PERSONAL NOTES

FINITE ELEMENT ANALYSIS

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2024

Finite Element Analysis.

1/2/2024.

~ Fundamentals of primal FEM.

1). Method of weighted residuals, . Galerkin's method & variational equations.

2). Linear eliptic boundary value problems in 1, 2, 3D (spatial dimensions)

$\Downarrow$

3). Applications in structural, solid, fluid mechanics & heat transfer.

4). Properties of standard element families & numerically integrated elements.

5). Implementation of FEM using MATLAB, assembly of equations, and element routines.

6). Lagrange multiplier & penalty methods for treatments of constraints.

Preparation Notes.

Chapter 1 Finite Elem. Meth. for Elliptic Problems in 1D

→ 2nd-order elliptic diff. eqn. for 1D:

$$-(k(x)u'(x))' + b(x)u'(x) + c(x)u(x) = f(x).$$

independent var. (space).

coefficients.

relevant function to be studied.

take $0 < x < L$, Ω is an interval,

$\Omega = (0, L)$. → $\begin{cases} \text{Dirichlet Prob.} \\ \text{Neumann Prob} \end{cases}$

→ Galerkin Method.

- Vector spaces of functions.
- Solution to this problem.

→ The Finite Element Method.

- Simplest C^0 Finite Element Space.

pter 2. Diffusion Problems in 2D.

- Strong Form of BVP,
- Galerkin method.
- Finite Element in 2D.
  - simplest C^0 finite element in 2D space
  - Barycentric coordinates & basis functions of P_1.
  - element stiffness matrix.
  - element load vector
  - Solving 2D diffusion problems with P_1 FE (Dirichlet case)
  - Solving problems with Neumann boundaries

pter 3. Numerical Analysis of the FEM for Elliptic Problems.

- Basic Idea.

  Approximability.
  Continuity.
  Coercivity.
  Strict Monotonicity.

- Abstract Error Estimate for Galerkin Method

- Normed Spaces.
- Coercivity
- Interpolation Errors.
- Convergence

Chapter 4 Linear Elasticity.

- The variational problem of linear Elasticity
- From variational form to weak form.
- Galerkin method.
- Finite element spaces for multifield problems 2D.
- Solving linear elasticity problems in 2D with P_1 Finite element.
- Variational method as minimum principle.
- Minimization problems and variational method

1/9/2024

2nd-order Problems

$$-(k(x)\, u'(x))' + b(x)\, u'(x) + c(x)\, u(x) = f(x) \quad (*)$$

Goal: find u $\qquad x \in \Omega = [0, L]$

$u \equiv$ unknown function

$k, b, c, f : \Omega \rightarrow \mathbb{R}$

"data coefficients"

$\begin{cases} \text{(a) } u \text{ should be smooth enough} \\ \text{(b) } u \text{ should satisfy } (*) \ \forall\, x \in [0, L]. \end{cases}$

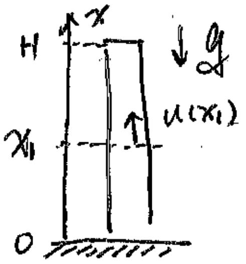
1/11/2024.

Differential and Variational Equations.

- Vector spaces of functions.
- consistency.
- classical variational equation.
- essential & natural BCs.
- other variational eqns. e.g., Nitsches eqn.

1.1.

$$[E(x)\, u'(x)]' = \rho(x)\, g$$



$\Omega = [0, H]$

$u : \Omega \rightarrow \mathbb{R}$.

1.2 Heat conduction.

$$-(k(x)\,u'(x))' = f(x).$$

$\hookrightarrow$ temperature.

1.3.

$$-u''(x)=0, \quad x \in (0.l).$$

$$u(x) = C_1 + C_2 x. \qquad C_1, C_2 \in \mathbb{R}.$$

BCs: $\begin{cases} \text{Dirichlet condition.} \rightarrow \text{impose } u. \\ \text{Neumann condition.} \rightarrow \text{impose } u' \end{cases}$

Closure of $\bar{\Omega}$: $\Omega \cap \partial\Omega$.

add BCs: $\begin{cases} u(0) = g_0 \\ u'(L) = d_L. \end{cases}$

$\rightarrow$ $u(x) = g_0 + d_L x.$

1.6. $-u''(x) + \dfrac{u(x)}{x^2} = 0. \quad \forall x \in (0.L).$

general sol'n: $u(x) = C_1 x^{(1+\sqrt{5})/2} + C_2 x^{(1-\sqrt{5})/2}.$

$C_1, C_2 \in \mathbb{R}. \longrightarrow$ if $u(0) = g_0 \in \mathbb{R} \rightarrow C_2 = 0$

$\Downarrow$

$u(x) = c_1 x^{(1+\sqrt{5})/2}$

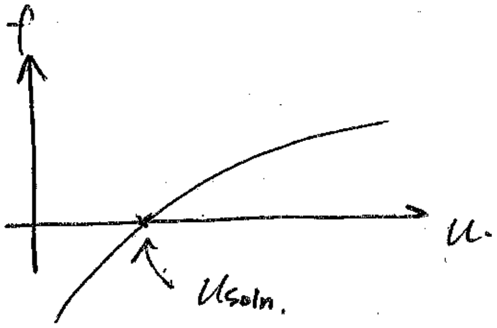
$\downarrow$

$u(0) = 0$

if $g_0 \neq 0 \rightarrow$ NO SOLUTION !!!

Variational Equations.

$f(u) \equiv u^2 + \ln u - 1 = 0 \quad (*)$.



$R(u, v) = (u^2 + \ln u - 1)\, v = 0.$
$\forall\, v \in \mathbb{R}.$

If u solves $(*) \Rightarrow R(u, v) = 0, \ \forall\, v \in \mathbb{R}$.

$$\Downarrow$$

$R(u, 1) = 0$
$R(u, 2) = 0$
$\vdots$

Definition of Vector Space. (Appendix).

~ V.S. of functions.

$\mathcal{V}$: Set of all real quadratic polynomials that are zero @ $x = 0$.

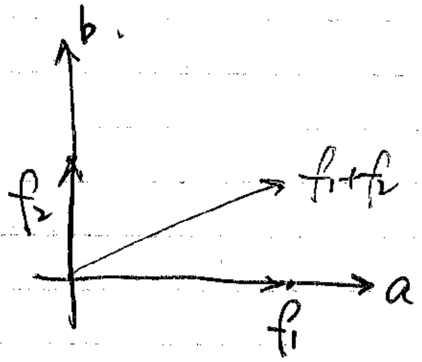
$f: \mathbb{R} \to \mathbb{R}, \ f \in \mathcal{V} \Leftrightarrow f(x) = ax^2 + bx.$
$a, b \in \mathbb{R}.$

$\left.\begin{array}{l} f_1(x) = 3x^2 \\ f_2(x) = x. \end{array}\right\} \in \mathcal{V}.$ $\quad f_1(x) + f_2(x) = 3x^2 + x \in \mathcal{V}.$
$3 f_2(x) = 3x \in \mathcal{V}.$

define the "+" & ".".

· $h(x) = f(x) + g(x)$.

· $w(x) = a f(x)$.



→ Smooth functions → all deriv. exists & all continuous.

Example (A.9)

$V_1 = \{ f : [a,b] \to \mathbb{R} \mid \text{smooth} \}$.

V_1 is a vector space.

A.10. $V_2 = \{ f : [a,b] \to \underbrace{\mathbb{R} \mid}_{\text{smooth}} \; f(a) = f(b) = 0 \}$.

→ Linear Combination & Spans. ...

Variational Equation.

Definition.

linearity: $R(u, v + \alpha w) = R(u, v) + \alpha R(u, w)$.

S be a set. V be a vector space.

$R : S \times V \to \mathbb{R}$.

Variational equation: $R(u, v) = 0. \ \forall v \in \mathcal{V}$

$\downarrow$

if satisfied, u is a sol'n to the variational equation.

Consistency. → variational eqn. consistent w/ BVP.

u solves Problem $\Leftrightarrow$ $R(u, v) = 0. \ \forall v \in \mathcal{V}$.

Problem: BVP.

$$-(k(x)\, u'(x))' + b(x)\, u'(x) + a(x)\, u(x) = f(x).$$

Classical Variational Equation

Pure diffusion problem:

$$-u''(x) = f(x). \qquad x \in \Omega.$$

$$u(0) = g(0).$$

$$u'(L) = d_L.$$

Step 1: build a residual. → (homogeneous eqn.)

$$r(x) = -u''(x) - f(x). \qquad r(x) = 0$$

Step 2: $v(x). \ r(x) = 0 \cdot v(x) = 0.$

$$v \in \mathcal{V}_1, \quad \mathcal{V}_1 = \{ f : [0, L] \to \mathbb{R}, \text{ smooth} \}.$$

Step 3: $$\int_{\Omega} v(x).\, r(x)\ dx = 0, \quad \forall v \in \mathcal{V},$$

$$R_1(u, v) = \int_0^L v(x).\, (-u''(x) - f(x))\ dx.$$

$$\forall v \in \mathcal{V},$$

$$= \int_0^L [-\ u''(x)\, v(x) - f(x)\, v(x).]\ dx$$

Recall " integration by part " ...

Step 4 $$= -\ u'(L).\, v(L) - u'(0) v(0) - \int_0^L (-u')\, v'$$

$R_2(u, v)$. $$- \int_0^L f\, v\, dx. \quad \forall v \in \mathcal{V},$$

$$u v|_L - u v|_0 = \int_0^L (u v)' = \int_0^L u' v + \int_0^L u v'$$

rearrange ...

Step 5. Replace terms w/ the BCs.

$$R_3(u, v) = -d_L v(L) + u'(0).\, v(L) + u'(0) v(0) + \int_0^L u' v' - \int_0^L f v, \quad \forall v \in \mathcal{V}.$$

Play some numerical "tricks".

$$R(u,v) = \int_0^L u'(x)\, v'(x)\, dx - d_L V(L) - \int_0^L f(x)\, v(x) dx = 0.$$

$$\mathcal{V} = \{ v: [0,L] \to \mathbb{R} \text{ smooth} \mid v(0) = 0 \}.$$

⇒ Natural & Essential BCs.

↓

any functions that satisfies.
& variational eqn. needs to sat.

↘ any BCs.
there are not
Natural BCs.

Nitsche's Method.

$$F(u, v) = 0, \forall v \in \mathcal{V}_1.$$

$$G(u, v) = 0, \forall v \in \mathcal{V}_2.$$

$$\alpha F(u, v) + \beta G(u, v) = 0, \forall v \in \mathcal{V}_1 \cap \mathcal{V}_2.$$

↘ Reformulate the variational problem.

combine to 2 var. eqns & R_3.

residual stabilize method.

Formulation on $\mathcal{V}$.

↑ weak form vs. strong form.

↓

Variational meth. → Integration of diff. eqn.

is just one way to and then solve it.
do weak form

(c). — prove analytical in part A. satisfy var eqn.

(d) — 1 const.

(e) — the other const.

think of a very simple test function.

$v(0)$, $v'(0)$. $u(0)$, $u(1)$. used in the variational eqn.

Lecture 3. 1/16/2024.

Review: variational equation. $R(u,v)=0$.

$$\mathcal{S} \times \mathcal{V} \to \mathbb{R}.$$

$\mathcal{S}$ be a set of $\mathcal{V}$.

Euler - Lagrange Equations

Example $u \in \mathbb{R}$.

$$R(u,v) = v(u^2 + \ln u - 1) = 0.$$

$$\forall v \in \mathbb{R}.$$

$$R(u,v)=0 \quad \forall v \in \mathcal{V}.$$

- $R(u,0)=0$ ← didnt learn anything
- $R(u,1) = u^2 + \ln u - 1 = 0$ (*).
- $R(u,1000) = 1000(u^2 + \ln u - 1) = 0$.

if (*) is satisfied $\Rightarrow R(u,v)=0. \ \forall v$.

Euler-Lagrange Eq.: $R(u,v)=0 \to EL(u,v)=0$

$$\forall x \in \omega \subseteq \overline{\Omega}$$

↙ Is called the Euler-Lagrange Eqn.

$$R(u,v) = \int_0^L u'v' - fv\,dx - d_L v(L) = 0$$

$$\mathcal{V} = \{ v : [0, L] \ \text{smooth} \mid v(0) = 0 \}.$$

$$R(u, v) = u' v \Big|_0^L - \int_0^L u'' v \, dx - \int_0^L f v \, dx$$

$$- d_L v(L) = 0.$$

$$= (u'(L) - d_L) v(L) - u'(0). v(0)$$

$$- \int_0^L (u'' + f) v \, dx = 0.$$

$$\forall v \in \mathcal{V}.$$

$\rightarrow \quad v \in \mathcal{V}. \quad$ s.t. $\quad v(0) = v(L) = 0$

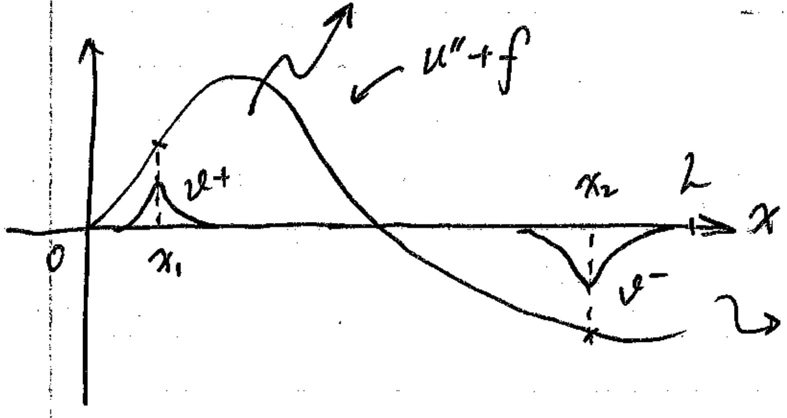
$$\Rightarrow - \int_0^L (u'' + f) v \, dx = 0 \quad \forall v \in \mathcal{V}.$$

$$v(0) = v(L) = 0$$

Assume $\exists x_1 \in (0, L)$. s.t. $u''(x) + f(x) > 0$

$$\int_0^L (u'' + f) v^+ \, dx > 0 \Rightarrow u'' + f \leq 0.$$

$$\forall x \in (0, L).$$



$\rightarrow u'' + f = 0, \quad \forall x \in (0, L).$

$\Rsh$ choose v^-, $u'' + f \geq 0$.

$$\Rightarrow R(u,v) = \left(u'(L) - d_L\right) v(L) - \cancel{u'(0)\, v(0)}$$

Choose $v \mid v(L)=1$. $\Rightarrow R(u,v) = u'(L) - \frac{0}{L} = 0$

$$\downarrow$$

or $u'(L) = d_L$.

$$EL(u,x) = \begin{cases} u''(x) + f(x) = 0. & x \in (0,L). \\ u'(L) = d_L. \end{cases}$$

Nitsche's Method.

$$\int_0^L u'(x)\, v'(x)\, dx + u'(0)\, v(0) + u(0)\, v'(0) + \mu u(0) v(0)$$
$$- \int_0^L f(x)\, v(x)\, dx - d_L v(L) - g_0 v'(0) - \mu g_0 v(0) = 0$$

$$u'(L)\, v(L) - u'(0)\, v(0) - \int_0^L u''(x)\, v(x)\, dx + u'(0)\, v(0)$$
$$+ u(0)\, v'(0) + \mu u(0)\, v(0) = \int_0^L f(x)\, v(x)\, dx + d_L v(L).$$
$$+ g_0 v'(0) + \mu g_0 v(0)$$

$$\int_0^L \underbrace{\left(u''(x) + f(x)\right)}_{\swarrow} v(x)\, dx = \left(u'(L) - d_L\right) v(L)$$
$$+ \left(u(0) - g_0\right) v'(0) + \mu\left(u(0) - g_0\right) v(0)$$

* strictly zero.

* Assumption & procedures: $u''+f=0$

and both BCs are natural and the BCs have to be satisfied ... why?

Affine Subspace.

$\mathcal{W}$ is a vector space. An affine subspace is s.t. $\mathcal{V}=\{s_2-s_1 \mid s_2\in\mathcal{S}\}$.

$\downarrow$

is a vector of $\mathcal{W}$.

Example 1 $\mathcal{W}=\mathbb{R}^2$:

$v=(-1,-1)$.

$S_1=\{\alpha v \mid \alpha\in\mathbb{R}\}$ ← v.s.

$S_2=\{\alpha v+(0,1) \mid \alpha\in\mathbb{R}\}$.

$\downarrow$

Affine subs. of L_2.

$s_1=\alpha_1 v+(0,1)$.

$s_2=\alpha_2 v+(0,1)$.

$s_1+s_2=(\alpha_1+\alpha_2)v+(0,2)$.

$\mathcal{V}=\{s_2-s_1 \mid s_2\in\mathcal{S}\}$.

$=\{(\alpha_2-\alpha_1)v \mid \alpha_2,\alpha_1\in\mathbb{R}\}$.

Example 2 $\mathcal{V}_3=\{w:[a,b]\to\mathbb{R}$ smooth $\mid w(a)=w(b)=1\}$.

$w_1\in\mathcal{V}_3$

$\mathcal{V}_2=\{w_2-w_1 \mid w_2\in\mathcal{V}_3\}$.

$=\{u:[a,b]\to\mathbb{R}$ smooth $\mid w(a)=w(b)=0\}$

$\mathcal{S}$ affine subspace. $s_1 \in \mathcal{S}$

$\mathcal{V} \to$ DIRECTION.

$$\mathcal{S} = \{ s_1 + v \mid v \in \mathcal{V} \}.$$

* Variational Problems and Weak Forms.

Abstract variational problem. $R(\,,\,) : \mathcal{S} \times \mathcal{V} \to \mathbb{R}$.

be an affine space, i.e., trial space.

Definition: Weak form.

Variational prob. $R(u, u) = 0$

* Discussion on the strong/weak form

"IFF" condition ... is it exact ???

Problem 1.3 $\to$ Problem 1.2

1.22

$\mathcal{V}_1 = \{ f : [0,1] \to \mathbb{R}$ smooth $\}$.

$$l(v) = \int_0^1 x^2 v(x)\, dx.$$

- $l(v)$ can be computed
- $l(v + \alpha w) = \int_0^1 x^2 (v + \alpha w)\, dx$

$$= \int_0^1 x^2 v\,dx + \alpha \int_0^1 x^2 w\,dx \quad = l(v) + \alpha l(w).$$

$$l(\cos x) = \int_0^1 x^2 \cos x\,dx = 2\cos(1) - \sin(1).$$

$$l(v) = \int_0^L f(x)\,v(x)\,dx$$

1.24. $V \equiv$ continuous functions over $\mathbb{R}$.

$$l(v) = v(0).$$

$$= \int_{\mathbb{R}} \delta(x)\,v(x)\,dx.$$

Lecture 4.   1/18/2024.

Rev: Affine subs.

Abstract variational prob. $\mathcal{S}$: trial space,

find $\mathcal{S}$: s.t. $R(u,v)=0$ $\langle$ $\mathcal{V}$: vector space.

$\forall v \in \mathcal{V}$.

$R(u,v)=0$, $\forall v \in \mathcal{V}$.

$\uparrow$

linear

Example: $v(u^2 + \ln u - 1)$.

linear function: $\mathbb{V} \to \mathbb{R}$. s.t. : $l(u+av) = l(u) + a\,l(v)$.

Ex. : $\int_0^L f(x) \cdot v(x) .\, dx$.

Bilinear form.

$\forall u, v \in \mathbb{W}$, $w, z \in \mathbb{V}$.

$$a(u+\alpha v, w) = a(u,w) + \alpha\, a(v,w).$$
$$a(u, w+\alpha z) = a(u,w) + \alpha\, a(u,z)$$

$\forall u, v \in \mathbb{V}$: $a(u,v) = a(v,u)$.

$\mathbb{W} \overset{\Downarrow}{=} \mathbb{V}$.

Ex. : $a: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ $a(u,v) = uv$.

(1.25)

$\mathbb{V}_1 = \{f : [0,1] \to \mathbb{R}$ smooth $\}$

means "well-defined".

$a: V_1 \times V_1 \to \mathbb{R}$.

Reverse example:

$v \equiv u = \frac{1}{\sqrt{x}}$.

$$a(u,v) = \int_0^1 u'(x)\, v'(x)\, dx.$$

$$a(u+\alpha w, v) = \int_0^1 (u' + \alpha w')\, v'$$

$$= \int_0^1 u'v' + \alpha \int_0^1 w'v'$$

$$= a(u,v) + \alpha\, a(w,v). \quad \checkmark$$

$$a(\sin x, x^2) = \int_0^1 \cos x\, 2x\, dx = 2(\sin(1) - \cos(1)).$$

$\downarrow$

"gives u a number."

linear variational Equations.

bilinear form: $a(\cdot,\cdot): \mathcal{W} \times \mathcal{V} \to \mathbb{R}$.

linear form: $l(\cdot): \mathcal{V} \to \mathbb{R}$

$$R(u,v) = a(u,v) - l(v)$$

$$\downarrow$$

linear var. $a(u,v) = l(v)$

* when $u = 0$, $R(0,v) \neq 0$, "affined"

how to construct / test linear variational Eqn.?

Combined u, v terms $\to a(u,v)$. v terms $\to l(v)$

linear Comb. / Span.

$$\text{span}(\mathcal{V}) = \left\{ \sum_i^n c_i e_i \mid n \in \mathbb{N},\ e_i \in \mathcal{V},\ c_i \in \mathbb{R} \right\}.$$

Ex. A.14.

$\mathcal{V}_1 = \{e_1, e_2\} \subset \mathbb{R}^3$.

$e_1 = (1, 0, 0)$, $e_2 = (1, 0, 1)$.

$$\text{Span}(\mathcal{V}_1) = \{c_1 e_1 + c_2 e_2 \mid (c_1, c_2) \in \mathbb{R}^2\}.$$

$$= \{(c_1 + c_2, 0, c_2) \mid (c_1, c_2) \in \mathbb{R}^2\}$$

A.15. $\mathcal{V}_2 = \{1, x, x^2\}$. ... direction.

$\text{Span}(\mathcal{V}_2) = \mathbb{P}_2$ 2nd-order Polynomials. $[0,1] \to \mathbb{R}$ smooth

example: $(3, 4, 5) \to 3x^2 + 4x + 5$

A.18 (follow-up. A.14).

$$c_1 e_1 + c_2 e_2 = 0 \Leftrightarrow c_1 = c_2 = 0$$

A.19. (fu. A.15).

$$p(x) = c_1 + c_2 x + c_3 x^2 = 0 \quad \forall x$$

"trick": $P(0) = 0 = c_1$.

