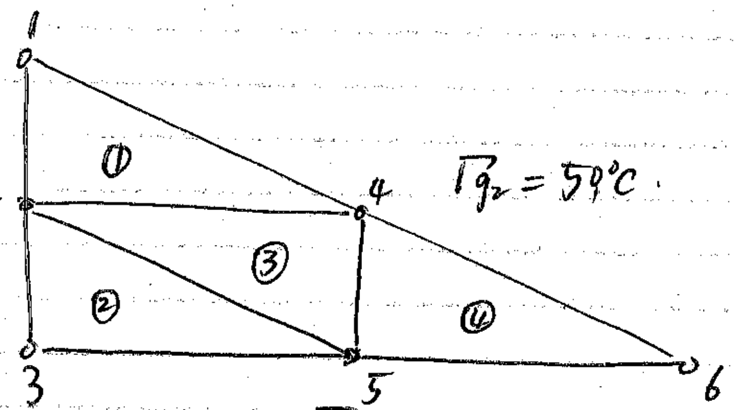
Problem Session #9. (Final Review).



$100°C = \Gamma_{g_1}$ $= \{(x_1, x_2) \ldots \Omega \mid x_1 = 0\}$

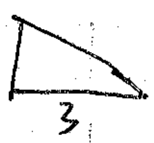
$\Gamma_{g_2} = 50°C$

$\Gamma_{g_2} = \partial\Omega \setminus (\Gamma_{g_1} \cup \Gamma_h)$

$\Gamma_h = \{(x_1, x_2) \in \partial\Omega \mid x_2 = 0,\ 0 < x_1 < 6\}$.

find temperature $T : \Omega \to \mathbb{R}$ s.t.

$$
\begin{aligned}
-\operatorname{div}(K(x)\nabla T) &= 0 && \text{on } \Omega \\
T &= 100°C && \text{on } \Gamma_{g_1} \\
T &= 50°C && \text{on } \Gamma_{g_2} \\
K(x)\nabla T \cdot \check{n} &= 0 && \text{on } \Gamma_h
\end{aligned}
$$



→ Construct variational equation of T:

$$-\int_\Omega \operatorname{div}(K\nabla T)\, v \, d\Omega = \int_\Omega (K\nabla T)\cdot \nabla v \, d\Omega$$

$$- \int_{\Gamma_g} \cancel{(K \nabla T)\cdot \check{n}}^{\,0}\ v \, d\Gamma_h$$

$$a(T, v) = \int_\Omega (K \nabla T) \cdot \nabla v \, d\Omega \quad , \forall v \in \mathcal{V}.$$

$$\ell(v) = 0$$

$$\mathcal{V} = \{ v : \Omega \to \mathbb{R} \text{ Smooth} \mid v = 0, \ \forall \underline{x} \in \Gamma_{g_1}, \Gamma_{g_2} \}.$$

... Define $\mathcal{W}_h$, $\mathcal{V}_h$ & $\mathcal{S}_h$ → state the FEM

$$\mathcal{W}_h = \text{span}(N_1, N_2, N_3, N_4, N_5, N_6).$$

$$\mathcal{V}_h = c N_5 \quad , \quad c \in \mathbb{R}.$$

$$\mathcal{S}_h = T_{g_1} (N_1 + N_2 + N_3) + T_{g_2} (N_4 + N_6) + c N_5.$$

* Find $T_h \in \mathcal{S}_h$ such that

$$a(T_h, v_h) = \ell(v_h) \quad \forall v_h \in \mathcal{V}_h.$$

$$a(T_h, v_h) = \int_\Omega (K \nabla T_h) \cdot \nabla v_h \, d\Omega.$$

$$\ell(v_h) = 0.$$

→ Find LV.

"anti-clockwise" ↺

$$LV = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 2 & 3 & 5 & 5 \\ 4 & 5 & 4 & 6 \end{bmatrix}$$

Assume thermal conductivity is constant for each element ($\underline{\underline{K}}(\underline{x}) \approx k^e \underline{\underline{I}}$, $\underline{x} \in \Omega^e$, $k^e \in \mathbb{R}$).

expressions of N_a^e & A are provided.

k^e	Value
k^1	14
k^2	27
k^3	45
k^4	27

$$N_1^e = \frac{1}{2A}\Big[-(X_2^3 - X_2^2)(x_1 - X_1^2) + (X_1^3 - X_1^2)(x_2 - X_2^2)\Big]$$

$$N_2^e = \frac{1}{2A}\Big[-(X_2^1 - X_2^3)(x_1 - X_1^3) + (X_1^1 - X_1^3)(x_2 - X_2^3)\Big]$$

$$N_3^e = \frac{1}{2A}\Big[-(X_2^2 - X_2^1)(x_1 - X_1^1) + (X_1^2 - X_1^1)(x_2 - X_2^1)\Big]$$

$$A = \frac{1}{2}(X_1^2 - X_1^1)(X_2^3 - X_2^1) - (X_2^2 - X_2^1)(X_1^3 - X_1^1)$$

Constrained index

$$\eta_g = \{1, 2, 3, 4, 6\} \qquad \ell(V_h) = 0$$

We can write out K and F

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} T_{\Gamma_{g_1}} \\ T_{\Gamma_{g_1}} \\ T_{\Gamma_{g_1}} \\ T_{\Gamma_{g_2}} \\ 0 \\ T_{\Gamma_{g_2}} \end{bmatrix}$$

