COURSE NOTES

Foundations of Solid Mechanics

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2021

Week 1: Mon. 8/29/2021

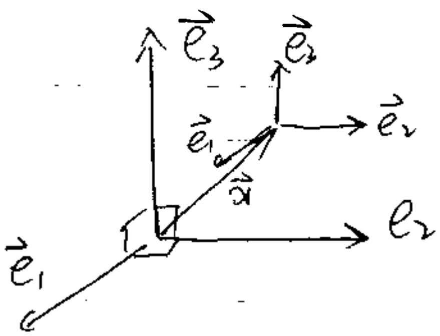
Vectors & tensors. → Cartesian.

in Physics, we are familiar with

$$\begin{cases} \nabla \times \vec{E} = 0 \\ \nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0} \end{cases} \Rightarrow \text{Electro-statics.}$$

which is, independent of coordinate system

in a RH coordinate,



$\|\vec{e}_i\| = 1.$

$$\vec{V} = \sum_{i=1}^{3} V_i \vec{e}_i$$

index subscript ← $\vec{e}_i \cdot \vec{e}_j \equiv \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$

⇓

Called Kronecker delta.

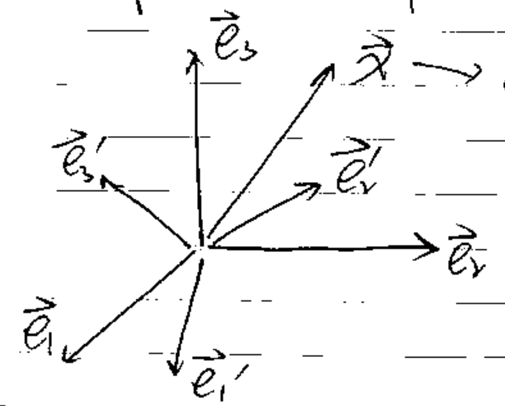
(in orthonormal basis).

Here, V_i is component of $\vec{V}$

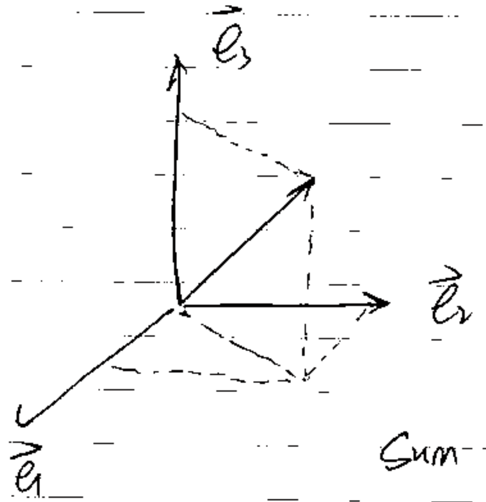
with a basis $\{\vec{e}_i\}$.

$$= \sum_{j=1}^{3} V_j' \vec{e}_j'$$

transformation of basis.



$\vec{x} \rightarrow$ cooridinate dependent.



Summation convection

↓

(indicial notation).

↓

Sum of $v_i e_i = v_i e_i = v_k e_k$

$\underbrace{\quad}_{\text{repeated indices.}}$

term: subscripts. $\underbrace{a_{ij} b_k c_{jm}}_{\text{summing over } j} \rightarrow$ free indices.

contraction, if $i = m$.

$a_{ip} b_k c_{pi} \rightarrow$ double summation.

*** A dummy index cannot repeat more than 2 !!!

$\delta_{ij}\delta_{jk} = d_{ik} = \delta_{ik}$.

e.g. $\delta_{ij}\delta_{jk} = \delta_{11}\delta_{1k} + \underset{0}{\delta_{12}}\delta_{2k} + \underset{0}{\delta_{13}}\delta_{3k}$

$= \delta_{1k}$.

$\delta_{2k}\ (i=2),\quad \delta_{3k}\ (i=3)$.

$\vec{v} \cdot \vec{w} = v_i \vec{e}_i \cdot w_j \vec{e}_j = v_i w_j \underbrace{(\vec{e}_i \cdot \vec{e}_j)}_{\delta_{ij}}$

$= v_i w_i = v_j w_j$.

usual dot product.