$P(1/2) = c_2/2 + c_3/4 = 0$. $\begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$

$P(1) = c_2 + c_3 = 0$

basis & dimension.

$\mathcal{V} = \{e_1, \dots, e_n\}$ is a basis of $\mathcal{V}$

if $\mathcal{V}$ lin. ind. & $\text{span}(\mathcal{V}) = \mathcal{V}$

Build numerical methods.

Variational numerical method.

Find $u_h \in \mathcal{S}_h$, s.t. $R_h(u_h, v_h) = 0$.

$\forall v_h \in \mathcal{V}_h$

Classical Galerkin Method.

Construct "BASE SPACE":

$$\mathcal{W}_h = \text{span}(\{1, x, \dots, x^p\}).$$

$w_h \in \mathcal{W}_h \Rightarrow w_h = w_0 + w_1 x + \dots + w_p x^p.$

$(w_0, \dots, w_p) \in \mathbb{R}^{p+1}$

$\mathcal{S}_h \subset \mathcal{W}_h$, $\mathcal{S}_h = \{w_h \in \mathcal{W}_h \mid w_h(0) = 3\}$

$\nwarrow$ enforce essential BCs.

$u_h \in \mathcal{S}_h$, $\quad u_h(x) = 3 + u_1 x + u_2 x^2 + \dots + u_p x^p.$

$\Downarrow$

$(u_1, \dots, u_p)$

$\mathcal{V}_h$ is direction of $\mathcal{S}_h$.

$$V_h = \{w_h \in W_h \mid w_h(0) = 0\}.$$

$$= \{w_1 x_1 + \dots + w_p x_p \mid (w_1, \dots, w_p) \in \mathbb{R}^p\}.$$

Side Note:

Integrating by part:

$$\int_0^1 \left[w'(x)u'(x) + \lambda w(x)u(x) - w(x)\cdot x^2\right] dx$$

$$= \int_0^1 w'(x)u'(x)\, dx + \int_0^1 \left[\lambda w(x)u(x) - w(x)\cdot x^2\right] dx$$

$$= \int_0^1 u'(x)\, d\, w(x) + \int_0^1 \left[\lambda w(x)u(x) - w(x)\cdot x^2\right] dx$$

$$= w(x)\, u'(x)\Big|_0^1 - \int_0^1 w(x)\, du'(x) + \int_0^1 \left[\lambda w(x)u(x) - w(x)\cdot x^2\right] dx$$

$$= w(1)\,u'(1) - w(0)\,u'(0) + \int_0^1 -u''(x)\,w(x) + \lambda w(x)u(x) - w(x)x^2\, dx$$

$$= w(1)u'(1) - \overset{=0 \text{ for sure}}{w(0)}u'(0) + \int_0^1 w(x)\left[-u''(x) + \lambda u(x) - x^2\right] dx.$$

$\int u\,dv = uv - \int v\,du$

Derivation

Problem 4 - 1. $\rightarrow$ 1st term

$$\int_0^1 w(x)\left[(1+x^2)u''(x) + xu'(x) + x^2u(x)\right]dx.$$

$$\int_0^1 \underbrace{w(x)(1+x^2)}_{a}\, d\overset{u'(x)}{u(x)} + \int_0^1 w(x)\left[xu'(x) + x^2u(x)\right]dx.$$

IBP

$$= w(x)(1+x^2)u'(x)\Big|_0^1 - \int_0^1 u'(x)\,d\left[w(x)(1+x^2)\right] + \int_0^1 w(x)\left[xu'(x) + x^2u(x)\right]dx.$$

$$= 2w(1)u'(1) - w(0)u'(0) - \int_0^1 u'(x)\left[w'(x)(1+x^2) + w(x)\cdot 2x\right]dx + \int_0^1 w(x)\left[xu'(x) + x^2u(x)\right]dx.$$

$$= 2w(1)u'(1) - w(0)u'(0) - \int_0^1 \left[u'(x)w'(x)(1+x^2) + u'(x)w(x)2x\right]dx. + \int_0^1 w(x)\left[xu'(x) + x^2u(x)\right]dx$$

Lecture 5. 1/23/2014.

Consistency : if u is a sol'n of a BVP.

$$\underset{\swarrow}{\text{discretize}} \quad R(u, v) = 0 \quad \forall v \in \mathcal{V}.$$

$$\rightarrow R_h(u, v_h) = 0, \quad \forall v_h \in \mathcal{V}_h.$$

"you have to satisfy this for every BCs".

Ex. 1.33. classical Galerkin Method.

we need to prove: $a(u, v_h) = l(v_h). \ \forall v_h \in \mathcal{V}_h.$

$$\mathcal{V}_h = \text{span}\{x, x^2, x^3\}.$$

$$S_h = \{3 + v_h \mid v_h \in \mathcal{V}_h\}.$$

we know $a(u, v) = l(v). \ \forall v \in \mathcal{V}.$

$$= \{v : [0,1] \rightarrow \mathbb{R} \text{ smooth} \mid v(0) = 0\}.$$

$$\mathcal{V}_h \subset \mathcal{V}.$$

1.35 (counter example)

$$\mathcal{V}_h = \{1, x, \dots, x^{p-1}\}. \rightarrow \text{Petrov-Galerkin meth.}$$

$\downarrow$

Sec. 1.3.3.

$\Downarrow$

this method is not consistent.

$$0 = R(u, v_h), \quad \forall v_h \in \mathcal{V}_h.$$

$$0 = \int_0^1 u'v_h' + bu'v_h + uv_h \, dx$$

$$= \int_0^1 (-u'' + bu' + u) v_h \, dx + u'v_h\Big|_0^1$$

$= - u'(0)\, v_h(0)$.

Consisteancy: whether v_h satisfy continuous v.

choice of v_h leads to inconsistency.

⇒ Patch Test Property

$u \in \mathcal{S}_h, \quad \Rightarrow \quad u_h = u$.

↖ has the patch test property.

Galerkin Condition.

$$\mathcal{V}_h = \{ v_h = w_h - z_h, \quad \dots \}.$$

- Bubnov - G, G & continuous - G.
- Discontinuous - G.
- Petrov - G: test space for

1.36. $\mathcal{S}_h = \{ 3 - x + w_1 x^2 + w_2 x^3 \mid w_1, w_2 \in \mathbb{R} \}$.

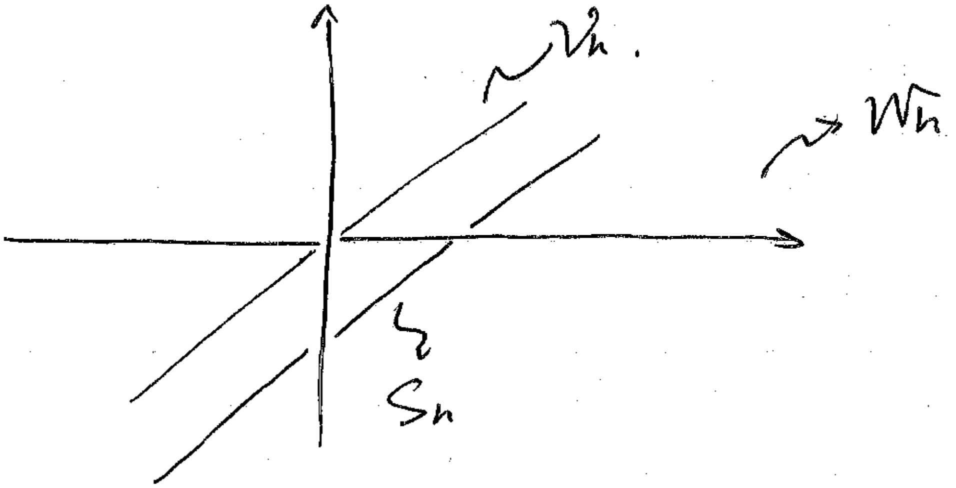
$$= \{ 3 - x + 10x^2 + 100x^3 - w_1 x^2 + w_2 x^3 \mid w_1, w_2 \in \mathbb{R} \}.$$

$\hookrightarrow \mathcal{V}_h = \{ w_1 x^2 + w_2 x^3 \mid w_1, w_2 \in \mathbb{R} \}$.

$\mathcal{S}_h = \{ w_h \in \mathbb{P}_3(\Omega) \mid w_h(0) = 3;\ w_h'(0) = -1 \}$.

$\mathcal{V}_h = \{ w_h \in \mathbb{P}_3(\Omega) \mid w_h(0) = 0,\ w_h'(0) = 0 \}$.

Classic Discrete Variational Problem.



1.38 · $\mathcal{W}_h = \mathbb{P}_4(\Omega) = \{w_h = w_0 + w_1 x + w_2 x^2 + \cdots w_4 x^4\}$
where . $w_i \in \mathbb{R}$

$\mathcal{S}_h = \{w_h \in \mathcal{W}_h \mid w_4 = 0,\ w_1 = -1,\ w_0 = 3\}$.

following 1.36

⇒ Classic Discrete Linear Variational Problem

1) Assuming a basis : $\{N_1, \dots, N_m\} \equiv$ Basis for $\mathcal{W}_h$.

$\mathcal{S}_h \subseteq \mathcal{W}_h, \quad \mathcal{V}_h \subseteq \mathcal{W}_h$.

$$u_h(x) = \sum_{b=1}^{m} u_b N_b(x). \quad \in \mathcal{V}_h.$$

$$v_h(x) = \sum_{a=1}^{m} v_a N_a(x). \quad \in \mathcal{V}_h.$$

2). $\underbrace{\underbrace{N_1, N_2, \dots, N_n}_{\text{basis for } \mathcal{V}_h}, N_{n+1}, \dots, N_m}_{\text{basis for } \mathcal{W}_h}$

3). $a(u_h, N_a) = \ell(N_a)$ $\quad 1 \le a \le n$.

we need m, we only have n DoF.

4). choose $\bar{u}_h \in \mathcal{S}_h$.

$$\text{s.t. } \bar{u}_h = \underbrace{\bar{u}_1 N_1 + \cdots + \bar{u}_n N_n}_{\in \mathcal{V}_h} + \underbrace{\cdots + \bar{u}_m N_m}_{\notin \mathcal{V}_h}.$$

$u_a = \bar{u}_a, \quad n+1 \le a \le m$.

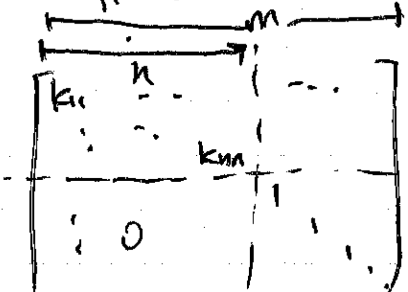
$$\ell(N_a) = a\left(\sum_{b=1}^{m} u_b N_b, N_a\right).$$

$$= \sum_{b=1}^{m} u_b \, a(N_b, N_a).$$

$F_a = \ell(N_a), \quad K_{ab} = a(N_b, N_a \mid 1 \le a \le n, 1 \le b \le m)$.

$$F_a = \bar{u}_a, \quad K_{ab} = \delta_{ab} = \begin{cases} 1 & a=b \\ 0 & a \ne b \end{cases}$$

$$\underbrace{\overbrace{\begin{bmatrix} k_{11} & \cdots & k_{1m} \\ & \ddots & \\ k_{m1} & & k_{mm} \end{bmatrix}}^{K}}_{\text{Stiffness matrix}} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \underbrace{\begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix}}_{\text{load vector}} \qquad KU = F.$$



$3+x+w_1x^2+w_2x^3$

$3+x+100x^2+1000x^3+v_1x^2+h_1x^3$

we select.

$W_h = \mathrm{span}(\{1, x, x^2, x^3\})$.

$S_h = \{x + v_h \mid v_h \in V_h\}$.

$V_h = \mathrm{span}(\{x, x^2, x^3\})$.

$\rightarrow m=4, \quad n=3.$

$N_1 = x \qquad N_2 = x^2 \qquad N_3 = x^3. \quad \dots \rightarrow N_4 = 1.$

$\bar{u}_h \in S_h, \quad \bar{u}_h = 3+x.$

$1 \le a \le 3 \qquad a(u_h, N_1) = l(N_1).$

$a(u_h, N_2) = l(N_2).$

$a(u_h, N_3) = l(N_3).$

$a=4. \qquad u_4 = \bar{u}_4 = 3.$

Remarks: the choice of basis for W_h.

$V_h = \{w_h \in W_h \mid w_h(x_0) = 0\}$.

$v_h \in V_h \iff \sum_{a=1}^{m} v_a N_a(x_0) = 0 \iff v_1 = 0.$

$x_0 = 0. \quad v_h \in V_h. \iff v_1 = 0.$

$x_0 = 2, \quad v_h \in V_h \iff v_1 \cdot 1 + v_2 \cdot 2 + v_3 \cdot 2^2 + v_4 \cdot 2^3 = 0$

the choice of V_h impacts the coefficients

Simpliest v_h of choice:

$$\bar{u}_h = \bar{u}_{n+1} N_{n+1} + \cdots + \bar{u}_m N_m$$

$$\bar{u}_h \in \mathcal{S}_h.$$

$$\bar{u}_h^* = \bar{u}_h + v_h \in \mathcal{S}_h \qquad N_u \in \mathcal{V}_h.$$

- -

HW 2. Derivation on the differences.
(Pb 3).

$$a(w,u) \rightarrow \int_0^1 2x\, w'(x)\, u(x)\, dx - \int_0^1 x\, u(x)\, w'(x)\, dx \quad \cdots (\Delta)$$

$$a(u,w) \rightarrow \int_0^1 2x\, u'(x)\, w(x)\, dx - \int_0^1 x\, w(x)\, u'(x)\, dx \quad \cdots (\Delta\Delta)$$

For Eqn. (Δ):

$$\rightarrow \int_0^1 2x\, u(x)\, d w(x) - \int_0^1 x\, u(x)\, d w(x).$$

$$2x\, u(x)\, w(x) \Big|_0^1 - \int_0^1 2x\, w(x)\, d u(x) - x\, u(x)\, w(x) \Big|_0^1 + \int_0^1 x\, w(x)\, d u(x).$$

$$x\, u(1)\, w(1) - \overbrace{x\, u(0)\, w(0)} + \left(-a(u,w)\right)$$

Expand the non-BCs terms.

$$-\int_0^1 u'(x)\,\underset{\checkmark}{w'(x)}\,(1+x^2)\,dx - \int_0^1 u'(x)\,w(x)\,x\,dx$$

$$+ \int_0^1 w(x).\,\underset{\checkmark}{u(x)}\,x^2\,dx$$

$$\Downarrow$$

let $a(u,w)$ = this form

... test bilinearity.

$$a(w,u) = -\int_0^1 w'(x)\,\underset{\checkmark}{u'(x)}\,(1+x^2)\,dx$$

$$-\int_0^1 w'(x)\,u(x)\,x\,dx + \int_0^1 u(x)\,\underset{\checkmark}{w(x)}\,x^2\,dx$$

... ?

→ How to show $w'(x)\,u(x) = u'(x)\,w(x)$?

assume relationship exists:

$$\int u(x)\,dw(x) \overset{?}{=} \int w(x)\,du(x)$$

$$u(x)w(x)\Big|_\Omega - \int w(x)\,du(x) = u(x)\,w(x)\Big|_\Omega - \int u(x)\,dw(x)$$

$$u(x)w(x)\Big|_\Omega = 2\int w(x)\,du(x)$$

$W_h = \text{span}\{1, x, x^2, x^3\}$.

trial space: $x.$ x

test space: 1

$$\begin{cases} N_1 = x \\ N_2 = x^2 \\ N_3 = x^3 \end{cases} \qquad N_4 = 1$$

↓ test space

active ind.

↘ trial

constrained. ind.

$\mathcal{V}_h = \text{span}(\{x, x^2, x^3\})$.

$\mathcal{S}_h = \{1 + v_h \mid v_h \in \mathcal{V}_h\}$.

$\mathcal{S}_h = \{w_h \in \mathcal{W}_h \mid w_h(0) = 1\}$.

$= \{w_h = 1 + w_1 x + w_2 x^2 + w_3 x^3 \mid (w_1, w_2, w_3) \in \mathbb{R}^3\}$.

$a(u_h, N_1) = \ell(N_1)$

$a(u_h, N_2) = \ell(N_2)$

$a(u_h, N_3) = \ell(N_3)$

$$F = \begin{bmatrix} \ell(N_1) \\ \ell(N_2) \\ \ell(N_3) \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$u_4 = 1$.

$$K = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 4/3 & 3/2 & 0 \\ 1 & 3/2 & 9/5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$U = K^{-1} F$.

Derivation for consistency check.

- For $W_k = 1$

$$\int_0^1 u'(x)\, 2x\, dx - \int_0^1 [x u'(x) + x^2 u(x)]\, dx - 6u(1) = 0$$

$$\int_0^1 2x\, du(x) - \int_0^1 x\, du(x) - \int_0^1 x^2 u(x)\, dx - 6u(1) = 0$$

$$\int_0^1 x\, du(x) - \int_0^1 x^2 u(x)\, dx - 6u(1) = 0$$

$$u(1) - \int_0^1 u(x)\, dx - \int_0^1 x^2 u(x)\, dx - 6u(1) = 0$$

$$-5u(1) - \int_0^1 (1 + x^2)\, u(x)\, dx = 0$$

$$-5u(1) - \int_0^1 u(x)\, d\left[\frac{x^3}{3} + x\right]$$

$$-5u(1) - \frac{x}{2}$$

$$\to \quad W_n' = 3ax^2+2bx+c \; .$$

$$W_n = ax^3+bx^2+cx+d.$$

$$\int_0^1 u'(x)\left[3ax^2+2bx+c\right](1+x^2)\,dx + \int_0^1 u'(x)\left[ax^3+bx^2+cx+d\right]2x\,dx$$

$$-\int_0^1 (ax^3+bx^2+cx+d)\,(xu'(x)+x^2u(x)]\,dx = 6W(1)\,u(1)$$

1.35 $$R(u,v_n) = \int_0^1 \left[u'v_n' + bu'v_n + uv_n\right]dx$$

$$= \int_0^1 (-u''+bu'+u)\,v_n\,dx + u'(0)\,v_n(0)$$

$$= u'(0)\,v_n(0).$$

$$\int_0^1 u'(x)(1+x^2)\,dW_n(x) + \int_0^1 u'(x)\,W_n(x)\,2x\,dx.$$

$$-\int_0^1 W_n(x)\left[xu'(x)+x^2u(x)\right]dx - 6W_n(1)u(1) = 0$$

$$W_n(x)\,u'(x)(1+x^2)\Big|_0^1 - \int_0^1 W_n(x)\,d\left[u'(x)(1+x^2)\right]$$

$$+\int_0^1 u'(x)\,W_n(x)\,2x\,dx - \int_0^1 W_n(x)\left[xu'(x)+x^2u(x)\right]dx$$

$$-6W_n(1)\,u(1) = 0$$

$$2W_n(1)u'(1) - W_n(0)\,u'(0) - 6W_n(1)u(1) - \int_0^1 W_n(x)\left[u''(1+x^2)+2xu'\right]dx$$

$$+\int_0^1 W_n(x)\,u'(x)\,2x\,dx - \int_0^1 W_n(x)\left[xu'+x^2u\right]dx.$$

$$\underbrace{2u'(1) - 6u(1)}_{=0}$$

$$2W_h(1)u'(1) - W_h(0)u'(0) - 6W_h(1)u(1)$$

$$-\int_0^1 W_h(x)\Big[\underbrace{u''(1+x^2) + \underbrace{2xu' - 2u'x}_{=0} + xu' + x^2u}_{=0 \text{ from problem.}}\Big]\,dx$$

Lecture #6 1/25/2024

essential BCs: S_h. test space: $\mathcal{V}_h$.

$a_h(u_h, v_h) = l_h(v_h)$. $\forall v_h \in \mathcal{V}_h$.

$\{N_1, \dots, N_n\}$. basis for $\mathcal{V}_h$.

Impose $a_h(u_h, v_a) = l(v_a)$, $a = 1, \dots, n$.

$$a_h(u_h, v_h) = a_h(u_h, \sum_{a=1}^{n} v_a N_a)$$

$\uparrow$ a is bilinear
"takes the sum out".

$$= \sum_{a=1}^{h} v_a \, a_h(u_h, N_a).$$

$$= \sum_{a=1}^{h} v_a \, l_h(N_a)$$

$$= l\left(\sum_{a=1}^{h} v_a N_a\right) = l(v_h)$$

"Shuffling functions".

$\{N_a\}_{a=1,\dots,m}$ basis for $\mathcal{V}_h$.

$\eta = \{1, \dots, m\}$. index set

$\eta_a \subset \eta$, η_a = active indices.

$\mathcal{V}_h = \text{span}\left(\bigcup_{a \in \eta_a} \{N_a\}\right)$.

define : $\eta_g = \overset{\curvearrowright}{\eta \setminus} \eta_a \equiv$ constrained indices.

$w_h \in \mathcal{V}_h$. $\Leftrightarrow$ $w_h = \sum_{a \in \eta_a} w_a N_a$

$a(u_h, N_a) = l(N_a)$. $a \in \eta_a$.

$\bar{u}_h = \sum_{a \in \eta_g} \bar{u}_a N_a$

$u_a = \bar{u}_a$, $a \in \eta_g$

Recall Notes (book).

$F_a = l_h(N_a)$, $K_{ab} = a_h(N_b, N_a)$.

before: $N_1 = x$, $N_2 = x^2$, $N_3 = x^3$, $N_4 = 1$.

Now: $N_1 = x$, $N_2 = 1$, $N_3 = x^2$, $N_4 = x^3$.

$\eta = \{1, 2, 3, 4\}$,

$\eta_a = \{1, 3, 4\}$. Remark: indices change → but the result for $\mathcal{V}$ should be the same.

$\eta_g = \{2\}$.

$\bar{u}_2 = 3$.

Ex. 1.44.

$-u'' + u' + u = -5 \exp(-2x)$. $x \in (0, \frac{\pi}{2})$.

$u(0) = 1$.

$u(\pi/2) = \exp(-\pi)$.

$W_h = \text{span}(\{1, \sin x, \sin 2x, \sin 4x\})$.

$N_1 = 1$, $N_2 = \sin x$, $N_3 = \sin 2x$, $N_4 = \sin 4x$.