$LV = LG$ (conformal) $\rightarrow K^e_{ab} \rightarrow K_{LG(a,e)\,LG(b,e)}$

therefore

$$K_{51} = 0$$

$$K_{52} = K^2_{31} + K^3_{21}$$

$$K_{53} = K^2_{32}$$

$$K_{54} = K^3_{23} + K^4_{21}$$

$$K_{55} = K^2_{33} + K^3_{22} + K^4_{22}$$

$$K_{56} = K^4_{23}$$

$A \rightarrow$ same for all elements $\rightarrow$ Assume k const.

$$K^e_{ab} = \int_{\Omega^e} k^e \nabla N^e_b \cdot \nabla N^e_a \, d\Omega^e = k^e \nabla N^e_b \cdot \nabla N^e_a \int_{\Omega^e} d\Omega^e$$

$$= k^e A \nabla N^e_b \cdot \nabla N^e_a$$

Write out the gradients:

$$\nabla N_1^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \nabla N_2^2 = \begin{bmatrix} -1/3 \\ -1 \end{bmatrix} \quad \nabla N_3^2 = \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}$$

$$\nabla N_1^3 = \begin{bmatrix} -1/3 \\ 0 \end{bmatrix} \quad \nabla N_2^3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \nabla N_3^3 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

$$\nabla N_1^4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \nabla N_2^4 = \begin{bmatrix} -1/3 \\ -1 \end{bmatrix} \quad \nabla N_3^4 = \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}$$

Substituting back into K_{ij}

$$K_{51} = 0$$

$$K_{52} = K_{31}^2 + K_{21}^3 = 0$$

$$K_{53} = K_{32}^2 = -3A$$

$$K_{54} = K_{23}^3 + K_{21}^4 = -72A$$

$$K_{55} = K_{33}^2 + K_{22}^3 + K_{22}^4 = 78A$$

$$K_{56} = K_{23}^4 = -3A$$

$$0 = T_{\Gamma_{g_1}}(K_{51} + K_{52} + K_{53}) + T_{\Gamma_{g_2}}(K_{54} + K_{56}) + u_5 K_{55}$$

$$0 = 100(0 + 0 - 3A) + 50(-72A - 3A) + u_5 78A$$

$$u_5 = 675/13$$

We have:

$$T_h = 100(N_1 + N_2 + N_3) + 50(N_4 + N_6) + \frac{675}{13} N_5$$

→ From T_h, find values @ centroid of element $\bar{x}^e$.

$\bar{x}^e$
$(1, 4/3)$
$(1, 1/3)$
$(2, 2/3)$
$(4, 1/3)$

$$T_h(\bar{x}^1) = \frac{1}{3}(100 + 100 + 50).$$

$$T_h(\bar{x}^2) = \frac{1}{3}\left(100 + 100 + \frac{675}{13}\right)$$

$$T_h(\bar{x}^3) = \frac{1}{3}\left(100 + 50 + \frac{675}{13}\right)$$

$$T_h(\bar{x}^4) = \frac{1}{3}\left(50 + 50 + \frac{675}{13}\right).$$

→ What convergence rates r_i would you expect $\|T - T_h\|_{0,2,\Omega}$ & $\|T - T_h\|_{1,2,\Omega}$

$r(\|T - T_h\|_{0,2,\Omega}) = k+1 = 2.$ ← 1^{st} order elem.

$r(\|T - T_h\|_{0,2,\Omega}) = k = 1$

Now lets switch to P_2 - element.

assume you have access to thermocouple allows you to get measurement T_{meas} @ $\underline{X}_{meas}$.

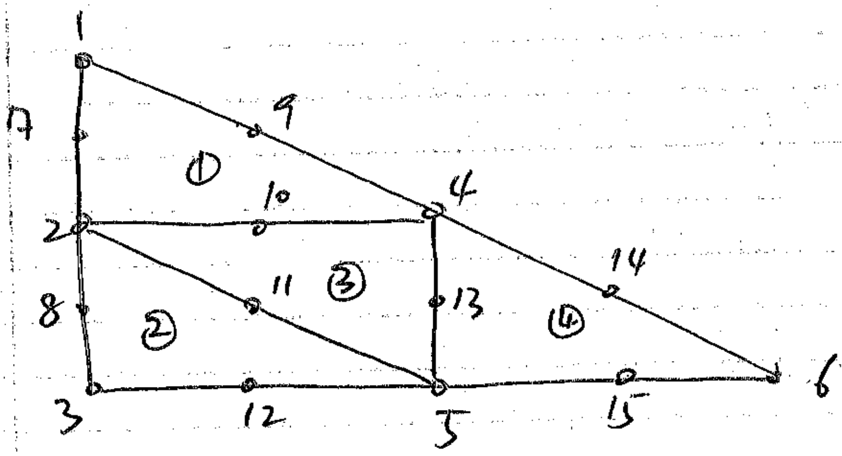
$$\hookrightarrow \quad -\operatorname{div}\left(k(\underline{x})\nabla T\right) = 0 \qquad \text{on } \Omega$$

$$T = 100°C \qquad \text{on } \Gamma_{g_1}$$

$$T = 50°C \qquad \text{on } \Gamma_{g_2}.$$

$$k(x)\,\nabla T\cdot\check{n} = 0. \qquad \text{on } \Gamma_h.$$

$$T(\underline{x}) = T_{meas}(\underline{X}_{meas})$$



$$T_h = 100(N_1 + N_2 + N_3 + N_7 + N_8) + 50(N_4 + N_6 + N_9 + N_{14}) + \sum_{j\in\eta_h} T^j_{meas} N_j .$$

$$\eta_h = \{5, 10, 11, 12, 13, 15\}$$