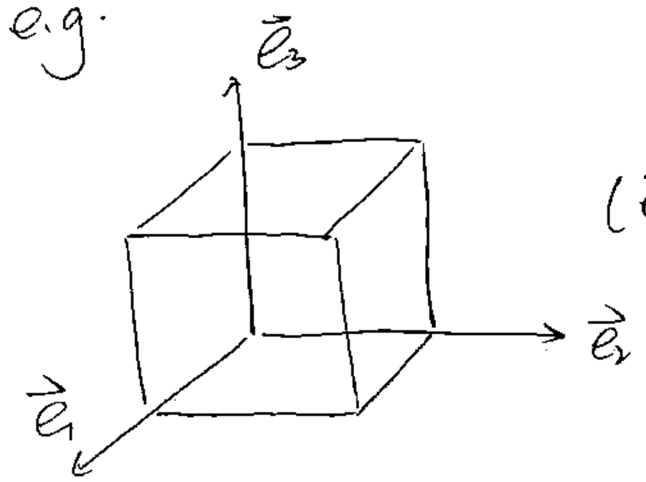
Cross product.

$\vec{v} \times \vec{w} = v_i \vec{e}_i \times w_j \vec{e}_j = v_i w_j\, \vec{e}_i \times \vec{e}_j \quad (a)$.

$\vec{e}_i \times \vec{e}_j = [\underbrace{(\vec{e}_i \times \vec{e}_j) \cdot \vec{e}_k}_{= \varepsilon_{ijk}}]\, \vec{e}_k$

free index on two sides of Eqn. must be equal !!!

$= \varepsilon_{ijk}$ (Permutation symbol) e.g.



$(\vec e_1 \times \vec e_2) \cdot \vec e_3 = 1.$

$\epsilon_{ijk} = 1$, $(1,2,3), (3,1,2), (2,3,1)$.

$= -1$, $(2,1,3), (1,3,2), (3,2,1)$.

$= 0$, otherwise.

Review ~~ang~~ undergrad linear algebra

det$(\sim)$.

eq. (a) writes. $v_i w_j \epsilon_{ijk} \vec e_k$

$\Rightarrow \epsilon_{ijk} v_i w_j \vec e_k$

$= \epsilon_{kij} v_i w_j \vec e_k$

Q. $\vec a \times (\vec b \times \vec c) = ?$

$= a_k \vec e_k \times (b_i c_j \, \vec e_i \times \vec e_j)$

$= a_k \vec e_k \times (b_i c_j \, \epsilon_{ijm} \vec e_m)$.

$= a_k b_i c_j \, (\delta_{iw}\delta_{jk} - \delta_{ik}\delta_{jw}) \vec e_w$

$= (a_i \vec e_i) \times [\epsilon_{pjk} b_j c_k \vec e_p]$

$= (\delta_{sj}\delta_{ik} - \delta_{sk}\delta_{ij}) a_i b_j c_k \vec e_s$

$= (b_s a_k c_k - b_i a_i c_s) \vec e_s$

$= [b_s(\vec a \cdot \vec c) - (\vec a \cdot \vec b) c_s] \vec e_s$

$= (\vec a \cdot \vec c)\vec b - (\vec a \cdot \vec b)\vec c$

· How vectors transform? (on basis)

$\vec v = v_i \vec e_i = v_j' \vec e_j'$

$v_j' = (\vec v \cdot \vec e_j') = (v_i \vec e_i \cdot \vec e_j')$.

$= v_i (\vec e_i \cdot \vec e_j')$.

$P_{ji} \equiv \vec e_j' \cdot \vec e_i$.

projection of one basis on another basis.

$\vec V_j' = P_{ji} \cdot \vec V_i$

$$\vec V' = \begin{pmatrix} V_1' \\ V_2' \\ V_3' \end{pmatrix} \quad \vec V = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

$\vec P = [P_{ji}]$

$\vec V' = \vec P \vec V$

$\vec V = \vec P^{-1} \vec V'$

☆☆☆ $\vec P^{-1} = \vec P^{t}$

Week 1:, Wed. 9/1/2021.

Second order Tensor $\underline{\underline{A}}$.

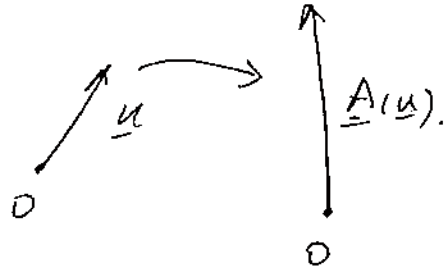
A 2^{nd} order tensor is a linear transformation from $\mathbb{E}^3$ to $\mathbb{E}^3$

LT = a special kind of mapping $\mathbb{E}^3 \to \mathbb{E}^3$

properties:

$\underline{\underline{A}}(\underline{u})$ to some vector

$\underline{\underline{A}} : (\underline{u}) \longrightarrow \underline{\underline{A}}(\underline{u})$.



$\mapsto \underline{\underline{A}}(a\underline{u}) = a\,\underline{\underline{A}}(\underline{u})$.

real No.

$\mapsto \underline{\underline{A}}(\underline{u} + \underline{w}) = \underline{\underline{A}}(\underline{u}) + \underline{\underline{A}}(\underline{w})$.