$S_h = \{ w_h \in W_h \mid w_h(0) = 1, \ w_h(\pi/2) = e^{-\pi} \}$.

$V_h = \{ w_h \in W_h \mid w_h(0) = 0, \ w_h(\pi/2) = 0 \}$.

$w_h(0) = 1 \rightarrow w_1 = 1$

$w_h(\pi/2) = e^{-\pi} \rightarrow w_1 + w_2 = e^{-\pi}$.

$w_2 = e^{-\pi} - 1$

$\bar{u}_h = 1 + (e^{-\pi} - 1) \sin x$.

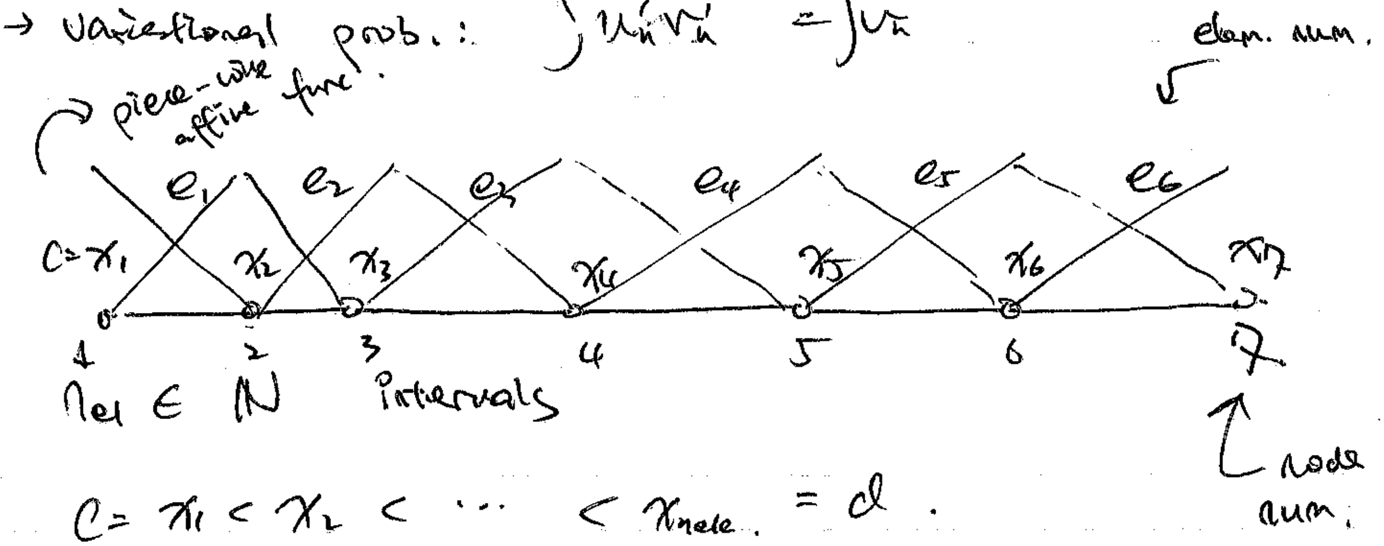
First Finite Element Method.

Deffusion problem.

$-u''(x) = 1$, $u(0) = 2$, $u'(1) = 0$.

$x \in (0, 1)$.

→ variational prob.: $\int u_h' v_h' = \int v_h$



$n_{el} \in \mathbb{N}$ intervals

$c = x_1 < x_2 < \cdots < x_{nele} = d$.

$$W_h = \text{Span}\left(\left\{N_1, \ldots, \overbrace{N_{n_{el}+1}}^{\text{element num.}}\right\}\right).$$ Finite element Space.

$$m = n_{el} + 1$$

1). $\displaystyle\sum_{a=1}^{n_{el}+1} N_a(x) = 1, \quad \forall x \in [c, d].$

2). $N_b(x_a) = \delta_{ba}$

$$w_h(x) = w_1 N_1(x) + \cdots + w_{n_{el}+1} N_{n_{el}+1}(x)$$

$$w_h(x_a) = w_1 N_1(x_a) + \cdots + w_a N_a(x_a) + \cdots + w \ldots$$

$$= w_a N_a(x_a) = w_a.$$

Lecture 7. 1/30/2024.

Finite Element Method in 1D.

- Integration by part of piecewise smooth functions.
- Consistency.

$$\begin{tikzcd}
\text{BVP} \arrow[r] & \text{Discrete Var. Eqn.} \arrow[l, bend left, "\text{consistency}"] \arrow[r] \arrow[r, bend right, leftrightarrow, "\text{trial space}"'] & \text{Var. Num. Meth.} \arrow[ll, bend left=60, "\text{find solution}"]
\end{tikzcd}$$

"Consistency of this piecewise function approach".

$$\int_a^b u'(x) v(x)\, dx = \sum_{i=0}^{k} [\![u(x) v(x)]\!]_{x=c_i} - \int_a^b u(x) v'(x)\, dx$$

$$[\![u]\!]_{x=c} = \underset{\text{from the LHS}}{\underbrace{\lim_{x\to c^-} u(x)}} - \underset{\text{From the RHS}}{\underbrace{\lim_{x\to c^+} u(x)}}$$

\# Consistency:

$$R_h(u_h, v_h) = 0. \quad \forall v_h = \mathcal{V}_h$$

$$R_h(u_h, v_h) = \int_0^1 u_h' v_h' \, dx - \int_0^1 v_h(x) \, dx$$

Need: $R_h(u, v_h) = 0. \quad \forall v_h \in \mathcal{V}_h.$

$$R_h(u_h, v_h) = R(u_h, v_h)$$

notation ⇔ for

$$R(u, v) = \int_0^1 u' v' \, dx - \int_0^1 v \, dx. \quad \forall v \in \mathcal{V} . \cup \mathcal{V}_h$$

$$\mathcal{V} = \{ v : [0, 1] \to \mathbb{R} \text{ smooth} \mid v(0) = 0 \}$$

→ test w/ smooth functions.

↳ what's the largest test set that can be selected?

$\mathcal{V}$: "$\mathcal{H}1$" space.

<u>Definition</u> $e = (K_e, \mathcal{N}^e)$. domain: $K_e \subset \mathbb{R}^d$.

finite set of basis functions $\underline{\underline{\mathcal{N}^e = \{N_1^e, \dots N_k^e\}}}$

↓

or to basis functions ← shape functions

$\mathcal{P}^e = \text{Span}(\mathcal{N}^e)$ = element space

k = degree of freedom, $f^e : K_e \to \mathbb{R}$

$$f^e(x) = \phi_1^e N_1^e(x) + \dots + \phi_k^e N_k^e(x).$$

$$(\phi_1^e, \dots, \phi_k^e) \in \mathbb{R}^k$$

Notation ...

$$e = (K_e, \mathcal{N}^e).$$

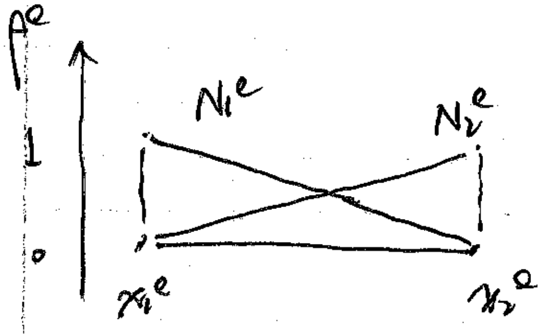
$$\downarrow$$

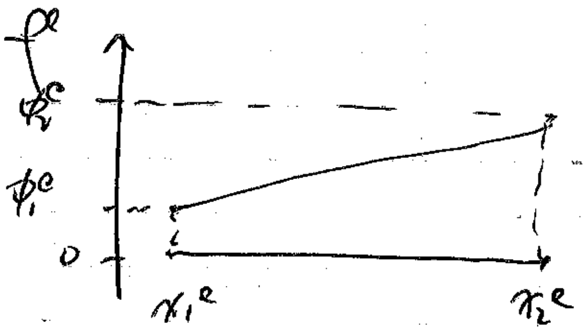
$$K_e = (\hat{K}_e, \mathcal{N}^e).$$

Remark: " K_e & $\hat{K}_e$ ad identical ingeneral ".

Example.

2.1. P₁ - element: $K_e = [x_1^e, x_2^e]$.





$$N_1^e(x) = \frac{x - x_2^e}{x_1^e - x_2^e}, \quad N_2^e(x) = \frac{x - x_1^e}{x_2^e - x_1^e}$$

$$f^e(x) \in P^e \quad \phi_1^e \frac{x - x_2^e}{x_1^e - x_2^e} + \phi_2^e \frac{x - x_1^e}{x_2^e - x_1^e} = 0 \quad \forall x \in K_e$$

"P_1 - Element",

$$P^e = \mathbb{P}_1(K_e).$$

$$N_1^e(x) + N_2^e(x) = 1, \quad \forall x \in K_e$$

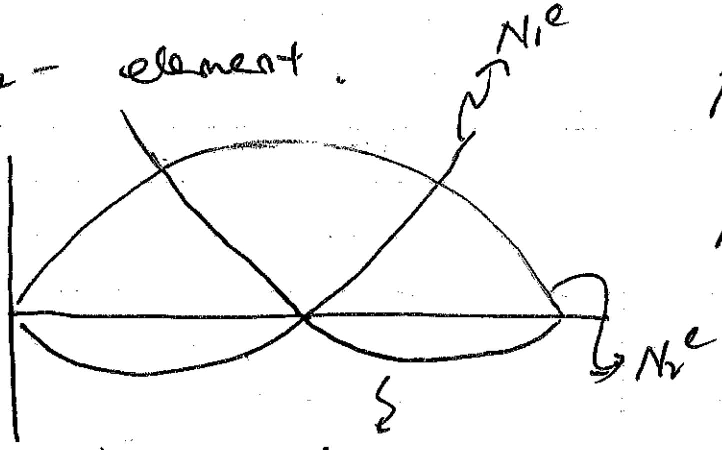
$$\underbrace{x_1^e N_1^e(x) + x_2^e N_2^e(x) = x.}_{\dots \text{ linear independence.}} \quad \forall x \in K_e.$$

$\Downarrow$

Equals to x everywhere.

P_1 - element — (check example!)

P_2 - element.

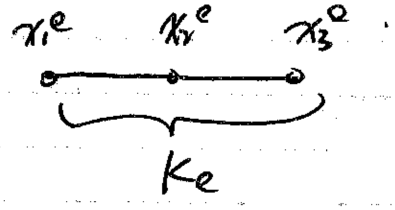


$$N_1^e(x) = \frac{(x-x_2^e)(x-x_3^e)}{(x_1^e-x_2^e)(x_1^e-x_3^e)}$$

$$N_2^e(x) = \frac{(x-x_1^e)(x-x_3^e)}{(x_2^e-x_1^e)(x_2^e-x_3^e)}$$

$$N_3^e(x) = \frac{(x-x_1^e)(x-x_2^e)}{(x^e-x_1^e)(x^e-x^e)}$$

For P_2 - Element.



P_k - element.

$$K_e = [z_1, z_2].$$

$$x_a^e = z_1 + (a-1)\cdot\frac{(z_2 - z_1)}{k}.$$

$$N_a^e(x) = \frac{\prod_{b=1,b\neq a}^{k+1}(x - x_b^e)}{\prod_{b=1,b\neq a}^{k+1}(x_a^e - x_b^e)}.$$

For P_k-element. $k \geq 1$.

$$\text{Span}(N^e) = \mathbb{P}_k(K_e).$$

$$N_a^e(x_b^e) = \Sigma_{ab}. \quad \cdots (*)$$

$$0 = f^e(x) = \underbrace{\phi_1^e N_1^e(x) + \cdots + \phi_a^e N_a^e(x) + \cdots + \phi_{k+1}^e N_{k+1}^e(x)}$$

applying property (*).

" show each of these equals to zero ..

$$f^e(x_b^e) = \phi_b^e.$$

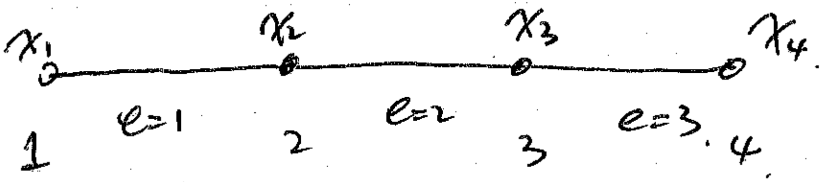
$$f \in \mathbb{P}_k(K_e). \quad g(x) = f(x_1^e)N_1^e(x) + \cdots + f(x_{k+1}^e)N_{k+1}^e(x)$$

"polynomial"

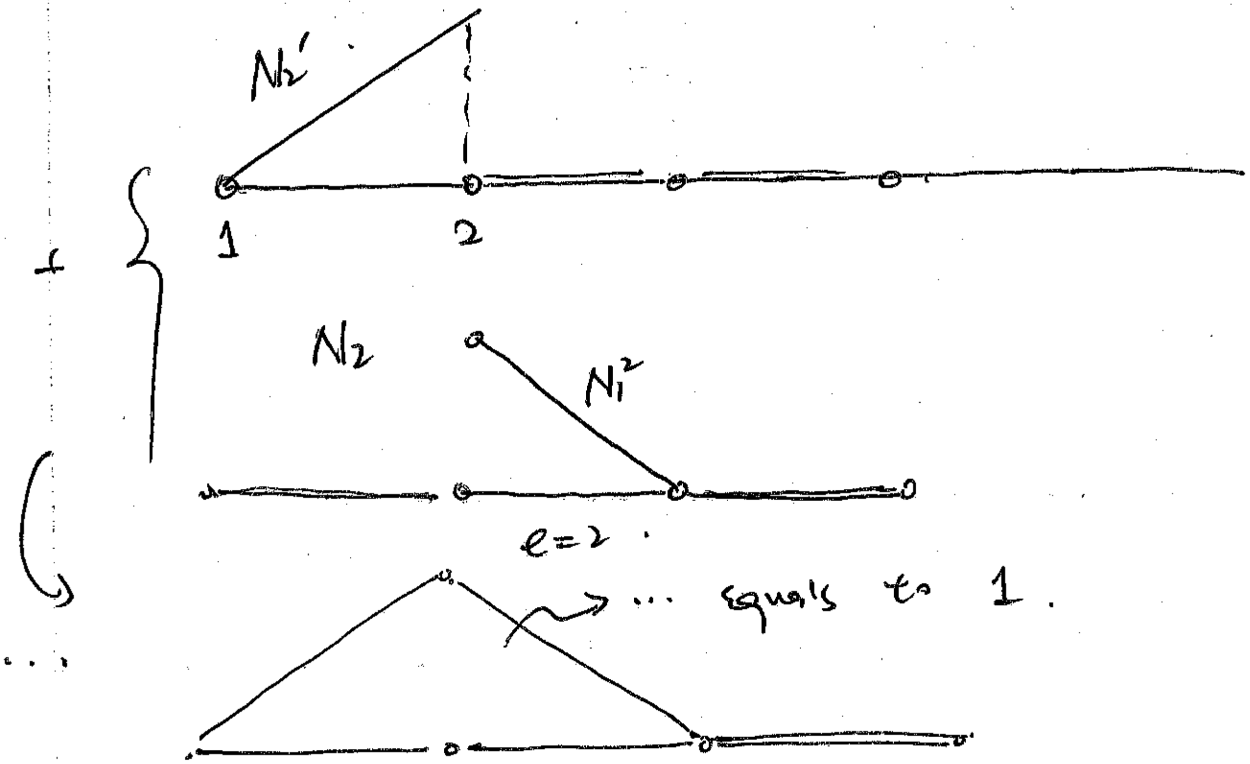
$$f(x) - g(x) = 0. \quad \forall x = x_b^e.$$

"Lagrange Elements". $\leftarrow$ DoF.

Example



P1- elements.

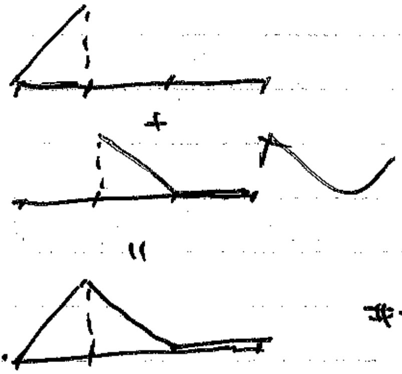


... equals to 1.

Lecture 8 · 2/1/2024.

Review: def'n : pair → (K_e, N^e).

D.F. → num. functions we can build within one element.



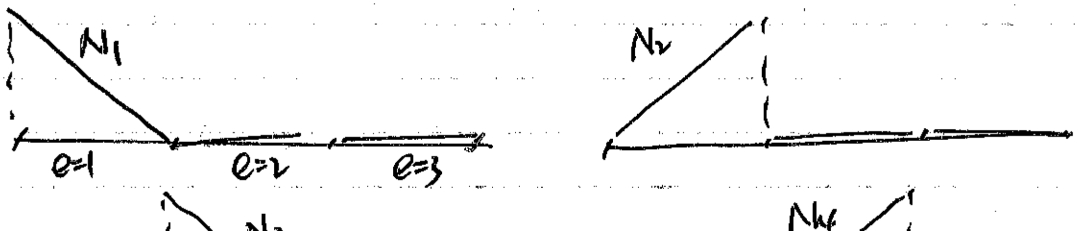
Def'n the values at the node

take the limit at the node

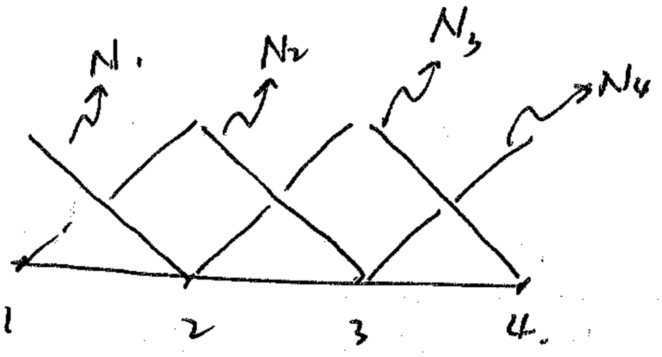
C0 - element. "not C1"

☆ Define a local-to-global map.

Defining LG (2.9).



2.10.



$$LG = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}.$$

#Broken Sum. $\mathcal{W}_h \times \mathcal{W}_h \to \mathcal{W}_h$.

$$(f_h \dot{+} g_h)(x) = f_h(x) + g_h(x). \quad x \neq x_i,$$

and

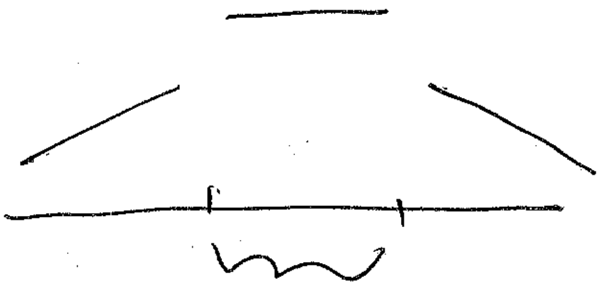
$$(f_h \dot{+} g_h)(x_i) = \lim$$

Continue on 2.9.

$U_n \in \mathcal{W}_n$.

$$U_n = 1 N_1 \dot{+} 2 N_2 \dot{+} 3 N_3 \dot{+} 3 N_4$$
$$\dot{+} 2 N_5 \dot{+} 0 N_6.$$

$$U = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 2 \\ 0 \end{bmatrix}$$



* Confused ...?

broken sum. i.e., function undefined

$N_1 = N_1^1$.

$N_2 = N_2^1$ & N_1^2.

Set: $\{(a,e) \mid LG(a,e)=3\} = \{(2,2), (1,3)\}$

locating indices in the LG matrix.

N_n^e → element index

n ↳ functions.

*Q: after add "&", the N would be discontinuous?

Local-to-Global DoF Map.

$$u_h = \sum_{A=1}^{m} u_A N_A .$$

DoF

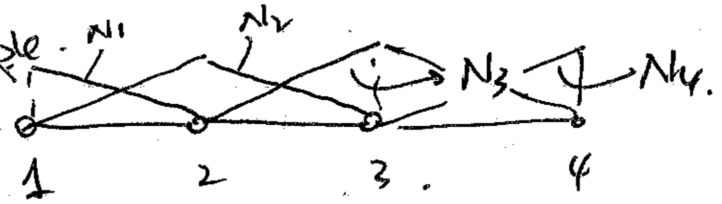
↳ $u_h = \sum^{nel} \sum^{k} u_{LG(a,e)} N_a^e$.

Element stiffness matrix & elem. load vec.

$$a_h(u_h, v_h) = \int_\Omega [k u_h' v_h' + b \cdot u_h' v_h + c u_h v_h] dx.$$

$$l_h(v_h) = k(L) d_L v_h(L). + \int_\Omega f(x). v_h(x) dx.$$

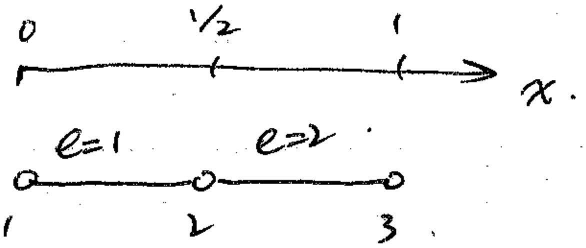
Example.



$$K_{33} = a_h(N3, N3).$$

$$a_h^e = \int_{Ke} \dots$$ ← element stiffness mat.

Example 2.12.



$$a_h^e = \int_{Ke} (u'v' + 3x\, uv) dx.$$

$$l_h^e = \int_{Ke} 10 v\, dx.$$

shape functions.

$$N_1^1(x) = \frac{1/2 - x}{1/2} \qquad N_1^2(x) = \frac{1-x}{1/2}$$

$$N_2^1(x) = x/(1/2). \qquad N_2^2(x) = \frac{x-1/2}{1/2}$$

$$K^1_{ab} = a^1_h(N^1_b, N^1_a)$$

$$= \int_{K_1} (N_a^1)'(N_b^1)' + 3\,x\,N_a^1 N_b^1 .$$

*Q: how is LG used here?

Boundary terms. - - -

Assembly:

*Q. for 2D case. LG → 3D matrix

K → 2D matrix

Lecture 9. 2/6/2024.

Fourth-Order Problems. (last lecture 1D).

Example 3.1. $(q(x)\, u''(x))'' + c(x)\, u(x) = f(x) \quad \forall x \in \Omega.$

$u(0) = g_0$

$u'(0) = d_0$

$u''(L) = m_L$

$u'''(L) = n_L$.

q & c piecewise smooth.

non-negative.

