(x)\,dx = \int_0^1 x\, v(x)\,dx$$

$$\rightarrow \int_0^1 u'(x)\, v'(x)\,dx = \int_0^1 x\, v(x)\,dx.$$

bilinear form.

$$a(u,v) = \int_0^1 u'(x) v'(x)\,dx.$$

linear functional

$$l(v) = \int_0^1 x\, v(x)\,dx$$

bilinear $a(\cdot,\cdot)$ ...

→ Continuity: $|a(u,v)| \le \|u'\|_{L^2(0,1)} \|v'\|_{L^2(0,1)}$

→ Coercivity: $\int_0^1 |u'(x)|^2\,dx \geq \alpha \|u\|^2_{H_0^1(0,1)}$.

for some $\alpha > 0$, Thus $a(u,v) \geq \alpha \|u\|^2$.

giving the strict positivity needed for invertibility

→ Céa's lemma.

$$\| u - u_h \|_{H_0^1} \leq \left(1 + \frac{M}{\alpha}\right) \min_{v_h \in V_h} \| u - v_h \|_{H_0^1}.$$

M: continuity constant.

α: coercivity constant.

→ in practice, $\min_{v_h} \| u - v_h \|$ is the "best approximation error" of u by FEM span V_h.

~ Convergence rate

For a Poisson-type problem, with P_k-element, with mesh size h.

→ exact sol'n of u is smooth

→ homogeneous Dirichlet B.C.s

H^1-seminorm: $\| u - u_h \|_{H^1(\Omega)} = O(h^k)$

L^2-norm: $\| u - u_h \|_{L^2(\Omega)} = O(h^{k+1})$.

L^1-norm: $\| u - u_h \|_{L^1(\Omega)} = O(h^{k+1})$ → L^r $1 \leq r \leq 2$

$$\|v\|_{L^1(\Omega)} \le \underbrace{\sqrt{|\Omega|}}_{\text{finite measure of the domain.}} \|v\|_{L^2(\Omega)}.$$

$\Downarrow$

$$\|u - u_h\|_{L^1(\Omega)} \le \sqrt{|\Omega|}\, \|u - u_h\|_{L^2(\Omega)} = \mathcal{O}(h^{k+1}).$$

Further questions to explore: what if the assumptions do not hold ... ?