$\Updownarrow$

$\underline{\underline{A}}(a\underline{u} + b\underline{w}) = a\underline{\underline{A}}(\underline{u}) + b\,\underline{\underline{A}}(\underline{w})$.

$\forall\ \underline{u}, \underline{w}$ w/ $\mathbb{E}^3$ and a, b.

$\underline{\underline{A}}(\underline{0}) = \underline{0}$.

Example rigid body rotation about a fixed point.

Defination gradient tensor.
Stress tensor

$\underline{x} = x_i \vec{e}_i$

$\underline{\underline{A}}(\underline{x}) = \underline{\underline{A}}(x_j \vec{e}_j)$.

$= \cancel{A_j}\ x_j\ \underline{\underline{A}}(\underline{e}_j)$

this tells us that a linear transformation is completely determined by its action on the basis vectors.

$\underline{\underline{A}}(\underline{e}_j) = a_{ij}\underline{e}_i$

$a_{ij} e_i (e_j \cdot \underline{x})$

$\underline{\underline{A}}(x_j e_j) = \cancel{a_{ij}}\ x_j \underline{\underline{A}}(\underline{e}_j) = a_{ij} x_j \underline{e}_i$

$\underline{\underline{A}}(\underline{e}_1)\cdot \underline{e}_1 = a_{11}$

$\underline{\underline{A}}(\underline{e}_1)\cdot \underline{e}_2 = a_{21}$

$\underline{\underline{A}}(\underline{e}_1)\cdot \underline{e}_3 = a_{31}$

$$A = \begin{bmatrix} \cancel{A} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\underline{\underline{A}}(\underline{x}) = a_{ij}\, \underline{e}_i (\underline{e}_j \cdot \underline{x})$

$= \underbrace{a_{ij}\, \underline{e}_i\, \underline{e}_j}_{\underline{\underline{A}}} \cdot \underline{x}$

Define $\underline{a}\underline{b}$ as the linear transformation.

$(\underline{a}\underline{b})(\underline{x}) = \underline{a}(\underline{b}\cdot\underline{x})$. ??

check: this is a LT.

$= \underline{\underline{A}} \cdot \underline{x}$ (we can skip the dot). ??

$\underline{a}\underline{b} \rightarrow$ dyad

*** Any linear transformation can be written as sum of dyad

$\underline{a}\underline{b} \Leftrightarrow \underline{a} \otimes \underline{b} \rightarrow$ linear transformation.

A simple representation: $\underline{\underline{A}} = a_{ij}\, \underline{e}_i\, \underline{e}_j$

$\underline{a}\underline{b} \neq \underline{b}\underline{a}$.

$\underline{e}_i \to \underline{e}'_k$.

$$\underline{\underline{A}} = a_{ij}\, \underline{e}_i \underline{e}_j = a'_{rs}\, \underline{e}'_r \underline{e}'_s$$

$$= a_{ij} \underbrace{(\underline{e}_i \cdot \underline{e}'_r)\underline{e}'_r}_{\underline{e}_i} \underbrace{(\underline{e}_j \cdot \underline{e}'_s)\underline{e}'_s}_{\underline{e}_j}$$

$$= a_{ij} \underbrace{(\underline{e}_i \cdot \underline{e}'_r)}_{P_{ri}} \underbrace{(\underline{e}_j \cdot \underline{e}'_s)}_{P_{sj}}$$

$$= a'_{rs}\, \underline{e}'_r \underline{e}'_s$$

$a'_{rs} = a_{ij} P_{rs} P_{ij}$.

$[PA]_{ij} [P^T]_{js}$.

$A' = PAP^T \qquad P^T A' P = A$.

$\Downarrow$

$[a'_{rs}]$.

$\underline{\underline{A}}(\underline{x}) = \underline{y}$

$\det \underline{\underline{A}} \equiv \det A$.

$$\det A = \det(P^T A' P) = \det P^T \det A' \det P$$

$$= \det \underbrace{(P^T P)}_{I} \det A'$$

$$= \det A'$$

** det is invariant

$(a\underline{\underline{A}} + b\underline{\underline{B}})(\underline{x}) \equiv a\underline{\underline{A}}(\underline{x}) + b\underline{\underline{B}}(\underline{x})$

composition $\underline{\underline{A}}^{-1}$ of mapping.

$$(\underline{\underline{A}} \circ \underline{\underline{B}})(\underline{x}) \equiv \underline{\underline{A}}(\underline{\underline{B}}(\underline{x}))$$

$$= \underline{\underline{A}} \cdot (\underline{\underline{B}} \cdot \underline{x}).$$