$\Downarrow$

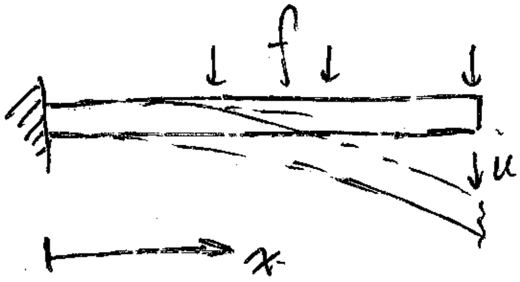
"well-defined".

→ general formulation:

source term ↓

$$\underbrace{(q(x)\, u''(x))''}_{\text{fourth-order term.}} - \underbrace{(b(x)\, u'(x))'}_{\text{diffusion term}} + \underbrace{c(x)\, u(x)}_{\text{reaction term}} = f(x) \quad x \in \Omega$$

→ Euler-Bernoulli Beam.



$$\left[E(x) I(x)\, u''(x)\right]'' = f(x).$$

3.3. Image denoising

$u_0 : \Omega \to \mathbb{R}$.

$[q(x)\, u(x)'']'' + u = u_0 \qquad x \in \Omega$.

#Need to specify a B.C.s. ← based on the order of prob.

$u(0) = g_0$.

$u'(0) = d_0$. → clamped.

$u''(L). = m_L$

$u'''(L) = n_L$. ← bending moment & shear force. i.e., applied load.

Build the residual: $r(x) = (q u'')'' + cu - f$.

$$\int_0^L r v \, dx = 0.$$

$$\int_0^L \underset{\uparrow}{(qu'')''} v + cuv - fv \, dx = 0$$

" IBP for twice ".

first IBP:

$$(qu'')'v\Big|_0^L - \int_0^L (qu'')'v' + \int_0^L cuv - f \, dx = 0$$

$$(qu'')'v\Big|_0^L - qu''v'\Big|_0^L + \int_0^L qu''v'' + cuv - f dx = 0$$

Classical Galerkin formulation.

$$R(u,v) = a(u,v) - l(v) = 0$$

$$a(u,v) = \int_0^L [q(x).u''v'' + cuv] \, dx$$

$$l(v) = \int_0^L f v \, dx - (q(L)n_L + q'(L).m_L)v(L)$$
$$+ q(L) \, m_L v'(L)$$

$$\mathcal{V} : \{ v : [0,L] \to \mathbb{R} \text{ smooth} \mid \underbrace{v(0) = 0 \ \& \ v'(0) = 0}\}$$

#Q: classical Galerkin requirement ???

$$\underbrace{m_L \ \& \ n_L}$$

natural boundary conditions.

$$\underbrace{S_0 \ \& \ d_0}$$

essential B.C.s

Consistency check.

Caso n_L & $M_L = 0$.

→ the derivatives of v_h has to be continuum.

Hermite element.

$$\Omega_e = [x_1^e, x_2^e].$$

$$N_1^e(x) = \left(\frac{x_2^e - x}{x_2^e - x_1^e}\right)^2 \left(1 + 2\frac{x - x_1^e}{x_2^e - x_1^e}\right).$$

$$N_3^e(x) = \left(\frac{x_1^e - x}{x_1^e - x_2^e}\right)^2 \cdot \left(1 + 2\frac{x - x_2^e}{x_1^e - x_2^e}\right)$$

$$N_2^e(x), \quad N_2^e(x), \quad \dots$$

Hermite elem: continuous & cont. deriv.

Lecture 10 2/8/2024.

Sec. 4.1. PDE.

↳ DIFFUSION EQUATION.

$$-\text{div}(K \nabla u) = f. \quad \text{in } \Omega \subset \mathbb{R}^2$$

$u: \Omega \to \mathbb{R}$. unknown.

K: positive definite matrix. $K \in \mathbb{R}^{2\times2}$.

IFF $\vec{x}^T K \vec{x} > 0, \ \forall x \in \mathbb{R}^2$

$$\nabla u = \frac{\partial u}{\partial x_1} \underline{e}_1 + \frac{\partial u}{\partial x_2} \underline{e}_2 \quad (\partial_1 u, \partial_2 u)$$

↳ i.e., gradient.

$v: \Omega \to \mathbb{R}^2 \Rightarrow \text{div } v = \partial_1 v_1 + \partial_2 v_2$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

e.g. $v(x_1, x_2) = x_1 x_2 \underline{e}_1 + (x_1 + x_2) \underline{e}_2$

$(x_1 x_2, \ x_1 + x_2)$.

$v_1 = x_1 x_2, \ v_2 = x_1 + x_2$.

$\text{div } v = x_2 + 1$.

$J = -k \nabla u, \quad J \equiv \text{flux}.$

↳ heat flux. ⇝ i.e., Fourier's law

For the 2D case, diff. eqn. writes,

$$- \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left[\sum_{j=1}^{2} k_{ij} \frac{\partial u}{\partial x_j} \right] = f.$$

$$\Updownarrow$$

$$(k_{ij} u_{,j})_{,i} = f.$$

Ex. 4.1. Poisson's Eqn.

$$K = \begin{pmatrix} k(x) & 0 \\ 0 & k(x) \end{pmatrix}. \qquad k(x) = k_0 \quad \text{const.}$$

$$J = -k \nabla u = - \begin{pmatrix} k_0 & 0 \\ 0 & k_0 \end{pmatrix} \cdot \begin{pmatrix} \partial_1 u \\ \partial_2 u \end{pmatrix} = -k_0 \begin{pmatrix} \partial_1 u \\ \partial_2 u \end{pmatrix}$$

$$\underbrace{-k_0 \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2} \right)}_{} = f$$

$$-k_0 \Delta u . = f \quad \leftarrow \text{Laplacian of } u.$$

$$\vdots$$

Poisson's Eqn.

$$u(x) = C_0 + C_1 x_1 + C_2 x_2$$

$$u(x_1, x_2) = \ln\left[(x_1 - X_1)^2 + (x_2 - X_2)^2 \right]$$

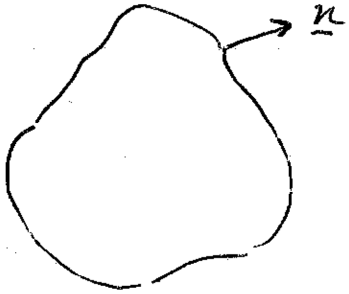
$$(x_1, x) \neq (X_1, X_2).$$

Ex 4.2 Elastic membrane.

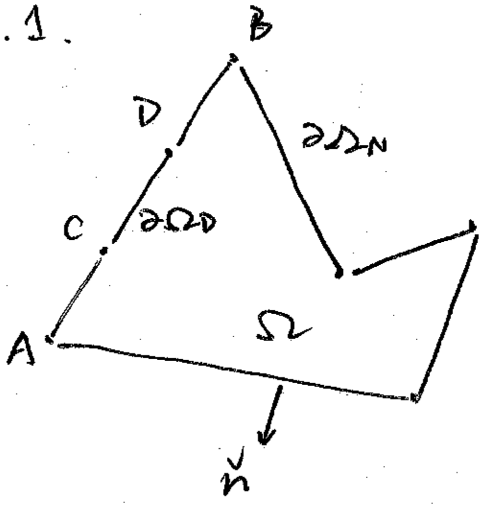
$$P = -\mathrm{div}\,(T\,\nabla u).$$

B.C.s: $-\mathrm{div}\,(k\nabla u) = f$ in Ω.

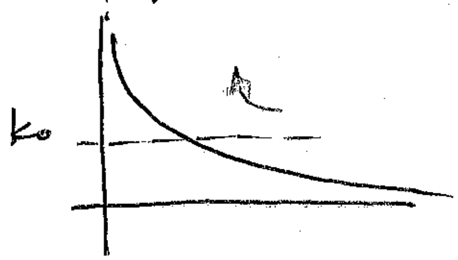
- Impose u at $x \in \partial\Omega$. (Dirich..)
- Impose $J\cdot\vec{n}$ at $x \in \partial\Omega$ (Neum..



\# Problem 4.1.



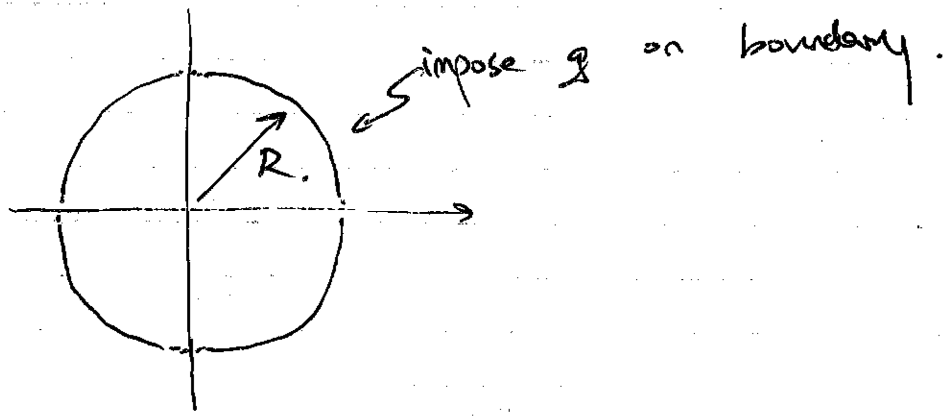
Reason why k has to be positive.



← k may blow up.

Ex. 4.4

$$u(x_1, x_2) = g - \frac{f}{4K}(x_1^2 + x_2^2 - R^2).$$



"Application of divergence theorem".

$$\int_\Omega v \,\mathrm{div}\, w \, d\Omega = \int_{\partial\Omega} v w \cdot \check{n} \, d\Gamma - \int_\Omega w \cdot \nabla v \, d\Omega.$$

"move the vector field to the scalar function".

$$\sum_{i=1}^{d} \left[\int_\Omega v \partial_i w_i \, d\Omega\right] = \sum_{i=1}^{d} \left[\int_{\partial\Omega} v w_i \check{n}_i \, d\Gamma - \int_\Omega w_i v \, d\Omega\right]$$

* look at the DIVERGENCE THEOREM.

$$\int_\Omega v(-\mathrm{div}\,(K\nabla u) - f) = 0$$

$$\int_\Omega -\mathrm{div}\,(K\nabla u) v - \int_\Omega f v = 0$$

$$-\int_{\partial\Omega} k\nabla u\cdot n\, v\, d\Gamma + \int_{\Omega} k\nabla u\cdot\nabla v\, d\Omega$$

$$-\int_{\Omega} fv = 0$$

$$-\int_{\partial\Omega_N} H v\, d\Gamma - \int_{\partial\Omega_D} k\nabla u\cdot n\, v\, d\Gamma$$

$$+\int_{\Omega} (k\nabla u)\cdot\nabla v\, d\Omega - \int_{\Omega} fv = 0$$

ce

$(0)=0$

$$\int_{\Omega} (k\nabla u)\cdot\nabla v\, d\Omega = \int_{\Omega} fv + \int_{\partial\Omega_N} H v\, dv$$

$$\forall v \in \mathcal{V} = \{\text{smooth} \mid v(0)=0\}.$$

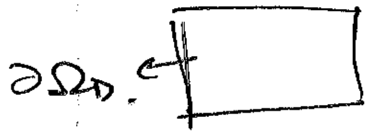
Euler-Lagrange Equations.

Find $u_h \in \mathcal{S}_h$. s.t.

$$a_h(u_h, v_h) = l_h(v_h). \quad \forall v_h \in \mathcal{V}_h.$$

$\mathcal{V}_h$ is the direction of $\mathcal{S}_h$.

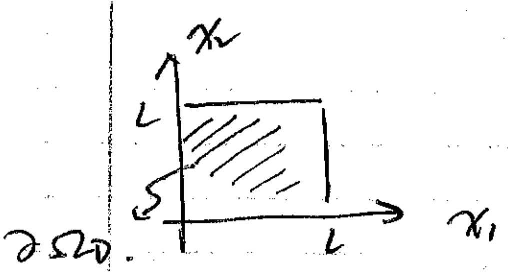
(Recall the 1D chapter)



$$\mathcal{V}_h = \{v_h \in \mathcal{W}_h \mid v_h(x)=0, \quad x \in \partial\Omega_D\}.$$

Ex. 4.9

Domain is a square. → $\Omega = [0, L)^2$



$$-\Delta u = \frac{f}{k}.$$ k. const. f. const.

$u = g$ on $\partial\Omega$.

"entire boundary: Dirichlet B.C.s".

→ Classical Galerkin.: $\mathcal{W}_h = \mathbb{P}_r(\Omega)$. $r=1$.

if $r=2$, $\mathbb{P}_2(\Omega)$.? $v(x_1, x_2) = \overbrace{C_1 + C_2 x_1 + C_3 x_2}^{\uparrow}$ $+ C_4 x_1^2 + C_5 x_1 x_2 + C_6 x_2^2$

if $r=3$ ... $+ C_7 x_1^3 + C_8 x_1^2 x_2 + C_9 x_1 x_2^2 + C_{10} x_2^3$.

$\mathcal{S}_h = \{ w_h \in \mathcal{W}_h \mid w_h = g. \text{ on } \partial\Omega \}$

$\mathcal{V}_h = \{ w_h \in \mathcal{W}_h \mid w_h = 0, \text{ on } \partial\Omega \}$.

$$\frac{\partial w_h}{\partial x_2}(x_2 = 0) = 0 = C_3. \qquad \forall x_1.$$

$$\frac{\partial w_h}{\partial x_1}(x_1 = 0) = 0 = C_2.$$

w_h on $\partial\Omega = g = C_1$. $\mathcal{S}_h$ identically "g"

if choose $\mathcal{W}_h = \mathbb{P}_1(\Omega)$. → $\mathcal{V}_h$ identically zero

$$\mathcal{W}_h = \{ C_1 + \underbrace{x_1(L - x_1) x_2 (L - x_2)}_{=0 \text{ on } \partial\Omega} p(x_1, x_2) \mid p \in \mathbb{P}_{r-4}, C_1 \in \mathbb{R} \}$$

Lecture 11. 2/13/2024.

Finite Element Spaces in 2D.

... following the $\mathcal{W}_h$ example.

$r=4$. $\mathcal{V}_h = \{ V_1 N_1(x_1, x_2) \mid V_1 \in \mathbb{R} \}$.

$$N_1(x_1, x_2) = x_1(L - x_1)\, x_2(L - x_2)$$

$$\mathcal{S}_h = \{ g + V_1 N_1(x_1, x_2) \mid V_1 \in \mathbb{R} \}.$$

we can then identify: $\overline{u}_h = g$ const.

$$u_h = g + u_1 N_1$$

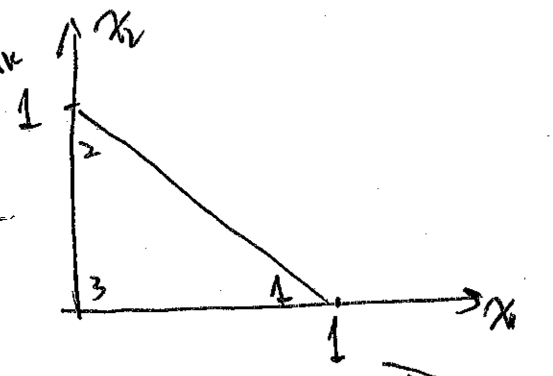
$$a(u_h, N_1) = \ell(N_1).$$

$$\int_\Omega \nabla(g + u_1 N_1) \nabla v_1 = \int_\Omega \frac{f}{k} N_1$$

$$\hookrightarrow u_1 = \frac{5f}{4kL^2}$$

$(x_b) = \delta_{ab}$.

$, x_2) = c_1 N_1 + \cdots + c_k N_k$



P_1 - element in 2D

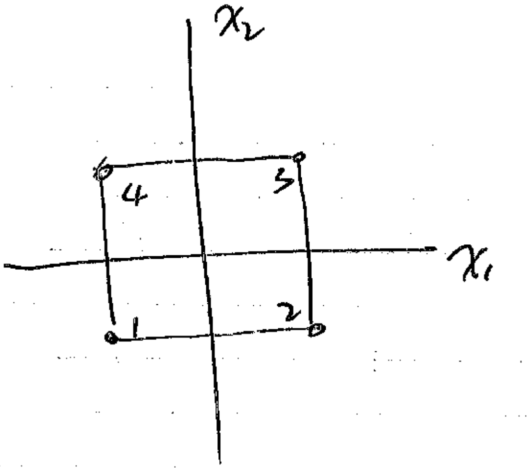
$$N_1(x_1, x_2) = x_1$$

$$N_2(x_1, x_2) = x_2$$

$$N_3(x_1, x_2) = 1 - N_1 - N_2$$

$$= 1 - x_1 - x_2$$

$$\int_0^1 dx_1 \int_0^{1-x_1} f(x_1, x_2)\, dx_2$$

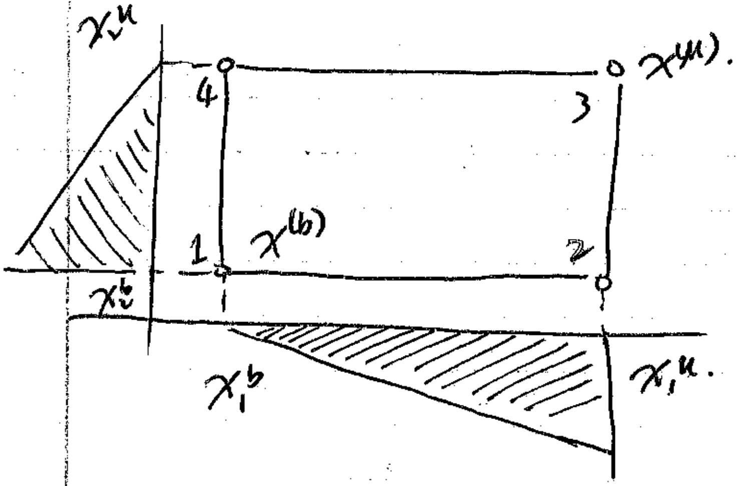


Q_1 - Element.

$$\int_{-1}^{1}\int_{-1}^{1} f(x_1, x_2)\, dx_1 dx_2$$

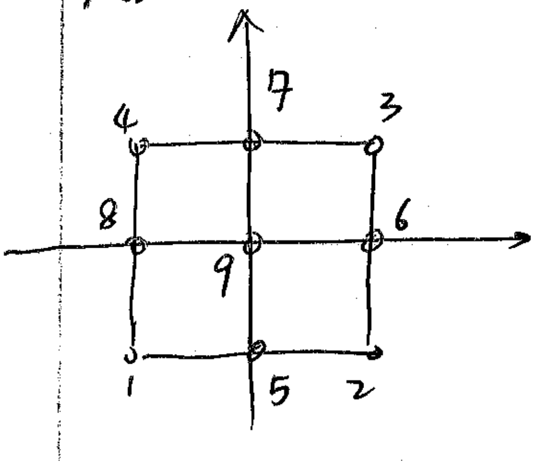
$$N_1(x_1, x_2) = \frac{1}{4}(1-x_1)(1-x_2).$$
$$N_2(x_1, x_2) = \frac{1}{4}(1+x_1)(1-x_2).$$
$$N_3(x_1, x_2) = \frac{1}{4}(1+x_1)(1+x_2).$$
$$N_4(x_1, x_2) = \frac{1}{4}(1-x_1)(1+x_2).$$

→ affine in each argument
↓
called "bilinear func."

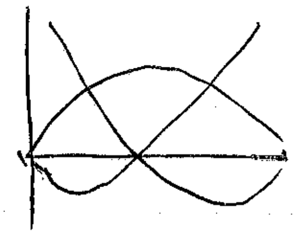


$$N_1(x_1, x_2) = \frac{x_1 - X_1^u}{X_1^b - X_1^u} \cdot \frac{x_2 - X_2^u}{X_2^b - X_2^u}$$
$$N_2(x_1, x_2) = \quad \dots$$
$$N_3(x_1, x_2) = \quad \dots$$
$$N_4(x_1, x_2) = \quad \dots$$

Q_2 - Element over a rectangle.



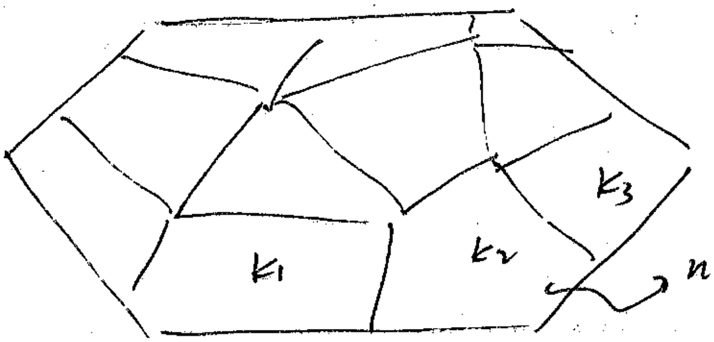
$$\mathcal{N} = \{M_i(x_1)\, M_j(x_2) \mid 1 \le i, j \le 3\}.$$
$$= \{N_1, \dots, N_9\}.$$



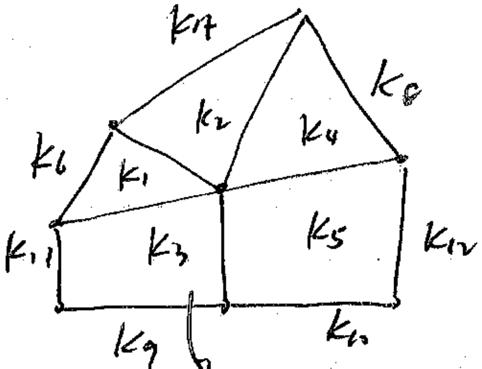
Definition of " Mesh"

mesh: $\mathcal{T} = \{K_1, \dots, K_{nel}\}$. $\Omega \subset \mathbb{R}^d$.

$\mathring{K}_i \cap \mathring{K}_j = \phi$. when $i \neq j$. & $\Omega = \cup_{i=1}^{Nel} K_i$.



↳ no intersect.



↳ rectangle - the boundary.

diameter of elem. dom. K.

$h_K = diam(K) = \max |x-y|$.

$h = \max_{K \in \mathcal{T}} h_K$

↳ mesh size.

↓

close set to include the boundary.

→ Polyhedral Meshes.

→ Conforming Mesh

→ Finite Element Mesh. $\hat{K}_i = (K_i, \mathcal{N}_i)$.

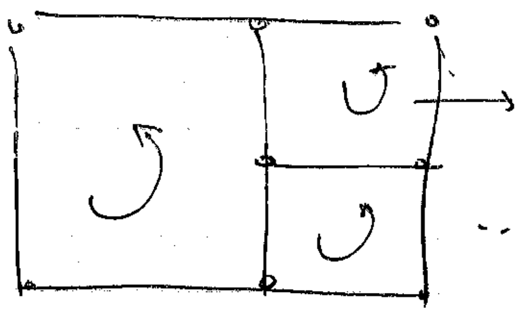
mesh for $\Omega \leftarrow \mathcal{T} = \{K_1, \dots, K_{nel}\}$

Sepecifying finite Element mesh.

$$X \left[\begin{array}{c} \cdots \;-\; \vec{x} \;-\; \cdots \\ \cdots \;-\; \vec{y} \;-\; \cdots \end{array} \right].$$

$$LV = \left.\begin{bmatrix} | & | & | \\ \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ | & | & | \end{bmatrix}\right\} \text{nodal labels}$$

connectivity



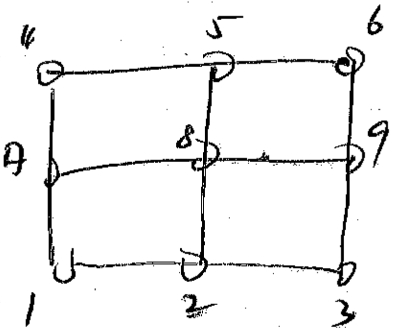
rotation convention.

... "RH rotation"

Lecture 12   2/15/2024.

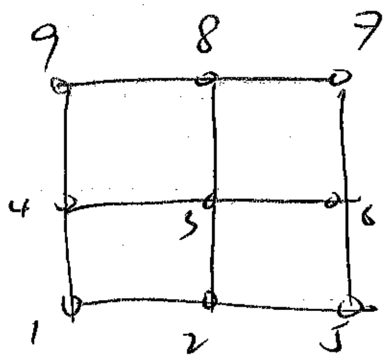
→ Finite Element Spaces

→ Bary. Coordinates.



Q_1 -element example.

$$LV = \begin{bmatrix} 1 & 2 & 7 & 8 \\ 2 & 3 & 8 & 9 \\ 8 & 9 & 5 & 6 \\ 7 & 8 & 4 & 5 \end{bmatrix}.$$

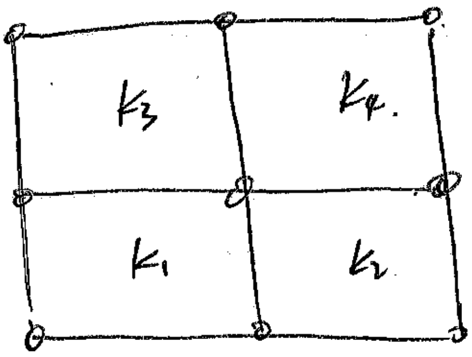


$$LG = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 5 & 3 & 6 \\ 3 & 3 & 8 & 7 \\ 4 & 6 & 9 & 8 \end{bmatrix} \times \quad \rightarrow \begin{bmatrix} 1 & 2 & & \\ 2 & & & \\ & & \ddots & \\ & & & \end{bmatrix}$$

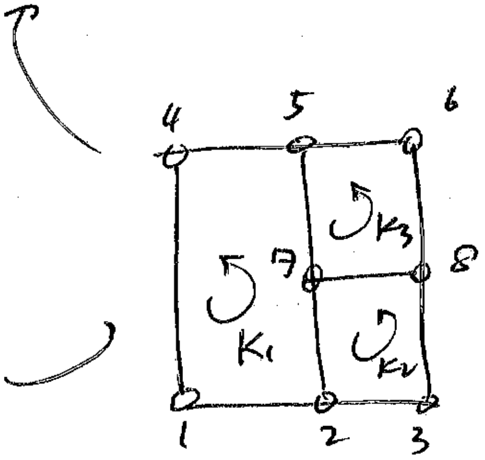
$$\downarrow$$

LG = LV

Q_2 - element.

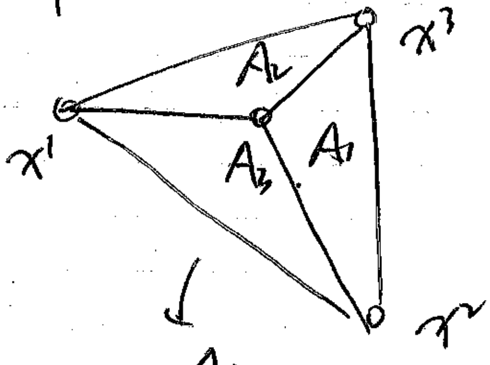


$N = N_3^1 + \frac{1}{2}N_4^2 + \frac{1}{2}N_1^3 + N_4^3$



Conformity in meshes.

Barycentric Coordinates.

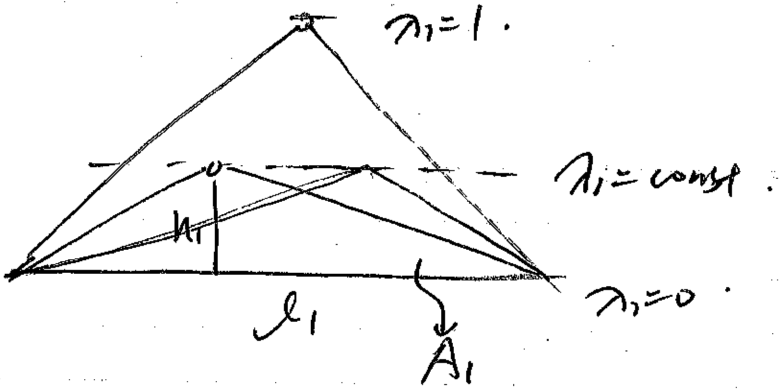


$$\lambda_j \equiv \frac{A_j}{A}$$

$$(x_1, x_2) \rightarrow (\lambda_1, \lambda_2, \lambda_3)$$

$$\downarrow$$

$$\sum_i \lambda_i = 1.$$



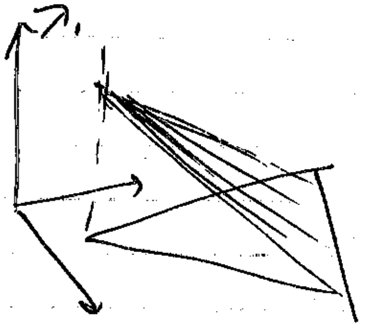
$$A_1 = \frac{l_1 h_1}{2}$$

$$\downarrow$$

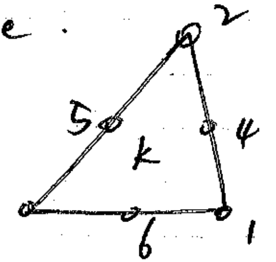
$$A = l_1 h / 2$$

$$\hookrightarrow \lambda_1 = h_1 / h.$$

λ is a linear function w.r.t. h_1.



P_2 - triangle.



$$N_5 = 2\lambda_2\lambda_3$$

$$N_3 = 2\lambda_3(\lambda_3 - 1/2)$$

Example Diffusion Problem.

$$\int_{\Omega} (k \nabla u_h) \cdot \nabla v_h \, d\Omega = \int_{\Omega} f v_h \, d\Omega + \int_{\partial\Omega_N} H v_h \, d\Gamma.$$

$$\forall v_h \in \mathcal{V}_h = \{ w_h \in \mathcal{W}_h \mid w_h(x) = 0, \ x \in \Omega_D \}$$

$$u_h \in \mathcal{S}_h = \{ w_h \in \mathcal{W}_h \mid w_h(x) = g(x), \ x \in \partial\Omega_D \}$$

$$K^e_{ab} = \int_{K_e} k \nabla N^e_b \cdot \nabla N^e_a \, d\Omega .$$

$\mathcal{W}_h \Rightarrow P_1$ - elements.

$$K^e_{ab} = \nabla N^e_b \cdot \nabla N^e_a \left(\int_{K_e} k \, d\Omega \right).$$

$$= k_e \nabla N^e_b \cdot \nabla N^e_a .$$

Define a matrix:

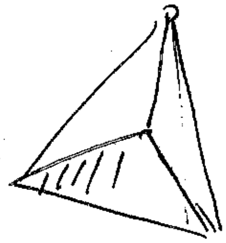
$$dN = \begin{bmatrix} \frac{\partial N^e_1}{\partial x_1} & \frac{\partial N^e_2}{\partial x_1} & \frac{\partial N^e_3}{\partial x_1} \\ \frac{\partial N^e_1}{\partial x_2} & \frac{\partial N^e_2}{\partial x_2} & \frac{\partial N^e_3}{\partial x_2} \end{bmatrix}$$

$K^e = k_e A_e \, dN^T dN$. ⟶ element stiffness matrix

$$F^e_a = \int_\Omega f N^e_a \, d\Omega .$$

$$= f_e \int_\Omega N^e_a \, d\Omega .$$

$$= f_e \frac{A_e}{3}.$$



Approximation. { functional discretization. → p-discretization

spatial discretization. ↓ h-discretization

Finite element method: hp- discretization.

Spectral discretization: ↗

LG = LTA → continuous test functions

conforming meshes.

General rule of thumb.

LT_1, LT_2, LT_3 (i76) → just types of elems.

there is no requirement for picking the Starting node

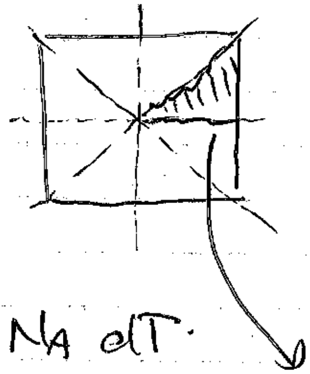
when we are labeling the nodes, the LV (LG) are constructed based on the notation convention. Both ways (1~4, 5~8) (15 26 ...) are correct. Just need to be consistent across the domain.

elements are labeled based on empirical usage. No general rule.

Lecture 13.

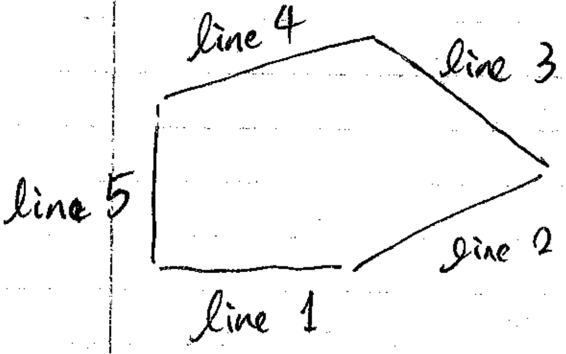
Neumann B.C.s.

$$F_A = \int_{\Omega} f N_A \, d\Omega + \int_{\partial\Omega_N} H N_A \, d\Gamma$$



symmetry property across the domain.

normal derivatives are zero across the internal boundaries.

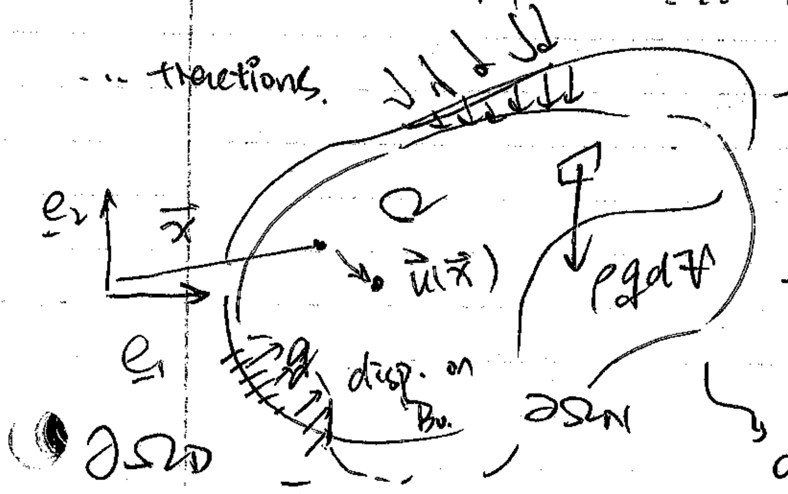


$\rightsquigarrow$ BE $\begin{bmatrix} \text{boundary nodes} \ldots \\ \text{— nodes} \ldots \\ \text{lines.} \ldots \end{bmatrix}$

Linear Elasticity Problem (Chap. 7).

H ... source term

... tractions.



$\to \partial\Omega = \partial\Omega_D \cup \partial\Omega_N$

2D. problem.

$\to$ body of gravity.

or "$\rho b \, dV$".

deformed config.

$\bar{g}$ "REFERENCE CONFIGURATION".

vector: displacement field.

$$\begin{cases} \bar{g} : \partial\Omega_D \to \mathbb{R}^2 \\ \bar{H} : \partial\Omega_N \to \mathbb{R}^2 \\ \vec{b} : \Omega \to \mathbb{R}^2 \end{cases}$$

reference configuration

$$\bar{u}(\bar{x}) = u_1(\bar{x})\bar{e}_1 + u_2(\bar{x})\cdot\bar{e}_2$$

Scalar field (functions).

→ u satisfies the principle of minimum potential energy.

... the concept of potential energy.

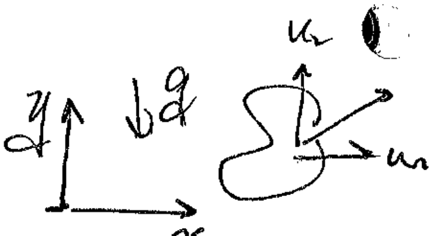
$$\mathcal{V}(u) = U(u) - \int_{\Omega} \vec{b}\cdot\vec{u}\cdot d\Omega - \int_{\partial\Omega_N} H\cdot u \, d\partial\Omega$$

$$\mathcal{V} : \mathcal{W} \to \mathbb{R}.$$

$$\mathcal{W} : \{\bar{u} : \Omega \to \mathbb{R}^2 \ \text{SMOOTH}\}.$$

<u>Example</u>: is the pot. ener.

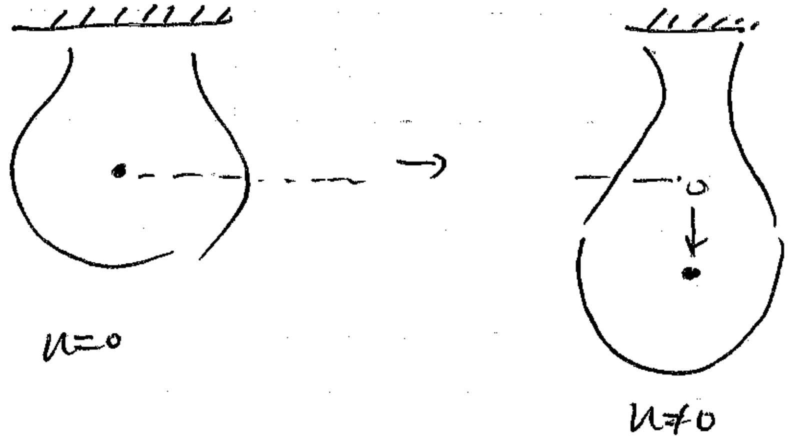
$$\int_{\Omega} \rho g u_2 \, d\Omega \quad \text{(gravity)}.$$



Q : 1). opt. alg.

2). current config.

Ex.



3). what's G

4) what's HH.

... the gradient of the displacement field.

$$\nabla \underline{u} = \begin{bmatrix} \partial_1 u_1 & \partial_2 u_1 \\ \partial_1 u_2 & \partial_2 u_2 \end{bmatrix}.$$

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

$$\varepsilon(\nabla u) = \frac{1}{2}(\nabla u + \nabla u^T)$$

$$\omega(\nabla u) = \frac{1}{2}(\nabla u - \nabla u^T)$$

↳ split into two components.

$$\nabla u(\vec{x}) = \underbrace{\varepsilon(\nabla u)}_{\hookrightarrow \text{symmetric}} + \underbrace{\omega(\nabla u)}_{\hookrightarrow \text{anti-symmetric}}$$

The strain Energy.

$$U(u) = \int_\Omega \frac{E}{2(1+\nu)} \left(\varepsilon : \varepsilon + \frac{\nu}{1-2\nu} \cdot (\mathrm{div}\, u)^2 \right\} d\Omega$$

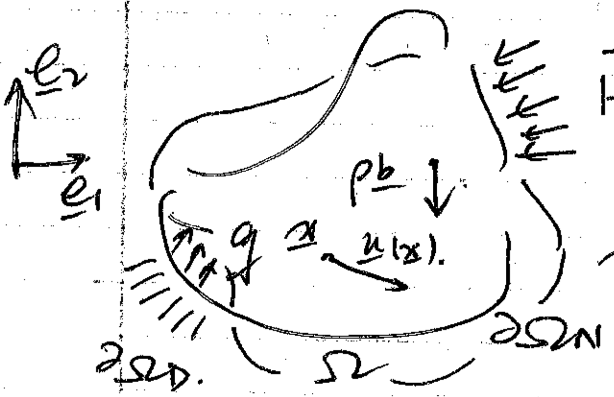
$$A : B = \sum_{ij} A_{ij} B_{ij}$$

.. Einstein note.

#Lecture 14 2/20/2024

* Minimum principle. → weak form.

↳ Multi-field problems.



linear elasticity problem.

⇝ "deformation".

$\bar{u}: \Omega \to \mathbb{R}^2$ (Recap).

$$\bar{u} = u_1(x_1, x_2)\,\underline{e}_1 + u_2(x_1, x_2)\,\underline{e}_2.$$

$$\nabla \bar{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{bmatrix}$$

$$\nabla \bar{u} = \underset{\substack{\uparrow \\ \text{symmetric}}}{\varepsilon(\nabla u)} + \underset{\substack{\uparrow \\ \text{anti-symmetric.}}}{\omega(\nabla u)}$$

$$\varepsilon(\nabla u) = \frac{1}{2}(\nabla u + \nabla u^T). \qquad \omega(\nabla u) = \frac{1}{2}(\nabla u - \nabla u^T)$$

We are looking functions:

$$\mathcal{W} = \{ \bar{w} : \Omega \to \mathbb{R}^2 \quad \text{SMOOTH} \}.$$

Minimization principle. equilibrium sol'n. $\bar{u}$

$$\mathcal{V}(\bar{u}) = U(\bar{u}) - \int_\Omega \underline{b} \cdot \underline{u} \, d\Omega - \int_{\partial\Omega_N} \underline{H} \cdot \underline{u} \, d\partial\Omega$$

- Strain Energy

$$U(u) = \int_\Omega \frac{E}{2(1+\nu)} \left(\varepsilon(\nabla \underline{u}) : \varepsilon(\nabla \underline{u}) + \frac{\nu}{1-2\nu} \cdot (\operatorname{div} \underline{u})^2 \right] d\Omega.$$

Primal Variational form.

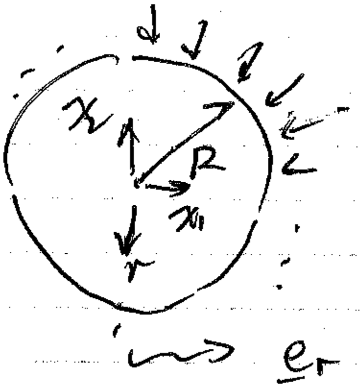
Expand the potential energy.

$$\mathcal{V}(\underline{w}) = \int_\Omega \frac{E}{2(1+\nu)} \left[\varepsilon(\nabla \underline{w}) : \varepsilon(\nabla \underline{w}) + \frac{\nu}{1-2\nu} (\operatorname{div} \underline{w})^2 \right] d\Omega$$

$$- \int_\Omega \underline{b} \cdot \underline{w} \, d\Omega - \int_{\partial\Omega_N} \underline{H} \cdot \underline{w} \, d\Gamma.$$

Ex 7.1

"Sphere"



const. pressure

$\underline{u}(0) = 0$

$\bar{H}(\bar{x}) = -p\,\bar{r}(\bar{x})$. $\bar{x} \in \partial\Omega$. $\rightsquigarrow = -p\,\underline{e}_r(x)$,

$\Omega = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leq 1 \}$.

$\bar{u}(\underline{x}) = \varphi(r) \cdot \underline{e}_r(\underline{x})$.

$\varepsilon(\nabla u) : \varepsilon(\nabla u) = \varphi'(r)^2 + 2\frac{\varphi(r)^2}{r^2}$.

$\mathrm{div}(u) = \varphi(r) + \frac{2\varphi(r)}{r}$

apply B.C.s $\varphi(0) = 0 \rightsquigarrow$ equation satisfies.

$\mathcal{S} = \{ \varphi : [0, R] \to \mathbb{R} \mid \varphi(0) = 0 \}$.

$$\int_{\partial\Omega} -p\,\underline{e}_r \cdot \varphi(r)\,\underline{e}_r = \int_{\partial\Omega} \varphi(r)\,p = -\varphi(R).\,p\,4\pi R^2$$

↑ plug in B.C.s

Assume $\varphi(r) = Ar$, $A \in \mathbb{R}$. $\varphi(R)$.

$V = \gamma A^2 + 4\pi p R^3 A$. ... solve for A

$A = -\frac{1-2\nu}{E} \cdot p$ $\rightsquigarrow$ hydrostatic pressure.

$\underline{u}(\underline{x}) = A\underline{x} = -\frac{1-2\nu}{E}\, p\, \underline{x}$

from variational to weak form

Theorem: minimize u to find $a(u,v) = l(v)$

is equivalent to solving the minimization prob.

Stress field

$$\underline{\underline{\sigma}} = \underbrace{\frac{E}{1+\nu}}_{2\mu} \underline{\underline{\varepsilon}}(\nabla \underline{u}) + \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda} \cdot \underbrace{(\mathrm{div}\, \underline{u})}_{\underline{\underline{I}} : \nabla \underline{u}} \cdot \underline{\underline{I}}$$

... linear elasticity:

$$\underline{\underline{\sigma}} = \lambda\, \mathrm{tr}(\underline{\underline{\varepsilon}}(\underline{u}))\, \underline{\underline{I}} + 2\mu\, \underline{\underline{\varepsilon}}(\nabla \underline{u})$$

From Variational to weak form.

Recall the PDE.