$\underline{\underline{B}} \cdot \underline{x} = b_{ij}\, \underline{e}_i \underline{e}_j \cdot \underline{x}$

$$\underline{\underline{A}} \cdot (\underline{\underline{B}} \cdot \underline{x}) = (a_{rs}\, \underline{e}_r \underline{e}_s) \cdot b_{ij}\, \underline{e}_i \underline{e}_j \cdot \underline{x}.$$

$$= a_{rs}\, \underline{e}_r\, b_{ij} \underbrace{(\underline{e}_s \cdot \underline{e}_i)}_{\delta_{si}} \underline{e}_j \cdot \underline{x}.$$

$$= a_{rs} b_{sj}\, \underline{e}_r \underline{e}_j \cdot \underline{x}.$$

$[\underline{\underline{C}}] = C \qquad C = AB$.

$$\underline{\underline{A}}^{-1} \circ \underline{\underline{A}} = \underline{\underline{I}}.$$

$$\underline{\underline{I}} = \delta_{ij}\, \underline{e}_i \underline{e}_j = \underline{e}_i \underline{e}_i$$

↓ identity tensor.

$\underline{\underline{A}}^T$: transpose of $\underline{\underline{A}}$.

$$\underline{v} \cdot \underline{\underline{A}}^T \cdot \underline{u} = \underline{u} \cdot \underline{\underline{A}} \cdot \underline{v}$$

for all $\underline{u}$ & $\underline{v}$ in $\mathbb{E}^3$

$\underline{u} = \underline{e}_j$

$\underline{v} = \underline{e}_i$

$$\underline{e}_j \cdot \underline{\underline{A}} \cdot \underline{e}_i$$

$$= \underline{e}_j \cdot a_{rs}\, \underline{e}_r \underbrace{\underline{e}_s \cdot \underline{e}_i}_{\delta_{si}}$$

$$= a_{rs}\, \delta_{jr} \delta_{si} = a_{ji}.$$

$$\underline{e}_i \cdot \underline{\underline{A}}^T \cdot \underline{e}_j = \underbrace{a_{ij}^T = a_{ji}}$$

⇓

true for 1 coor.

true for all ~

$$(\underline{\underline{A}} + \underline{\underline{B}})^T = \underline{\underline{A}}^T + \underline{\underline{B}}^T$$

$$(\underline{\underline{A}}^T)^T = \underline{\underline{A}} \qquad (\underline{\underline{A}}^{-1})^T = (\underline{\underline{A}}^T)^{-1}$$

$$(\underline{\underline{A}} \circ \underline{\underline{B}})^T = \underline{\underline{B}}^T \circ \underline{\underline{A}}^T$$

$\underline{\underline{A}}^T = \underline{\underline{A}}$ symmetric

$\underline{\underline{A}}^T = -\underline{\underline{A}}$ asymmetric → mechanics of solids.

*** Eigenvalue of asymmetric tensor

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) \rightarrow \text{independent of basis.}$$

$$= \det(A' - \lambda I)$$

$$= \det[\underbrace{PAP^T}_{A'} - \lambda \underline{I}]. \qquad PP^T = I$$

$$= \det[P(A - \lambda I)P^T]$$

$$= \underbrace{\det[PP^T]}_{I} \det(A - \lambda I)$$

$$\equiv \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}).$$

$$\det(A - \lambda I)$$

$$= P_3(\lambda) = (-\lambda)^3 + I_1 \lambda^2 - I_2 \lambda + \det A.$$

$$I_1 = \tfrac{1}{2}\left[(\mathrm{tr}(A))^2 - \mathrm{tr}(A^2)\right] \qquad I_2 = A_{ii} = \mathrm{tr}(A).$$

$\hookrightarrow a_{11} + a_{22} + a_{33}.$

I_1, I_2, $\det A$

are scalar movement of the Tensor $\underline{\underline{A}}$

$$\underline{\underline{A}} = \lambda_1 E_1 E_1 + \lambda_2 E_2 E_2 + \lambda_3 E_3 E_3$$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix} \qquad \{E_i\}$$

$\uparrow$ eigenvectors of $\underline{\underline{A}}$

$$\det A = \lambda_1 \lambda_2 \lambda_3$$

$$I_2 = \lambda_1 + \lambda_2 + \lambda_3 .$$

$$I_1 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3 .$$

☆ how to diagonize 3x3 matrix.

(Labor day - Monday).

Week 2: Wed.

(Review)

$$\underline{\underline{A}}(\underline{v}) - \underline{\underline{A}} \cdot \underline{v}$$

$$\underline{\underline{A}} = a_{ij} \underline{e_i} \underline{e_j}$$

$$\underline{\underline{A}}(\underline{v}) = a_{ij} v_j \underline{e_i}$$

$$= a_{ij} \underline{e_i} \underbrace{\underline{e_j} \cdot \underline{v}}_{v_j}$$

$$(\underline{\underline{A}} \circ \underline{\underline{B}})(\underline{v}) = \underline{\underline{A}}(\underline{\underline{B}}(\underline{v}))$$

$$= (a_{ij} \underline{e_i} \underline{e_j})(b_{kl} v_l \underline{e_k}).$$

$$= a_{ij} \underline{e_i} b_{kl} v_l \delta_{jk}$$

$$= a_{ij} b_{jl} v_l \underline{e_i}$$

in other words,

$$\underline{\underline{A}} \circ \underline{\underline{B}} = a_{ij} b_{jl} \underline{e_i} \underline{e_l}$$

$$\wedge$$

$$= \underline{\underline{A}} \otimes \underline{\underline{B}} =$$

$$\underline{\underline{A}} = a_{ij} \underline{e_i} \underline{e_j}, \quad \underline{\underline{B}} = b_{kl} \underline{e_k} \underline{e_l}$$

$$\underline{\underline{A}} \cdot \underline{\underline{B}} = a_{ij} \underline{e_i} \underline{e_j} \cdot b_{kl} \underline{e_k} \underline{e_l} \qquad \underline{\underline{A}}\underline{\underline{B}} = \underline{\underline{A}} \circ \underline{\underline{B}}$$

$\underline{\underline{A}}\,\underline{\underline{B}}$

$\underline{a}\,\underline{b} : \underline{c}\,\underline{d} \equiv (\underline{a}\cdot\underline{c})(\underline{b}\cdot\underline{d})$

$\underline{a}\,\underline{b} \cdot\cdot\, \underline{c}\,\underline{d} \equiv (\underline{a}\cdot\underline{d})(\underline{c}\cdot\underline{b})$

$\underline{\underline{A}} = a_{ij}\,\underline{e}_i\,\underline{e}_j$

$\underline{\underline{B}} = b_{kl}\,\underline{e}_k\,\underline{e}_l$

$\underline{\underline{A}} : \underline{\underline{B}} = (a_{ij}\,\underline{e}_i\,\underline{e}_j) : (b_{kl}\,\underline{e}_k\,\underline{e}_l)$

$\underline{\underline{A}} : \underline{\underline{B}} = a_{ij} b_{kl} \underbrace{(\underline{e}_i\cdot\underline{e}_k)}_{\delta_{ik}} \underbrace{(\underline{e}_j\cdot\underline{e}_l)}_{\delta_{jl}}$

$= a_{kj} b_{kj} \rightarrow$ scalar

↳ can be extended to 2 angular relations of linear transformations

e.g. For vectors $\vec{a}$, $\vec{b}$ $\quad \underline{a}\cdot\underline{b}$

$tr(\underline{\underline{A}})$

$= a_{ij}\,\underbrace{(\underline{e}_i\cdot\underline{e}_j)}_{\delta_{ij}} = a_{ii}$

$tr(\underline{\underline{A}} + \underline{\underline{B}}) = tr\,\underline{\underline{A}} + tr\,\underline{\underline{B}}$

$tr(a\underline{\underline{A}}) = a\,tr(\underline{\underline{A}})$

$tr(\underline{\underline{A}}^T) = tr(\underline{\underline{A}})$

$tr(\underline{\underline{A}}\cdot\underline{\underline{B}}) = tr(\underline{\underline{B}}\cdot\underline{\underline{A}})$

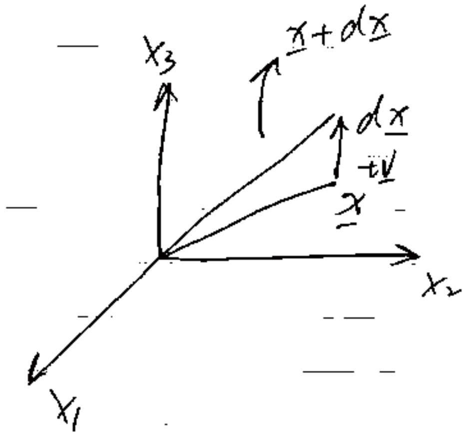
Tensor Field

scalar field $\quad f(\underline{x})$

$dx_i = d\underline{x}\cdot\underline{e}_i$

$df = \frac{\partial f}{\partial x_i} dx_i$

$= \frac{\partial f}{\partial x_i}(\underline{e}_i \cdot d\underline{x})$



$\nabla f \equiv \frac{\partial f}{\partial x_i}\underline{e}_i$

$= \nabla f \cdot d\underline{x}$

$\nabla \underline{u} = ?$

$\lim\left(\frac{\underline{u}(\underline{x}+t\underline{v}) - \underline{u}(\underline{x})}{t}\right) \Longleftarrow (\nabla\underline{u})\cdot\underline{v} \equiv \lim \frac{\underline{u}(\underline{x}+t\underline{v}) - \underline{u}(\underline{x})}{t}$