Problem: Find $\underline{u} \in \mathcal{S}$ s.t.

$$a(\underline{u},\underline{v}) = \ell(\underline{v}). \quad \forall \underline{v} \in \mathcal{V}.$$

$$a(\underline{u},\underline{v}) = \int_\Omega \underline{\underline{\sigma}}(\nabla u) : \underline{\underline{\varepsilon}}(\nabla v)\, d\Omega$$

recall: $\varepsilon(\nabla u) + \omega(\nabla u) = \nabla u$.

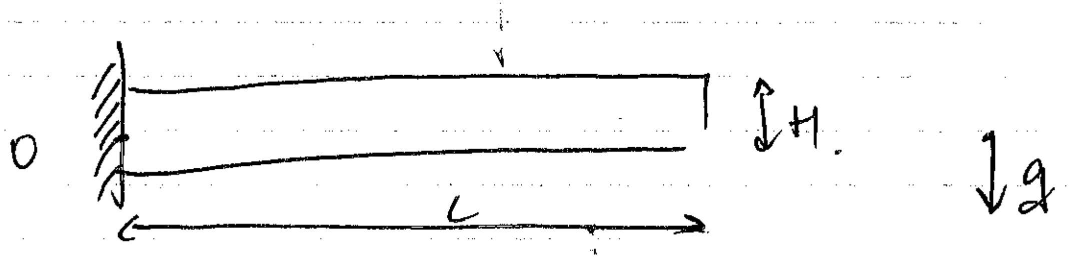
$$\int_\Omega \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}(\nabla \underline{v})\, d\Omega = \int_\Omega \underline{\underline{\sigma}} : \nabla \underline{v}\, d\Omega.$$

Variational Numerical Method.

$$a(\underline{u}_h, \underline{v}_h) = \ell(\underline{v}_h)$$

Solving linear elasticity problem.

Ex 7.5



#Problem Session. ~ FireDrake

mesh. geo. syn ~ .

~ . addPhysical Group

\# ~> label the group

"CG" - Lagrange Polynomials

↳ Element Order

↳ P1 - element

Lecture #15 2/27/2024

Linear Elasticity

Review → principle of minimum potential energy.

$$\mathcal{F}(\vec{u}) = \underbrace{U(\vec{u})}_{\text{elastic energy}} - \underbrace{\int_{\Omega} \vec{b}\cdot\vec{u}\, d\Omega}_{\text{body force}} - \underbrace{\int_{\partial\Omega_N} \vec{H}\cdot\vec{u}\, d\underset{\Gamma}{\partial\Omega}}_{\text{B.C.s}}$$

$$U(\vec{u}) = \frac{1}{2}\int_{\Omega} \sigma(\nabla\vec{u}) : \varepsilon(\nabla\vec{u})\, d\Omega .$$

Theorem: $\mathcal{F}(u) = \frac{1}{2}a(u,u) - l(u)$.

if $u \in \mathcal{S}$, satisfying:

$$\mathcal{F}(u) < \mathcal{F}(w), \quad \forall w \in \mathcal{S},\ w \neq u.$$

$$\Updownarrow$$

$$a(u,v) = l(v), \quad \forall v \in \mathcal{V}.$$

$\mathcal{V}$ is the direction of $\mathcal{S}$.

- apply shear force, change $\vec{X}$ to impose Neumann B.C.s.

Example on constrained index

$$N_A = \begin{bmatrix} 0 \\ x_1 x_2 \end{bmatrix} \qquad \nabla N_A = \begin{bmatrix} 0 & 0 \\ x_2 & x_1 \end{bmatrix}$$

$$\varepsilon(\nabla N_A) = \begin{bmatrix} 0 & x_2/2 \\ x_2/2 & x_1 \end{bmatrix} \rightarrow B^i$$

div for matrices: <u>sum of diagonals</u>

$\downarrow$

D^i are the diagonal summation ∇N_i

xample

$$a(N_5, N_6) = \int_0^L dx_1 \int_0^H dx_2 \; \Sigma_i^6 : B^5$$

subs the Σ_i^i & B^i $\longrightarrow$

$$= \int_0^L dx_1 \int_0^H dx_2 \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 2\mu x_1 \end{bmatrix} : \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}}$$

$$= LH \cdot 0 = 0$$

"contraction": element-wise
multiplication
summation

$$l(v) = \int bv + \int Hv = \int -\rho g\, \underline{e}_2 \cdot (v_1 \underline{e}_1 + v_2 \underline{e}_2)$$

$$= - \int_\Omega \rho g \, v_i \, d\Omega .$$

$$F_i = \ell(N_i) = - \int_\Omega \rho g \, N_i^i(x_1, x_2) \, dx_1 dx_2$$

Choice of basis, if you rotate the space the approximation of the vector shouldn't change.

$$\underline{w}_h = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \longrightarrow .$$

How to build finite element spaces?

$$\mathcal{V}_h = \left\{ w_h = \begin{bmatrix} w_{h1} \\ w_{h2} \end{bmatrix} \,\middle|\, w_{h1} \in \mathcal{V}_{h1} , \; w_{h2} \in \mathcal{V}_{h2} \right\}$$

$$= \mathcal{V}_{h1} \times \mathcal{V}_{h2}$$

$$\mathcal{N}^\Omega = \Big\{ \underbrace{\begin{bmatrix} N_1 \\ 0 \end{bmatrix}, \begin{bmatrix} N_2 \\ 0 \end{bmatrix}, \cdots \begin{bmatrix} N_{m_1} \\ 0 \end{bmatrix}}_{\underline{N}_1, \, \underline{N}_2, \, \cdots \, \underline{N}_{m_1}}, \underbrace{\begin{bmatrix} 0 \\ M_1 \end{bmatrix}, \cdots \begin{bmatrix} 0 \\ M_{m_2} \end{bmatrix}}_{\underline{N}_{m_1+1}, \, \cdots \, \underline{N}_{m_1+m_2}} \Big\}$$

In the language of finite element:

$$\mathcal{N}^e = \Big\{ \underbrace{\begin{bmatrix} N_1^e \\ 0 \end{bmatrix}, \begin{bmatrix} N_2^e \\ 0 \end{bmatrix}, \cdots \begin{bmatrix} N_{\ell_1}^e \\ 0 \end{bmatrix}}, \underbrace{\begin{bmatrix} 0 \\ M_1^e \end{bmatrix}, \cdots \begin{bmatrix} 0 \\ M_{\ell_2}^e \end{bmatrix}} \Big\}$$

$$LG(a,e) = \begin{cases} LG1(a,e) & 1 \le a \le l_1 \\ LG2(a-l_1, e) + m_1 & l_1+1 \le a \le l_1+l_2 \end{cases}$$

coding: LG = [LG1; LG2 + m1];

Example: Apply to P1 - elasticity.

$$V_h = W_h \times W_h = \{ \underline{w}_h = [w_{h1}, w_{h2}]^T \}$$

Element Stiffness matrix

$$K^e_{ab} = a^e(N^e_b, N^e_a) = \int_{K^e} \Sigma^e_a : B^e_a \, d\Omega.$$

$$dN = \begin{bmatrix} \frac{\partial N_1}{\partial x_1} & \frac{\partial N_2}{\partial x_1} & \frac{\partial N_3}{\partial x_1} \\ \frac{\partial N_1}{\partial x_2} & \frac{\partial N_2}{\partial x_2} & \frac{\partial N_3}{\partial x_2} \end{bmatrix}$$

"B is a vector func."

$$B^a = \varepsilon(\nabla N^e_a).$$

$$\bar{N}_1 = \begin{bmatrix} N_1 \\ 0 \end{bmatrix}$$

$$\nabla \bar{N}_1 = \begin{bmatrix} N_{1,1} & N_{1,2} \\ 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} dN_{11} & dN_{12} \\ 0 & 0 \end{bmatrix}$$

e.g.

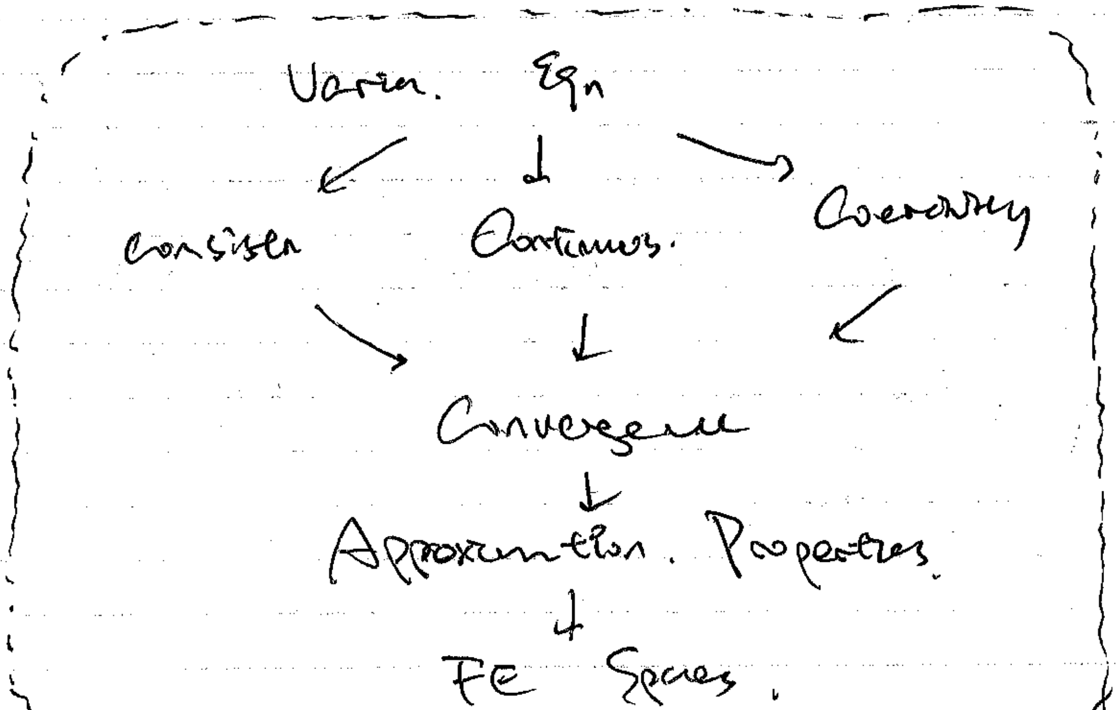
$$B^1 = \begin{bmatrix} dN_{11} & \frac{1}{2} dN_{21} \\ \frac{1}{2} dN_{21} & 0 \end{bmatrix}.$$

#order of convergence.

→ not indicating that the soln is converging to the correct soln. (could be wrong).

Lecture #16 2/29/2024.

- Variational Method as Minimum Principle.
- Numerical Analysis.
- Order of Convergence
- Norm
- Convergence in 1D
- Fundamental approx. – Res.



Recap ▷ minimum potential energy. $\to$ principle : $\mathcal{F}$ functional.

▽ Quadratic functional.

▽ Variational $\to$ Weak form.

Formulate a discrete variational problem

Find $u_h \in \mathcal{S}_h$ s.t.

$$\mathcal{F}(u_h) \le \mathcal{F}(w_h) \quad \forall w_h \in \mathcal{S}_h .$$

chose my space:

$\mathcal{W}_h$ base space.

$\mathcal{S}_h \subset \mathcal{W}_h$ affine space,

s.t. $\mathcal{S}_h \subset \mathcal{S}$.

Constrained optimization

{ using discretized $\mathcal{S}_h$ to approximate $\mathcal{S}$.

{ $u_h, w_h \in \mathcal{S}_h$, $\to u_h - w_h \in \mathcal{V}_h \subset \mathcal{V}$

hypothesis.

☆ If we know that there is a minimizer u

for the exact problem $\mathcal{V}$. Then the exact soln is bounded.

in our particular formulation:

$$\mathcal{F}(u) = \frac{1}{2} a(u, u) - l(u)$$

$$\mathcal{F}(\lambda u) = \frac{\lambda^2}{2} a(u, u) - \lambda l(u)$$

Numerical Analysis

~ Order of Convergence

$$-ku'' = f \quad \text{in } \Omega$$

$$\downarrow$$

$$a(u, v) = l(v)$$

$$a_h(u_h, v_h) = l_h(v_h)$$

is consistent

$$a_h(u, v_h) = l_h(v_h) \quad \forall v_h \in \mathcal{V}_h$$

→ how to guarantee that a_h is invertible

coercivity ...

Norm & Normed Space.

$\|\cdot\| : \mathcal{V} \to \mathbb{R}$.

$$\begin{cases} \|v\| \geq 0, \quad \|v\| = 0 \text{ iff } v = 0 \\ \|\alpha v\| = |\alpha| \, \|v\| . \\ \|v + u\| \leq \|v\| + \|u\| . \quad \text{triangular inequality}. \end{cases}$$

$(V, \|\cdot\|)$. def. $\|\cdot\| \to$ Normed space

<u>Examples</u> :
- L^{∞} - norm.
- L2 - norm.
- H1 - seminorm.
- H1 - norm

B.1

$\mathcal{V} = \mathbb{R}^3$. $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^3}$.

B.2

$\mathcal{V}_1 = \{ f : [a,b] \to \mathbb{R} \text{ Smooth} \}$.

$v \in \mathcal{V}_1 \quad \|v\| = \max_{x \in [a,b]} |v(x)|$.

Example $[a,b] = [0, \pi]$

$v(x) = \cos x$.

$\|v\|_{0,\infty} = 1$.

B.3. $\|v\|_{0,2} = \left[\int_0^{\pi} (\cos x)^2 \, dx \right]^{1/2} = \sqrt{\pi/2}$.

B.4 $\mathcal{V}_2 = \{ f : [a,b] \to \mathbb{R},$ SMOOTH, $v(a) = v(b) = 0 \}$.

★ " semi-norm ".

↓

Not a norm for $\mathcal{V}_1$ space.

e.g., const. functions.

B.6. $(\mathbb{R}^N, \|\cdot\|)$.

B.7 $(\mathcal{V}^1, \|\cdot\|_{0,\infty})$

$\mathcal{V}^1$ continuous

$\Downarrow$

bounded.

B.8 $(\mathcal{V}^1, \|\cdot\|_{0,2})$.

B. 10. $\Omega \subset \mathbb{R}^M$,

$$\|v\|_{0,2} = \left[\int_\Omega v^2 dx \right]^{1/2}$$

$$L^2(\Omega) = \{ v : \Omega \to \mathbb{R} \mid \|v\|_{0,2} < \infty \}$$

Main property for L2:

all func. smooth: you can

approx. any functions in L2-norm.

$$L^\infty(\Omega) = \{ v : \Omega \to \mathbb{R} \mid \|v\|_{0,\infty} < +\infty \}.$$

$$H^1(\Omega) = \{ v : \Omega \to \mathbb{R} \mid \|v\|_{1,2} < +\infty \}$$

7/6/2014 Lecture #17 (18).

Fundamental Approximation Result. - Cea's Lemma.

$\downarrow$

exact consist.

Domain of Norm,

$\to$ coeff. exists

Continuity. $|a_h(u-w_h, v_h)| \le M\|u-w_h\|\|v_h\|$

$|l_h(v_h)| \le m\|v_h\|.$

Coercivity $a_h(v_h, v_h) \ge a\|v_h\|^2$

$|f(x) - f(y)| \to$ as $x \to y$

$|a(u, v_h) - a(u_h, v_h)| \to 0$

as $u_h \to u$. $\forall v_h \in V_h$.

$\to |a(u-u_h, v_h)| \le M\|u-u_h\|\|v_h\|.$

$l(v_{h_1}) - l(v_{h_2})$

$= |l(v_{h_1} - v_{h_2})| \le m\|v_{h_1} - v_{h_2}\|$

$u_h, w_h \in S_h$

Prove the theorem,

$$\|u_h\| \le \frac{m}{a} + \left(1 + \frac{M}{a}\right) \min_{w_h \in S_h} \|w_h\|.$$

① $\quad a_h(u_h, v_h) = l_h(v_h) \quad \forall v_h \in V_h.$

$$a_h(u_h - w_h,\ u_h - w_h) \geq \alpha \|u_h - w_h\|^2$$

$$\|$$

$$a_h(u_h,\ u_h - w_h) - a_h(w_h,\ u_h - w_h)$$

$$\Big\downarrow ①$$

$$l_h|u_h - w_h| - a_h(w_h, u_h - w_h)$$

$$\|u_h - w_h\|^2 \le \frac{1}{\alpha}\left[l_h(u_h - w_h) - a_h(w_h, u_h - w_h)\right]$$

$\le$

replace $\frac{1}{\alpha}\left| l_h(u_h - w_h) - a_h(w_h, u_h - w_h)\right|$ by abs.

$$\le \frac{1}{\alpha}\left[|l_h(u_h - w_h)| + |a_h(w_h, u_h - w_h)| \right]$$

$$\le \frac{1}{\alpha}\left[m\|u_h - w_h\| + M\|w_h\|\|u_h - w_h\| \right]$$

implies.

$$\|u_h - w_h\| \le \frac{m}{\alpha} + \frac{M}{\alpha}\|w_h\|. \quad \forall w_h \in \mathcal{V}_h.$$

$$\|u_h\| \le \|u_h - w_h\| + \|w_h\|$$

$$\le \frac{m}{\alpha} + \left(1 + \frac{M}{\alpha}\right)\|w_h\|$$

$$\hookrightarrow \|u_h\| \le \frac{m}{\alpha} + \left(1 + \frac{M}{\alpha}\right)\min_{w_h \in \mathcal{S}_h} \|w_h\|$$

Recall rank-nullity theorem

$$KU = 0$$

$$\hookrightarrow a_h(u_h, v_h) = 0, \quad \forall v_h \in \mathcal{V}_h.$$

$$KU = F.$$

$$\hookrightarrow KU = 0$$

$$\Updownarrow$$

$$a_h(u_h, v_h) = 0 \quad \forall v_h \in \mathcal{V}_h$$

$$u_h \in \mathcal{V}_h \qquad \downarrow$$

$$0 \le \alpha\|u_h\|^2 \le a(u_h, u_h) = 0$$

$$\hookrightarrow \|u_h\| = 0. \quad \longleftrightarrow \quad u_h = 0$$

Fundamental Approximation Result.

$a(u_h, v_h) = \ell(v_h)$. $\forall v_h \in \mathcal{V}_h$.

$u_h \in \mathcal{S}_h$

Consistency. $a_h(u, v_h) = \ell(w_h)$. $\forall v_h \in \mathcal{V}_h$

$$\rightsquigarrow \underbrace{a_h(u - u_h, v_h) = 0}_{\text{Galerkin Orthogonality.}} \quad \forall v_h \in \mathcal{V}_h$$

$$\alpha \|u_h - w_h\|^2 \le a(u_h - w_h, u_h - w_h) \qquad w_h \in \mathcal{V}_h$$

$$\le a(u_h - u, u_h - w_h) \rightsquigarrow = 0$$

$$+ a(u - w_h, u_h - w_h)$$

$$\le M \|u - w_h\| \, \|u_h - w_h\|$$

$$\|u_h - w_h\| \le \frac{M}{\alpha} \|u - w_h\|$$

$$\|u - u_h\| \le \|u - w_h\| + \|w_h - u_h\| \le \|u - w_h\| + \frac{M}{\alpha}\|u - w_h\|$$

RHS:

$$= \left(1+\frac{M}{\alpha}\right) \|u - w_h\|.$$

$\Rightarrow$: $\|u - u_h\| \leq \min_{w_h \in \mathcal{S}_h} \left(1+\frac{M}{\alpha}\right) \|u - w_h\|$

Second-order problem in 1D.

Find $u_h \in \mathcal{S}_h$ s.t.

$a(u_h, v_h) = l(v_h).$

$\forall v_h \in \mathcal{V}_h.$

$\mathcal{S}_h = \{w_h \in \mathcal{W}_h \mid w_h(0) = g_0\}$

$\mathcal{V}_h = \{w_h \in \mathcal{W}_h \mid w_h(0) = 0\}$

$u_h \in \mathcal{W}_h$, $\rightarrow$ u_h is $C^1(k_e)$. $\forall k_e$

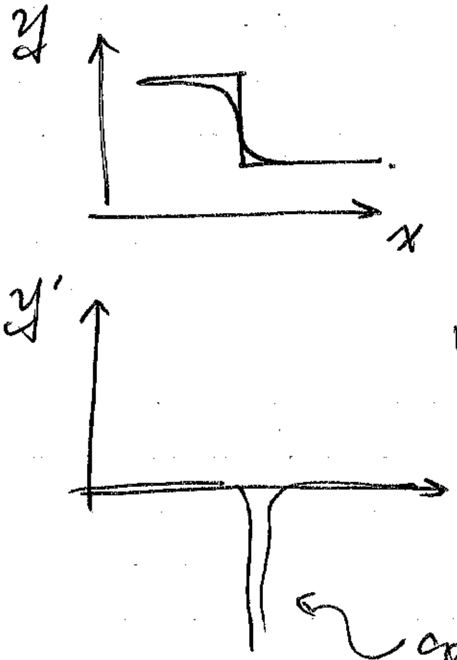
3/12/2024. Lecture #19.

$f \in H^k(\Omega)$ IFF $\exists\ f_1, f_2, \dots, f_n, \dots \in C^\infty(\Omega)$

infinitely differentiable

S.t. $\| f_i - f \|_k \to 0$

we are thinking about just the smooth functions



weak derivative 'cuz there's no derivative

approximate the delta function.

Consistency. $r_h(u, v_h) = 0$ ← exactly.

Coercivity. $\exists\ \alpha > 0$, S.t.

$$a_h(v_h, v_h) \geq \alpha \|v_h\|^2, \quad \forall\ v_h \in V_h.$$

↳ guarantee convergence. ... (why?)*

$$a_n(v_n, v_n) = \int_0^L \left[k(x)\, v_n'(x)^2 + c(x)\, v_n(x)^2\right] dx$$

if I want to have coercivity in L2:

$$\hookrightarrow \geq \left| \int_0^L c(x)\, v_n(x)^2\, dx \right|.$$

$$= \int_0^L |c(x)|\, |v_n(x)^2|\, dx$$

↳ everything positive

$$\geq C_{min} \int_0^L |v_n(x)|^2\, dx.$$

$$= C_{min} \, \|v_n\|_0^2$$

↖ coercivity in L2.

$$a_n(v_n, v_n). \geq \int_0^L \left(\min_x |c(x)|\right) v_n'^2 + \left(\min_x c(x)\right) v_n^2\, dx$$

$$\geq \int_0^L \min_x (k, c) . v_n'^2 + \min_x (k, c)\, v_n^2\, dx$$

$$= \min_x \{k(x), c(x)\} . \int_0^L v_n'^2 + v_n^2\, dx$$

$$= \underbrace{\min\{K_{min}, c_{min}\}}_{\alpha} \, \|v_n\|_1^2 . \swarrow$$

H^1-norm

$C_{min} = 0$

$\uparrow$

$c(x) = 0 \implies a_h(v_h, v_h) \geq k_{min} | v_h |_1^2$

Poincare's inequality.