$= \frac{\left[u(\underline{x}) + \frac{\partial u}{\partial x_i} + v_i\right] - u_i(\underline{x})}{t}$

$$= \frac{\partial \underline{u}}{\partial x_k} v_k$$

$$= \frac{\partial (u_i \underline{e}_i)}{\partial x_k} v_k = \frac{\partial u_i}{\partial x_k} \underline{e}_i (\underline{e}_k \cdot \underline{v})$$

$$= \underbrace{\left(\frac{\partial u_i}{\partial x_k} \underline{e}_i \underline{e}_k \right)}_{\nabla \underline{u}} \cdot \underline{v}$$

$\nabla \underline{u} = \frac{\partial u_i}{\partial x_k} \underline{e}_i \underline{e}_k \longrightarrow$ bump up by 1 order (w/ gradients).

$$\nabla \underline{\underline{P}} = \frac{\partial \underline{\underline{P}}}{\partial x_k} \underline{e}_k$$

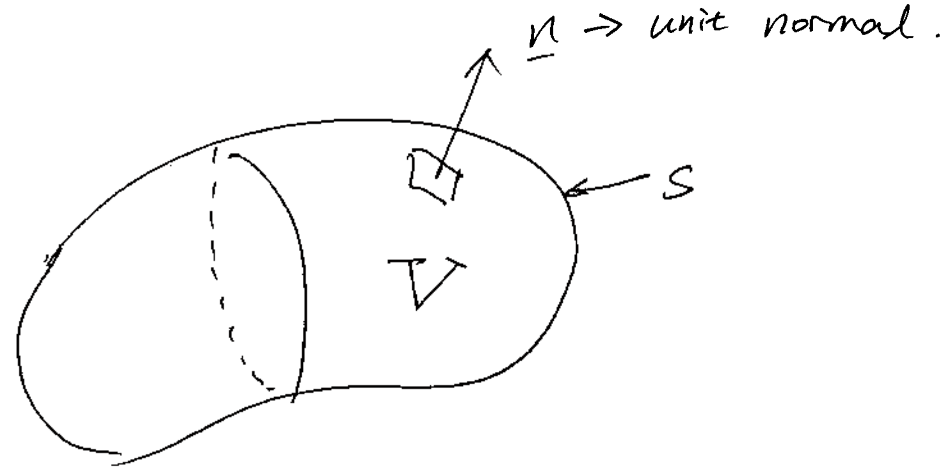
$$= \frac{\partial P_{ij}}{\partial x_k} \underline{e}_i \underline{e}_j \underline{e}_k$$

$$\mathrm{tr}(\nabla \underline{u}) = \frac{\partial u_i}{\partial x_k} (\underline{e}_i \cdot \underline{e}_k)$$

$$= \frac{\partial u_i}{\partial x_i}$$

$$= \nabla \cdot \underline{u}.$$

$$\nabla \cdot \underline{\underline{P}} = \frac{\partial P_{ij}}{\partial x_k} \underline{e}_i \underbrace{(\underline{e}_j \cdot \underline{e}_k)}_{\delta_{jk}} = \frac{\partial P_{ij}}{\partial x_j} \underline{e}_i$$



$$\iiint_V f_{,i} \, dV = \iint_S f \, n_i \, dS.$$

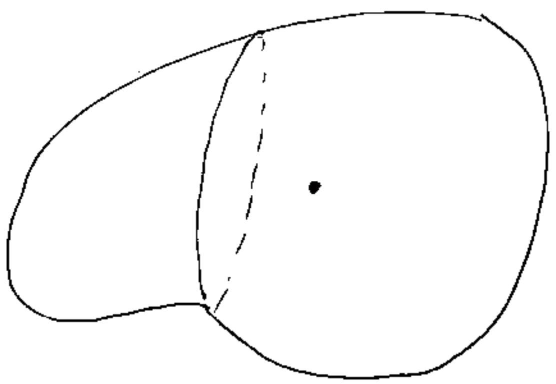
$\longrightarrow$ Green's theorem.

$$f_{,i} = \frac{\partial f}{\partial x_i}$$

$$\left. \begin{aligned} \int_V u_{j,i} \, dV &= \int_S u_j n_i \, dS \\ \int_V T_{kl,i} \, dV &= \int_S T_{kl} n_i \, dS. \end{aligned} \right\}.$$

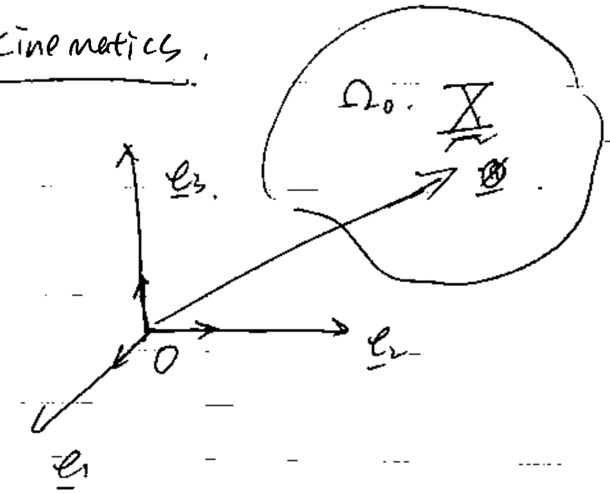
Mechanics.

- Kinematics ✓ (geometry)
- Balance laws ✓ (forces, moments, ...)
- Constitutive laws ← "Research" — connect. k.



Week 3: Mon.

Kinematics.



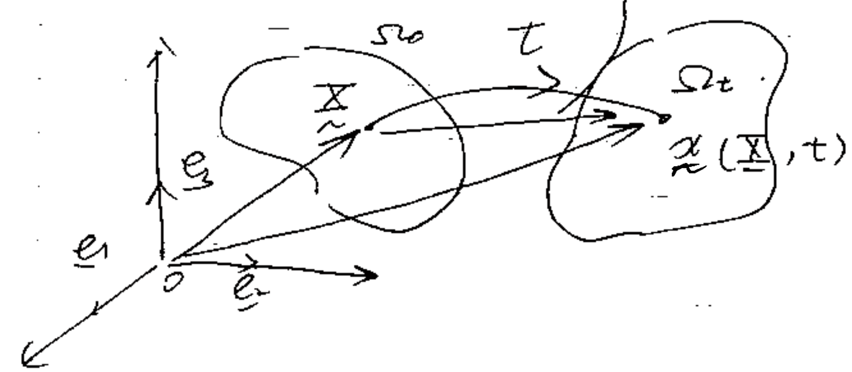
Material point is labeled by its coordinate X_i or $\underset{\sim}{X}$.

(Ref. config.)

Ω_0 is the configuration of the Body at $t=0$.

Normally we choose Ω_0 to be the undeformed State of the body.

$\rightarrow \underset{\sim}{x} - \underset{\sim}{X} = \underset{\sim}{u}(\underset{\sim}{X}, t)$



$t>0$. Body deforms at occupies Ω_t.

$\underset{\sim}{X} \rightarrow \underset{\sim}{x}$

mapping

function "kai"

this is a mapping: $\underset{\sim}{x} = \underset{\sim}{\chi}(\underset{\sim}{X}, t)$

$= \underset{\sim}{x}(\underset{\sim}{X}, t)$.

$\underline{u} = \underline{x} - \underline{X} \rightarrow$ displacement vector.

Always assume mapping χ is one-one points

$\downarrow$

given one point

$\downarrow$

another point associated w/ it

*** Some simple examples: (motion).

$\underline{x} = \underline{X} + \underline{c}(t)$. Rigid body translation.

X interesting cuz no defamation.

*** e.g. 2

let Ω_0 be a bar, (straight bar) — rectangular cross section.

We define $\underline{x} = \underline{X} + \sum_{k=1}^{3} (\lambda_k - 1) \underline{X}_k \underline{e}_k$

(remember)

$\underline{u} = \underline{x} - \underline{X}$

$\underline{u} = \sum_{k=1}^{3} (\lambda_k - 1) \underline{X}_k \underline{e}_k$

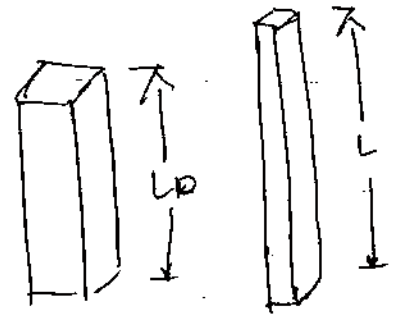
$\downarrow$

real positive numbers.

$\lambda_k = 1$: no displacements $\rightarrow$ body remain initial state.

$\lambda_k \neq 1$: strech & compress in $\underline{e}_1, \underline{e}_2, \underline{e}_3$ directions

$\hookrightarrow$ stretch ratios

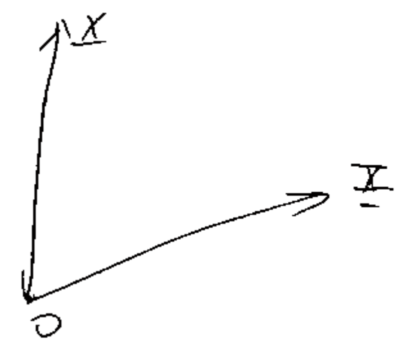


$\lambda_k = \frac{L}{L_0}$.