$\exists c_1 > 0$, s.t. $\forall u \in H^1(\Omega)$, $u(0) = g_a$

$$\| u \|_0 \leq c_1 | u |_1 .$$

$$\frac{k_{min}}{2} | v_h |_1^2 + \frac{k_{min}}{2} \cdot \frac{\| u \|_0^2}{c_1^2}$$

$$\geq \min \left\{ \frac{k_{min}}{2}, \frac{k_{min}}{2c_1^2} \right\} \| v_h \|_1^2 .$$

$R_h(u_h, v_h) = u (u(0) - g_0) v_h .$

Nitsche

$\downarrow$

this is not coercive

So we are not solving this variational eqn.

Continuity - $\exists M > 0$, s.t.

$$| a_h(u - w_h, v_h) | \leq M \| u - w_h \| \| v_h \| .$$

$$\forall v_h \in V_h, \quad \forall w_h \in S_h .$$

we will use the fact:

→ $\left| \int f(x) \right| \le \int |f|$ $|x+y| \le |x| + |y|$.

→ Cauchy - Schwartz ineq.

Vectors: $|\underline{x} \cdot \underline{y}| \le |\underline{x} \cdot \underline{x}|^{1/2} \, |\underline{y} \cdot \underline{y}|^{1/2}$.

For integrals: $f, g \in L^2(\Omega)$.

(Hilbert space's properties).

$$\left| \int_\Omega fg \, d\Omega \right| \le \left[\int_\Omega f^2 d\Omega \right]^{1/2} \left[\int_\Omega g^2 d\Omega \right]^{1/2}$$

$$\le \|f\|_0 \, \|g\|_0$$

Interpolation Result.

Fund. thm of Approx.

$$\|u - u_h\|_2 \le \min \|u - w_h\|_2$$

Convergence.

$$\|u - u_h\|_1 \le C h^k \|u^{(k+1)}\|_0.$$

Proof.

$$\mathcal{I}u = \sum_{a=1}^{m} u(x_a) N_a . \quad \rightsquigarrow \text{hat func.}$$

$$\| u - \mathcal{I}u \|_1 \leq C_{\mathcal{I}} \, h \, \| u'' \|_0$$

$$\| u - \mathcal{I}u \|_0 \leq C_{\mathcal{I}} \, h^2 \, \| u'' \|_0$$

$\nwarrow$ proving

for the interpolate.

- $\mathcal{I}u \in \mathcal{S}_h$, since

$$\mathcal{I}u(0) = \sum_{a=1}^{m} u(x_a) N_a(0).$$

$$= u(x_0) = u(0) = g_0$$

- $$\| u - \mathcal{I}u \|_0^2 = \sum_{e=1}^{n_{el}} \| u - \mathcal{I}u \|_{0,e}^2$$

$$\int_0^L (u - \mathcal{I}u)^2 = \int_0^{x_1} \cdots + \int_{x_1}^{x_2} \cdots + \cdots \int_{x_{n_{el}-1}}^{x_{n_{el}}}$$

- $$\| u - \mathcal{I}u \|_1^2 = \sum_{e=1}^{n_{el}} \| u - \mathcal{I}u \|_{0,e}^2 + \| u - \mathcal{I}u \|_{1,e}^2$$

Now consider

• $\eta(x) = u(x) - \mathcal{I}u(x)$.

a). $\eta(x_a) = \eta(x_{a+1}) = 0$

Interpolating on nodes.

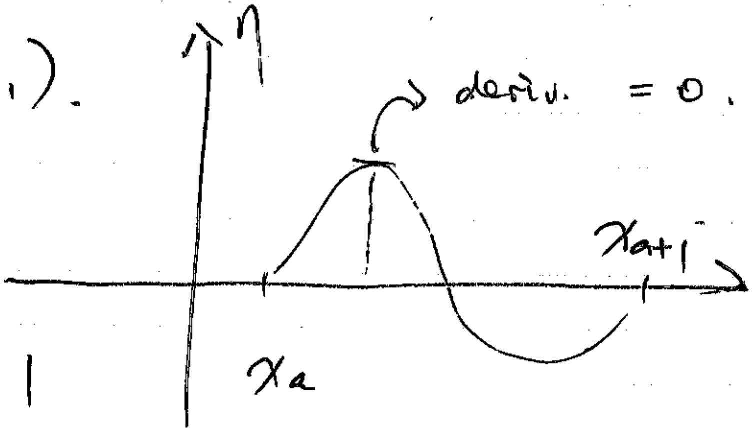
b). $\eta''(x) = u''(x)$. $x \in (x_a, x_{a+1})$.

$$\|u - \mathcal{I}u\|_{0,e}^2 = \int_{x_e}^{x_{e+1}} \eta(x)^2 \, dx$$

$|\eta'(x)|$

$\exists\, z \in (x_n, x_{n+1})$.

s.t. $\eta'(z) = 0$



$$|\eta'(x)| = \left|\int_{\xi}^{x} \eta''(x)\, dx.\right|$$

dummy var.

$$\le \|1\|_0 \|u''\|_0 = h_e^{1/2} \|u''\|_0$$

$$\le \int_{\xi}^{x} |\eta''(x)|\, dx = \int_{\xi}^{x} |u''(x)|\, dx$$

$$|\eta(x)| = \left|\int_{x_e}^{x} \eta'(x)\, dx\right| \leq \int_{x_e}^{x} |\eta'(x)|\, dx$$

$$\leq h_e^{3/2} \|u''\|_0.$$

$$\int_{x_e}^{x_{e+1}} \eta(x)^2\, dx \leq \int_{x_e}^{x_{e+1}} (h_e^{3/2})^2 \|u''\|_0^2$$

$$= h_e^3 \|u''\|_0^2\, h_e$$

$$\Rightarrow \|u - \mathcal{I}u\|_{0,e}^2 \leq h_e^4 \|u''\|_0^2$$

$$\|u - \mathcal{I}u\|_0^2 = \sum_e \|u - \mathcal{I}u\|_{0,e}^2$$

$$\leq \sum_e h_e^4 \|u''\|_{0,e}^2$$

$$\leq \left(\max_e h_e^4\right) \sum_e \|u''\|_{0,e}^2$$

$$\leq h^4 \cdot \|u''\|_0^2$$

Lecture 20. 3/14/2014.

$$\eta(x) = u(x) - \mathcal{I}u(x).$$

$$|\eta(x)| \le h_e^2 \|u'^2\|_{0,e}.$$

1) $$|u(x) - u_h(x)| \le \left| \int_0^x u'(x) - u_h'(x)\, dx \right|$$

triangular inequality. ↳

$$\le \int_0^x |u'(x) - u_h'(x)|\, dx$$

$$\le \|u' - u_h'\|_0 \|1\|. \curvearrowright L^{1/2}$$

$$\le \|u - u_h\|_1 L^{1/2}$$

$$\le C(u) h^k \cdot L^{1/2}.$$

↑

in 1D. H_1-convergence implies the u_h convergence.

2). $$\mathcal{I}[u] = \int_0^L \rho g u(x)\, dx$$

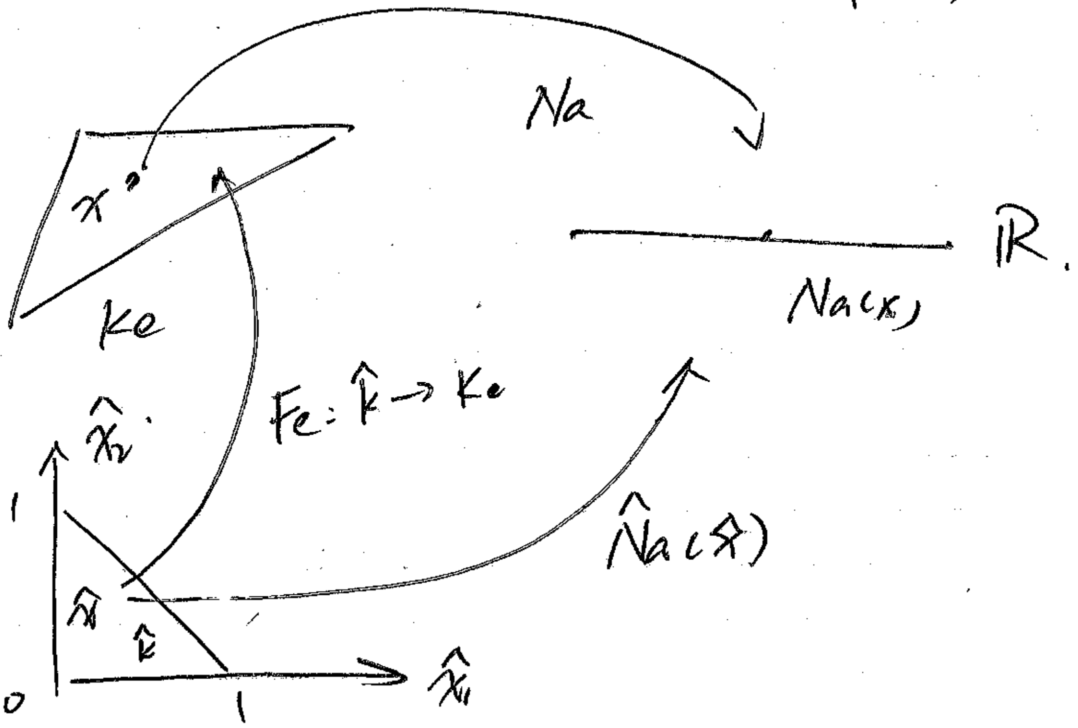
$$\mathcal{I}[u_h]$$

$$|\mathcal{I}[u] - \mathcal{I}[u_h]| \le \mathcal{O}(h^k). \sim \mathcal{O}(h^{2k})$$

c). $\|u - u_h\|_1 = \mathcal{O}(h^k)$.

$\|u - u_h\|_0 = \mathcal{O}(h^{k+1})$

↳ general case (dimensionality has no effect)



$$Na(x) = \hat{Na}(Fe^{-1}(x)).$$

$$Na(Fe(\hat{x})) = \hat{Na}(\hat{x})$$

$$(\hat{K}, \hat{N}), Fe \to (K, N).$$

$$K = Fe(\hat{K}).$$

$$Na \circ Fe = \hat{Ne}. \quad \forall \hat{Na} \in \hat{N}$$

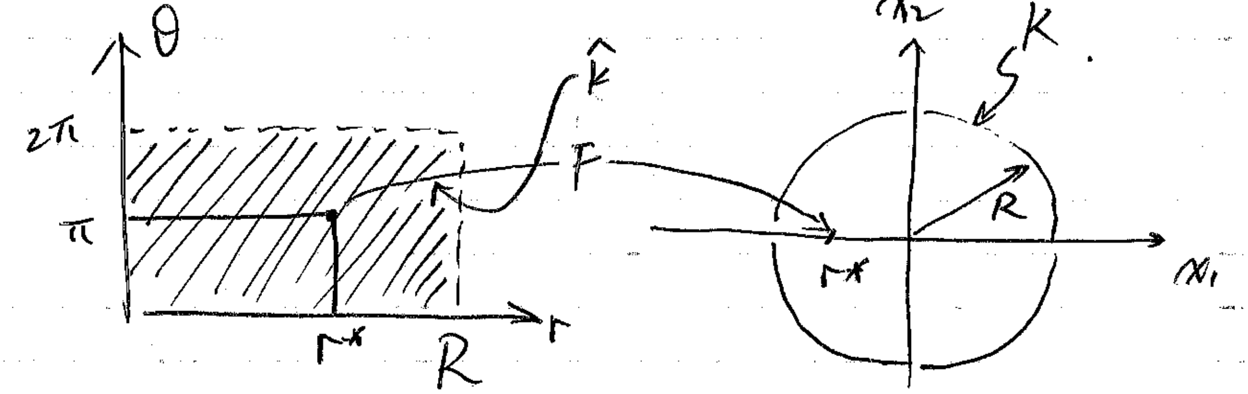
or $N_a = \hat{N}_a \circ F_e^{-1}$.

Domain map.

$F_e : \hat{K} \to \mathbb{R}^d$. 1-to-1, smooth.

Example 1.

Polar-coordinate map.



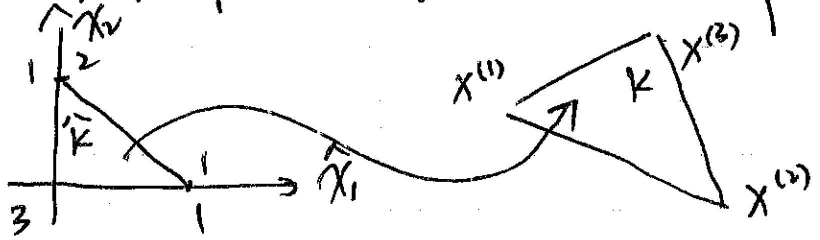
$\hat{K} = [0, R] \times [0, 2\pi]$.

$F(r,\theta) = (F_1(r,\theta), F_2(r,\theta))$.

$$\begin{cases} x_1 = F_1(r,\theta) = r\cos\theta \\ x_2 = F_2(r,\theta) = r\sin\theta \end{cases}$$

Example 2

Ref. triangle to any triangle.



To construct the map, use the Barycentric coordinate, on $\hat{K}$.

$$F(\hat{x}_1, \hat{x}_2) = \hat{\lambda}_1(\hat{x}_1, \hat{x}_2)\, x^{(1)} + \hat{\lambda}_2(\hat{x}_1, \hat{x}_2)\, x^{(2)} + \hat{\lambda}_3(\hat{x}_1, \hat{x}_2)\, x^{(3)}$$

$$= \hat{N}_1(\hat{x})\, x^{(1)} + \hat{N}_2(\hat{x})\, x^{(2)} + \hat{N}_3(\hat{x})\, x^{(3)}.$$

$$a^e \triangleq \int_{Ke} f(x).\, dV.$$

$$\int_{Ke} f(x)\, dx = \int_{\hat{K}} f(Fe(\hat{x})).\ |Je(\hat{x})|\, d\hat{x}$$

$$\nabla F = \begin{bmatrix} F_{1,r} & F_{1,\theta} \\ F_{2,r} & F_{2,\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$J = \det \nabla \underline{\underline{F}} = r\cos^2\theta + r\sin^2\theta = r$$

$$\int_{\text{circle}} f(x)\, dx = \int_0^R dr \int_0^{2\pi} f(F(r,\theta))\, r\, dr\, d\theta$$

P_1-element by composition.

$$F(\hat{x}_1, \hat{x}_2) = \cdots = A\hat{x} + b = x$$

$$\hat{x} = A^{-1}(x-b)$$

$$N_a(x) = \hat{N}_a(A^{-1}(x-b))$$

composing linear functions with linear functions

Finite Element Review 3/16/2024.

$$-(k u'(x))' + b u'(x) + c u(x) = f(x),$$

↳ general form for elliptic problem in 1D.

$$u(0) = g_0 \qquad \partial\Omega_D.$$
$$u'(L) = d_L \qquad \partial\Omega_N.$$

\# Derivation of Variational Equation.

→ Example on diffusion problem.

$$-u''(x) = f(x), \qquad x \in \Omega$$
$$u(0) = g_0,$$
$$u'(L) = d_L.$$

1. Integrating over test functions:

$$\int_0^L u''(x) v(x) + f(x) v(x) dx. = 0$$

2. Integration by part:

$$u'(L) v(L) - u'(0) v(0) - \int_0^L u'(x) v'(x) dx$$
$$+ \int_0^L f(x) v(x) dx$$

3. Substitute the B.C.s and require $v(0)=0$ (Galerkin formulation to find weak sol'n),

$$0 = d_L\, v(L) - u'(0)\, 0 - \int_0^L u'(x) v'(x)\, dx$$

$$+ \int_0^L f(x) . v(x)\, dx .$$

$$\rightarrow \int_0^L u'(x) v'(x) dx - d_L\, v(L) = \int_0^L f(x) . v(x) dx .$$

formulated test space : $\mathcal{V} = \{ w : [0, L] \to \mathbb{R}$ smooth $\mid v(0) = 0 \}$.

Remark : we formulate the problem in such a way s.t. it has the same number of derivatives required from u & v. & no evaluations of derivatives of on the boundary .

Recipe of obtaining variational equations

1. Form the residual

$$r = -[k u'(x)]' + b u'(x) + c u(x) - f(x)$$

→ for strong form:

$$r(x) = 0, \qquad x \in (0, L).$$

2. Multiply test function & integrate.

$$\int_0^L r(x)\, v(x)\, dx = 0$$

also weight functions.

$$- \int_0^L \left(-[k(x) u'(x)]' + b(x) u'(x) + c(x) u(x) - f(x) \right) v\, dx = 0$$

3. Integrate residual by parts.

$$\int_0^L k(x) u'(x) v'(x) + b(x) u'(x) v(x) + c(x) u(x) v(x) - f(x) v(x)\, dx$$

$$- k(L) u'(L) v(L) + k(0) u'(0) v(0) = 0$$

4. Substitute the boundary conditions

$$\uparrow u'(L).$$

$$\int_0^L k(x)u'(x)v'(x) + b(x)u'(x)v(x) + c(x)u(x)v(x) - f(x)v(x)\,dx$$

$$- k(L)d_L v(L) + k(0)u'(0)v(0) = 0$$

Request $v \in \mathcal{V}$, $v(0) = 0$.

$$\Rightarrow \int_0^L ku'v' + bu'v + cuv - fv\,dx$$

$$- k(L)\,d_L v(L) = 0$$

5. State the variational equation.

$$\int_\Omega [ku'v' + bu'v + cuv]\,dx - k(L)d_L v(L)$$

$$= \int_\Omega f(x)v(x)\,dx$$

for any $v \in \mathcal{V}$.

$$\mathcal{V} = \{w : \Omega \to \mathbb{R} \text{ smooth} \mid w(0) = 0\}.$$

Vector spaces of Functions.

1. Closure. $u + v \in V$, $\alpha \cdot u \in V$
2. Commutativity. $u + v = v + u$.
3. Associativity. $u + (v + w) = (u+v) + w$.

   $\alpha \cdot (\beta \cdot u) = (\alpha\beta) \cdot u$
4. Identity. $u + 0 = u$ & $1 \cdot u = u$
5. Additive Inverse. $\forall\ u \in V,\ \exists\ v \in V.$

   $v + u = 0$
6. Distributivity. $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u$.

   $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$

Vector Subspace $V \subset W$

Affine Subspace . $V = \{ s_2 - s_1 \mid s_2 \in S \}$

$$S \subset W$$

Span. $V \to$ vector space.

$U \subset V$ set of vectors

lear
binations

$$\operatorname{span}(U) = \left\{ \sum_{i=1}^{n} c_i e_i \mid n \in N,\ e_i \in U,\ c_i \in R \right\}.$$

$$u, v \in \mathcal{V}.$$

Linear Functional. $\mathcal{V} \to \mathbb{R}$.

$$l(u+\alpha v) = l(u) + \alpha l(v).$$

Bilinear Form. $\mathcal{V} \times \mathcal{V} \to \mathbb{R}$.

$$u, v, w \in \mathcal{V}.$$

$$a(u+\alpha v, w) = a(u, w) + \alpha a(v, w).$$

$$a(w, u+\alpha v) = a(w, u) + \alpha a(w, u).$$

if "a" symmetric bilinear:

$$a(u, v) = a(v, u)$$

Variational Eqn. $\mathcal{W} \times \mathcal{V} \to \mathbb{R}$.

$$F(u, v+\alpha w) = F(u, v) + \alpha F(u, w).$$

$$\forall \; u \in \mathcal{W}, \quad v, w \in \mathcal{V}, \quad \alpha \in \mathbb{R}$$

test space

$$F(u, v) = 0$$

variational eqn.

Linear Variational Eqn.

$$0 = F(u, v) = a(u, v) - l(v).$$

$$\Downarrow$$

$$a(u, v) = l(v), \quad \forall v \in \mathcal{V}.$$

Variational Methods.

Recall variational eqn.

$$F(u, v) = 0, \quad \forall v \in \mathcal{V}.$$

variational meth. → finite dimensional function spaces $\mathcal{V}_h$ & $\mathcal{S}_h$ → define u_h approx. u → find $u_h \in \mathcal{S}_h$ s.t.

$$F(u_h, v_h) = 0, \quad \forall v \in \mathcal{V}_h.$$

$\mathcal{S}_h$ → trial space. an affine space where u_h is sought.

∴ Problem formulation

Find $u_h \in \mathcal{S}_h$. s.t.

$$a(u_h, v_h) = l(v_h), \quad \forall v_h \in \mathcal{V}_h.$$

Consistency. require $\mathcal{V}_h \subseteq \mathcal{V}$. s.t.

$$F(u, v_h) = 0, \quad \forall v_h \in \mathcal{V}_h.$$

→ Said to be consistent. consistency condition.

$F(u,v)$

→ Summary

- $\mathcal{S}_h = \{u_h \in \mathcal{W}_h \mid . \; u_h$ satisfies essential B.C.s$\}$
- $\mathcal{V}_h \rightarrow$ Direction of $\mathcal{S}_h$.
- Consistency: $\mathcal{V}_h \subseteq \mathcal{V}$

Solution to Variational Method.

$$u_h(x) = \sum_{b=1}^{m} u_b N_b(x).$$

$$v_h(x) = \sum_{a=1}^{m} v_a N_a(x).$$

basis functions:

$$\underbrace{\underbrace{N_1, \dots, N_n,}_{\text{basis } \mathcal{V}_h} \dots, N_m}_{\text{basis for } \mathcal{W}_h.}$$

implying $v_a = 0$

$n < a \leq m$.

We will select basis functions of $\mathcal{V}_h$ as test functions:

$$\ell(N_a) = a(u_h, N_a).$$

linear system of equations

$$\underset{\text{Stiffness matrix}}{K} U = \underset{\text{load vector}}{F}.$$

$$K = \underbrace{\begin{bmatrix} k_{11} & \cdots & K_{1(n+1)} & \cdots & k_{1m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{n1} & \cdots & K_{n(n+1)} & \cdots & k_{nm} \\ 0 & & 1 & & \vdots \\ \vdots & \ddots & \vdots & \ddots & \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}}_{\delta_{ab} \text{ part}}$$

⇝ Arbitrarily-ordered basis.

$\eta = \{1, \dots, m\}$ → set of indices of all basis functions in $\mathcal{V}_h$.

basis functions for $\mathcal{V}_h$ → subset of η.

active indices ← $\eta_a \subset \eta$.

$$w_h \in \mathcal{V}_h \leftrightarrow w_h = \sum_{a \in \eta_a} w_a N_a.$$

$\eta_g = \eta \setminus \eta_a$, constrained indices.

testing each basis function in $\mathcal{V}_h$.

$$l(N_a) = a(u_h, N_a). \qquad a \in \eta_a.$$

we label:

$$\begin{cases} F_a = l(N_a); \quad K_{ab} = a(N_b, N_a). \\ \qquad\qquad\qquad \hookrightarrow a \in \eta_a, \ b \in \eta. \\ F_a = \bar{u}_a, \quad K_{ab} = \delta_{ab}. \\ \qquad\qquad\qquad \hookrightarrow a \in \eta_a, \ b \in \eta. \end{cases}$$

Euler-Lagrange Equations.

find a functional EL s.t.

$$F(u, v) = 0, \quad \forall v \in \mathcal{V}$$

$$\Updownarrow$$

$$EL(u, x) = 0, \quad \forall x \in \omega \subseteq \bar{\Omega}$$

Right-to-left implication:

$$EL(u, x) = 0, \qquad \forall x \in \omega = (0, L]$$

$$\Updownarrow$$

$$F(u, v) = 0. \qquad \forall v \in \mathcal{V}$$

General Steps to Obtain Euler-Lagrange.

$$\int_\Omega [k u' v' + b u' v + c u v]\, dx - k(L)\, d_L\, v(L) = \int_\Omega f v\, dx$$

$\forall\ v \in \mathcal{V}, \quad \mathcal{V} = \{w : [0, L] \to \mathbb{R} \text{ sm} \mid w(0) = 0\}$.

1. Integration by parts to eliminate all derivs.

$$0 = \int_0^L [k u' v' + b u' v + c u v - f v] dx - k(L)\, d_L\, v(L)$$

$$= \int_0^L [-k u'' v + b u' v + c u v - f v]\, dx + [k(L) u'(L) - k(L) d_L]\, v(L) - k(0) u'(0) v(0)$$

2. Group the v terms. & use conditions in $\mathcal{V}$. $\rightarrow$

$$0 = \int_0^L [-k u'' + b u' + c u - f]\, v\, dx + k(L)[u'(L) - d_L]\, v(L).$$

$$\boxed{\int u\, dv = uv - \int v\, du}$$

3. Obtain the differential Equation & potential boundary conditions.

$$0 = -ku'' + bu' + cu - f. \qquad x \in (0, L)$$

$$0 = k(L)\left[u'(L) - d_L\right].$$

Weak & Strong Form.

$$\mathcal{S} = \{ w : \Omega \to \mathbb{R} \text{ smooth} \mid w(0) = g_0 \}.$$

$$\mathcal{V} = \{ w : \Omega \to \mathbb{R} \text{ smooth} \mid w(0) = 0 \}.$$

$$\int_\Omega \left[ku'v' + bu'v + cuv \right] dx - k(L) d_L v(L) = \int_\Omega fv \, dx$$

weak form.

→ weak solution: $u(0) = g_0$

Abstract Weak Form.

$\mathcal{W}$ → vector space. $a : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ bilinear.

$\ell : \mathcal{W} \to \mathbb{R}$: linear functional.

find $u \in \mathcal{S}$, $a(u, v) = \ell(v)$. $v \in \mathcal{V}$.

#. C^0 Finite Element Space.

variational eqn.

$$\int_0^1 u'v' \, dx = \int_0^1 v \, dx.$$

$$\forall \; v \in \mathcal{V} = \{ w : [0, 1] \to \mathbb{R} \text{ smooth} \;|\; w(0) = 0 \}.$$

1. Build mesh of domain

$$c = x_1 < \ldots < x_{nel+1} = d.$$

x_i : Vertex, $i \to$ vertex number

2. Build basis func. $N_a(x)$
3. Build $\mathcal{V}_h$ & $\mathcal{S}_h$.

$$\mathcal{S}_h = \{ u_h \in \mathcal{W}_h \;|\; u_h(0) = 2 \}.$$

$$\mathcal{V}_h = \{ v_h \in \mathcal{W}_h \;|\; v_h(0) = 0 \}$$

e.g., $\mathcal{V}_h = \{ v_2 N_2 + \cdots + v_{nel+1} N_{nel+1} \;|\; v_2, \ldots v_{nel+1} \in \mathbb{R} \}.$

$$= \text{Span}(\{ N_2, \ldots, N_{nel+1} \}).$$

$$\mathcal{S}_h = \{ u_h \in \mathcal{W}_h \;|\; u_1 = 2 \}.$$

$$= \{ 2N_1 + v_h \;|\; v_h \in \mathcal{V}_h \}.$$

4. Compute K & F.

$l(N_a)$, $a(N_b, N_a)$

5. Solve Finite Element Sol'n.

Consistency.

if $\mathcal{V}_h$ is not a subset of the test space $\mathcal{V}$, we cannot guarantee consistency.

→ we need to check $F(u, v_h) = 0 \quad \forall v_h \in \mathcal{V}_h$

Substitute the exact sol'n → $F(u, v_h)$.

(following EL procedure) $\overset{\parallel}{0}$

→ State: $F(u, v) = 0, \quad \forall v \in \mathcal{V} + \mathcal{V}_h$.

where $\mathcal{V} + \mathcal{V}_h = \{ w = v + v_h \mid v \in \mathcal{V}, v_h \in \mathcal{V}_h \}$

Def'n of Finite Element.

a pair $e = (\Omega_e, \mathcal{N}^e)$.

Ω_e ↓ domain

$\mathcal{N}^e$ → basis functions:

$\mathcal{N}^e = \{ N_1^e, \ldots, N_k^e \}$.

Space of funcs. $\mathcal{P}^e = \text{span}\{ N_1^e, \ldots, N_k^e \}$

general defn for P_k- element.

$$N_a^e(x) = \frac{\prod_{b=1, b\neq a}^{k+1} (x - x_b^e)}{\prod_{b=1, b\neq a}^{k+1} (x_a^e - x_b^e)}$$

* elements → all D.o.F are values of the function at predefined locations in the elem. are called Lagrange elements.

↓

e.g., P_k- element.

Construction of Finite Elem. Space.

1. Extend shape fncs. by zero.
2. Define Local-to-Global Map.

$$LG = \begin{bmatrix} \xrightarrow{\text{element index}} & \\ & \end{bmatrix} \Big\downarrow \text{shape func. index}$$

$$LG(\underset{\text{shape func.}}{a}, \underset{\text{element index}}{e}) = \text{basis func. index}$$

3 Add Shape Functions.

with the set: $\{(a,e) \mid LG(a,e) = A\}$

→ practice some examples on assembly of Stiffness matrix of load vectors.

→ Some comments on symmetrization of Stiffness matrix for efficient calc.

Elliptic fourth-order Problem.

Variational eqn.

$$r(x) = [q(x)\, u''(x)]'' + c(x)\, u(x) - f(x)$$

$$\rightarrow 0 = \int_0^L r(x)\, v(x)\, dx = \int_0^L \left[[q(x)\, u''(x)]'' + c(x)\, u(x) - f(x) \right] v(x) dx$$

Natural B.C.s: $u''(L) = m_L$ & $u'''(L) = n_L$.

essential B.C.s: $u(0) = g_0$ & $u'(0) = d_0$.

$$\rightarrow a(u,v) = \int_0^L \left[q(x)\, u''(x)\, v''(x) + c\, u(x)\, v(x) \right] dx.$$

$$l(v) = \int_0^L f v dx - [q(L)\, n_L + q'(L)\, m_L]\, v(L) + q(L)\, m_L v'(L)$$

Diffusion Problem in 2D

Definitions: $\begin{cases} \text{Dirichlet boundary} : \partial\Omega_D \\ \text{Neumann boundary} : \partial\Omega_N \end{cases}$

Integration by parts for 2D or 3D.

$$\int_\Omega v \operatorname{div} w \, d\Omega = \int_{\partial\Omega} v w \cdot \check{n} \, d\Gamma - \int_\Omega w \cdot \nabla v \, d\Omega$$

with a d - dimensional problem.

$$\sum_{i=1}^{d} \left[\int_\Omega v \, \partial_i w_i \, d\Omega \right] = \sum_{i=1}^{d} \left[\int_{\partial\Omega} v w_i \check{n}_i \, d\Gamma - \int_\Omega w_i \partial_i v \, d\Omega \right]$$

$w \rightarrow (w_1, w_2, \ldots, w_d)$.

applying IBP for heat diffusion problem.

$$- \int_\Omega \operatorname{div} (K \nabla u) v \, d\Omega = \int_\Omega f v \, d\Omega$$

$$\rightarrow \int_\Omega (K \nabla u) \cdot \nabla v \, d\Omega = \int_\Omega f v \, d\Omega + \int_{\partial\Omega_N} v K \nabla u \cdot \check{n} \, d\Gamma$$

$$+ \int_{\partial\Omega_D} v \, k \nabla u \cdot \check{n} \, d\Gamma$$

$\downarrow$

we donot know this value on $\partial\Omega_D$.

Hence, the test space:

$$\mathcal{V} = \{ v : \Omega \to \mathbb{R} \text{ smooth} \mid v(x) = 0 \quad \forall x \in \partial\Omega_D \}.$$

$\rightarrow$ weak form:

$$a(u, v) = l(v), \quad \forall v \in \mathcal{V}.$$

$$a(u,v) = \int_\Omega (k \nabla u) \cdot \nabla v \, d\Omega.$$

$$l(v) = \int_\Omega f v \, d\Omega + \int_{\partial\Omega_N} H v \, d\Gamma.$$

Weak form for 2D diffusion:

$$\mathcal{S} = \{ v : \Omega \to \mathbb{R} \text{ smooth} \mid v(x) = g(x) \quad \forall x \in \partial\Omega_D \}$$

Find $u \in \mathcal{S}$, s.t. $a(u, v) = l(v) \quad \forall v \in \mathcal{V}$.

Nitsche's Method for high-dimensions.

$$\int_\Omega (k \nabla u) \cdot \nabla v \, d\Omega - \int_{\partial\Omega_D} v k \nabla u \cdot \check{n} \, d\Gamma$$

$$= \int_\Omega f v \, d\Omega + \int_{\partial\Omega_N} v H \, d\Gamma$$

impose the Dirichlet B.C.s:

$$\int_{\partial\Omega_D} (g-u).K\nabla v.\check{n}\, d\Gamma = 0$$

$$\int_{\partial\Omega_D} \mu(u-g)\, v\, d\Gamma = 0$$

$$\to \int_{\Omega} (k\nabla u).\nabla v\, d\Omega - \int_{\partial\Omega_D} (vk\nabla u + uk\nabla v).\check{n}\, d\Gamma$$

$$+ \int_{\partial\Omega_D} \mu u v\, d\Gamma = \int_{\Omega} fv\, d\Omega + \int_{\partial\Omega_N} vH\, d\Gamma$$

$$- \int_{\partial\Omega_D} gK\nabla v.\check{n}\, d\Gamma + \int_{\partial\Omega_D} \mu g v\, d\Gamma$$

Variational Numerical Methods.

- Spaces $\mathcal{S}_h$ & $\mathcal{V}_h$ composed of functions take values over Ω-dimensional domain
- domain boundary is a closed line. assume polygon for simplicity.
- Consistent & test space $\mathcal{V}_h$ are continuous.

Mesh. $\mathcal{T} = \{K_1, \underset{\substack{\uparrow \\ \text{finite domains}}}{\dots}, K_{ne_1}\}$.

$\mathring{K}_i \cap \mathring{K}_j = \phi$ and $\Omega = \bigcup_{i=1}^{Ne_1} K_i$.

Continuous P_1 finite element space.

↓

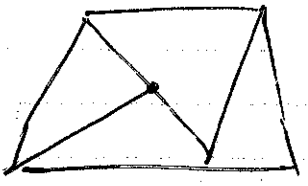
we want to uniquely define the vertices

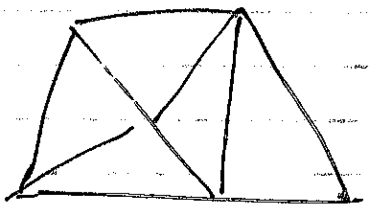
↓

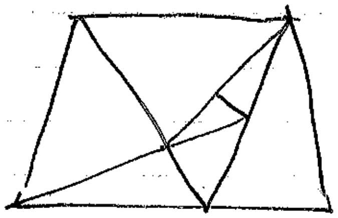
conforming triangulations.

Conforming triangulation.

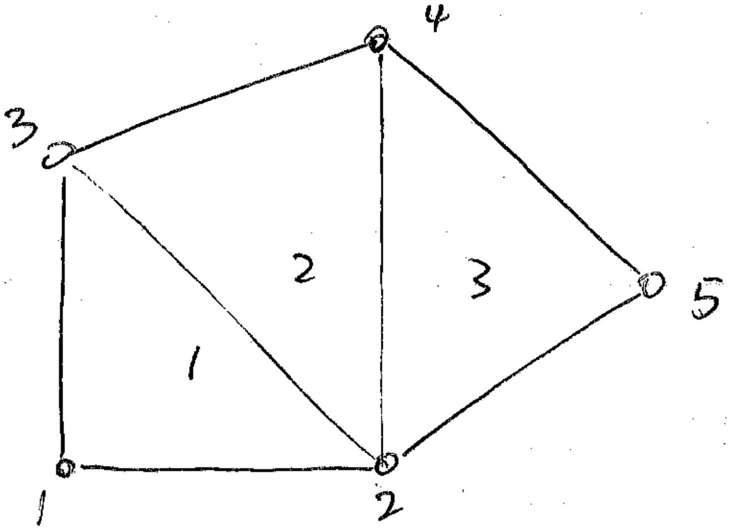
Polygonal domain Ω is a mesh for Ω s.t. intersection of 2 $\triangle$'s K & K'. is either (a) empty; (b) whole edge; or (3) vertex of both K & K'.







Example.



$$\underline{X} = \begin{bmatrix} 4 & 8 & 4 & 8 & 12 \\ 2 & 2 & 6 & 8 & 4 \end{bmatrix}$$

$$I\forall = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

Barycentric or Area Coordinates.

geometry of P_1 triangle $K \rightarrow \underline{X}^1, \underline{X}^2, \underline{X}^3$.

$$K = \mathcal{C}(\{\underline{X}^1, \underline{X}^2, \underline{X}^3\})$$

$$= \left\{ \sum_{j=1}^{d+1} \lambda_j \underline{X}^j \;\middle|\; \underbrace{0 \le \lambda_j \le 1 \;\; \forall j, \;\; \text{and} \; \sum_{j=1}^{d+1} \lambda_j = 1}_{\text{reference triangle} \leftarrow \hat{K}} \right\}$$

$$x = \sum_{j=1}^{3} \lambda_j \underline{X}^j$$ unique triplet $(\lambda_1, \lambda_2, \lambda_3) \in \hat{K}$

$$x \in K \iff (\lambda_1, \lambda_2, \lambda_3) \in \hat{K}.$$

λ_i : barycentric coordinates.

barycentric coordinates satisfy:

$$\lambda_i = \frac{A_i}{A}.$$

A_i → triangle formed by x

area of triangle K ↙ A

☆ the inverse map from $(\lambda_1, \lambda_2, \lambda_3)$

$$\downarrow$$

$$x = \sum_{j=1}^{3} \lambda_j \underline{X}^j.$$

$$\begin{cases} \lambda_1(x_1, x_2) = \frac{1}{2A}\left[-(\underline{X}_2^3 - \underline{X}_2^2)(x_1 - \underline{X}_1^2) + (\underline{X}_1^3 - \underline{X}_1^2)(x_2 - \underline{X}_2^2)\right] \\ \lambda_2(x_1, x_2) = \frac{1}{2A}\left[-(\underline{X}_2^1 - \underline{X}_2^3)(x_1 - \underline{X}_1^3) + (\underline{X}_1^1 - \underline{X}_1^3)(x_2 - \underline{X}_2^3)\right] \\ \lambda_3(x_1, x_2) = \frac{1}{2A}\left[-(\underline{X}_2^2 - \underline{X}_2^1)(x_1 - \underline{X}_1^1) + (\underline{X}_1^2 - \underline{X}_1^1)(x_2 - \underline{X}_2^1)\right] \end{cases}$$

where

$$2A = (\underline{X}_1^2 - \underline{X}_1^1)(\underline{X}_2^3 - \underline{X}_2^1) - (\underline{X}_2^2 - \underline{X}_2^1)(\underline{X}_1^3 - \underline{X}_1^1)$$

P_1 - element & LG map.

for triangular finite elements,

$$N_1^e = \lambda_1, \quad N_2^e = \lambda_2, \quad N_3^e = \lambda_3.$$

$$\nabla N_1^e = \frac{1}{2A}\begin{pmatrix} X_2^2 - X_2^3 \\ X_1^3 - X_1^2 \end{pmatrix}$$

$$\nabla N_2^e = \frac{1}{2A}\begin{pmatrix} X_2^3 - X_2^1 \\ X_1^1 - X_1^3 \end{pmatrix}.$$

$$\nabla N_3^e = \frac{1}{2A}\begin{pmatrix} X_2^1 - X_2^2 \\ X_1^2 - X_1^1 \end{pmatrix}$$

if $LG = LV$ → conform

↓

Vertices as nodes &

triangles as element domains.

Example

$$LG = LV = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix}.$$

$$N_1 = N_1^1$$

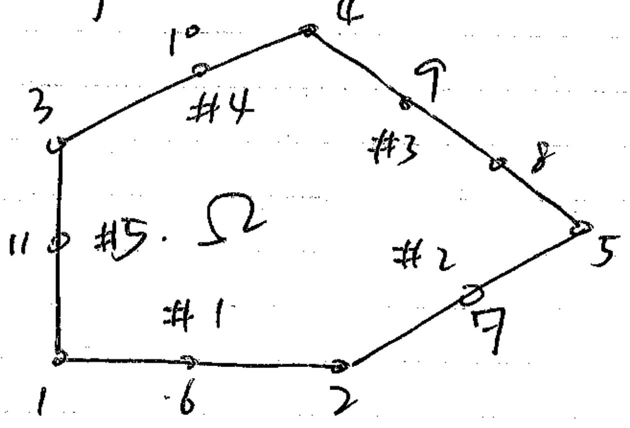
$$N_2 = N_2^1 + N_2^2 + N_2^3$$

$$N_3 = N_3^1 + N_1^2$$

$$N_4 = N_3^2 + N_1^3$$

$$N_5 = N_3^3$$

Boundary arrays of triangulation.



$$BE = \begin{bmatrix} 1 & 6 & 2 & 7 & 5 & 8 & 9 & 4 & 10 & 3 & 11 \\ 6 & 2 & 7 & 5 & 8 & 9 & 4 & 10 & 3 & 11 & 1 \\ 1 & 1 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 5 & 5 \end{bmatrix}$$

Handling Dirichlet Boundaries.

$$\tilde{\mathcal{S}}_h = \{ w_h \in \mathcal{W}_h \mid w_h(x) = g(x), \ \forall x \in \partial\Omega_D \}$$

$$\tilde{\mathcal{V}}_h^{\ell} = \{ w_h \in \mathcal{W}_h \mid w_h(x) = 0, \ \forall x \in \partial\Omega_D \}$$

$$\rightsquigarrow \mathcal{S}_h = \{ w_h \in \mathcal{W}_h \mid w^a = g(\underline{X}^a) \ \forall \text{ vertex } \underline{X}^a \in \partial\Omega_D \}$$

$$\mathcal{V}_h = \{ w_h \in \mathcal{W}_h \mid w^a = 0 \ \forall \text{ vertex } \underline{X}^a \in \partial\Omega_D \}$$

Neumann B.C.s

$$\int_{\partial\Omega_N} H N_A \, d\Gamma \longrightarrow \int_{\partial\Omega} \overset{\text{H vector}}{\vec{H}} N_i \, d\Gamma$$

Numerical Analysis for Elliptic Problem.

finite element space $\mathcal{W}_h$ → mesh over Ω.

provide a set of basis functions.

$\{N_a, \ a = 1, 2, \dots, n\}$.

$$w_h \in \mathcal{W}_h \iff W_h(x) = \sum_{a=1}^{n} C_a N_a(x).$$

trial & test space $\mathcal{S}_h$ & $\mathcal{V}_h$:

$$\mathcal{S}_h = \{ w_h \in \mathcal{W}_h \mid w_h \text{ essential B.C.s} \}.$$

$\mathcal{V}_h$ = Direction of $\mathcal{V}_h$

Fundamental Approximation

☆ Cea's Lemma → Exact consistency.

$$a(u, v_h) = \ell(v_h). \quad \forall v_h \in \mathcal{V}_h.$$

1. Domain of the Norm:

$$\|u\| < +\infty, \quad \|w_h\| < +\infty, \quad \forall w_h \in \mathcal{V}_h.$$

2. Continuity: exists $M > 0$ & $m > 0$

s.t. $|a(u - w_h, v_h)| \le M \|u - w_h\| \|v_h\|$.

$\forall v_h \in \mathcal{V}_h, \ \forall w_h \in \mathcal{S}_h$.

$|l(v_h)| \le m \|v_h\|$. $\forall v_h \in \mathcal{V}_h$.

3. Coercivity. exists $\alpha > 0$ s.t.

$a(v_h, v_h) \geq \alpha \|v_h\|^2$. $\forall v_h \in \mathcal{V}_h$.

" If 1. 2. 3 are satisfied, then.

a) finite element soln exists. & unique.

satisfying stability:

$$\|u_h\| \le \frac{m}{\alpha} + \left(1 + \frac{M}{\alpha}\right) \min_{w_h \in \mathcal{S}_h} \|w_h\|$$

b) a priori approximation result.

$$\|u - u_h\| \le \left(1 + \frac{M}{\alpha}\right) \min_{w_h \in \mathcal{S}_h} \|u - w_h\|$$

$\swarrow$

Norm:
$$\begin{cases} \|v\| \geq 0, \quad \|v\| = 0 \ \text{IFF} \ v = 0 \\ \|\beta v\| = |\beta| \ \|v\| \\ \|v + u\| \le \|v\| + \|u\| \end{cases}$$