☆ You can always impose a displacement field on a body.

Rigid body rotation

$\underline{x} = \underline{R}(\underline{X})$.

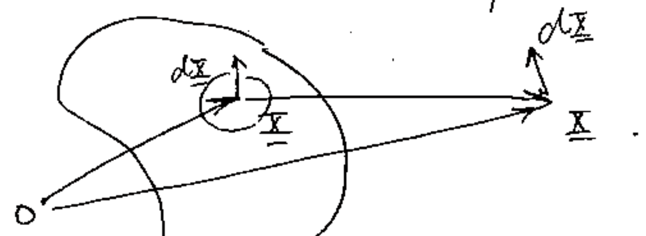


$\underline{e}_k \longrightarrow \underbrace{\underline{R}(\underline{e}_k)}_{\underline{n}_k}$

$\boxed{\underline{R} = \underline{n}_k \, \underline{e}_k}$ $\longleftrightarrow$ rotation.

linear trans. $\rightarrow$ completely det. by action on its basis

does not increase any defination



$d\underline{x} = \frac{\partial x}{\partial X_j} \, dX_j$ (def of grad.).

$= \underbrace{\nabla_{X} x}_{\underline{\underline{F}}} \cdot d\underline{X}$.

$\underline{\underline{F}} \equiv \nabla_{X}$ → deformation gradient tensor.

↳ contains all the information on local deformation.

$\underline{\underline{F}}(\underline{X}, t)$

$\underbrace{\nabla_{X} \underline{x}}_{\underline{\underline{F}}} = \frac{\partial (x_i \underline{e}_i)}{\partial X_j} \underline{e}_j$

$\underline{\underline{F}} = \frac{\partial x_i}{\partial X_j} \underline{e}_i \underline{e}_j$

$$[\underline{\underline{F}}] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \vdots & \vdots & \vdots \end{bmatrix}.$$

respect to the basis $\underline{e}_i, \underline{e}_j$

$d\underline{x} = \underline{\underline{F}} \cdot d\underline{X}$.

Change of length (fiber)

$d\underline{x} \cdot d\underline{x} - d\underline{X} \cdot d\underline{X}$

$= (\underline{\underline{F}} \cdot d\underline{X}) \cdot (\underline{\underline{F}} \cdot d\underline{X}) - d\underline{X} \cdot d\underline{X}$

$= d\underline{X} \cdot \underbrace{(\underline{\underline{F}}^T \cdot \underline{\underline{F}})}_{C} \cdot d\underline{X} - d\underline{X} \cdot d\underline{X}$

$\underline{\underline{C}}$ is the Cauchy-Green Tensor

$= d\underline{X} \cdot \underbrace{(\underline{\underline{C}} - \underline{\underline{I}})}_{\underline{\underline{E}}} \cdot d\underline{X}$.

$\underline{\underline{I}} \cdot d\underline{X} = d\underline{X}$.

$\frac{\|d\underline{x}\|^2}{\|d\underline{X}\|^2} = \left(\frac{\|d\underline{x}\|}{\|d\underline{X}\|} \right)^2$ Lagrangian Strain Tensor

$\frac{d\underline{x} \cdot d\underline{x}}{\|d\underline{X}\|^2} - 1$

$= \frac{d\underline{X}}{\|d\underline{X}\|} \cdot \underline{\underline{C}} \cdot \frac{d\underline{X}}{\|d\underline{X}\|} - 1$

$= \underline{N} \cdot \underline{\underline{C}} \cdot \underline{N} - 1$ stretch ratio

$\underline{N}$: unit vector

$\underline{X}, t$.

Solid: $\underline{X}$ → reference configuration.

Fluid: $\underline{x}$ ↳ spatial → current coordinates.

don't use summation with λ !!!

Ex 1: $\underline{u} = \sum_{k=1}^{3} (\lambda_k - 1) X_k \underline{e}_k$ (3k).

$$[\underline{F}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \underline{F} = \lambda_1 \underline{e}_1\underline{e}_1 + \lambda_2 \underline{e}_2\underline{e}_2 + \lambda_3 \underline{e}_3\underline{e}_3$$

$\underline{F} = \lambda_1 \underline{e}_1\underline{e}_1 + \lambda_2 \underline{e}_2\underline{e}_2 + \lambda_3 \underline{e}_3\underline{e}_3$.

$\lambda_1 = \lambda_2 = \lambda_3 > 1$.

uniform expansion.

$\lambda_1 = \lambda_2 = \lambda_3 < 1$

Uniform compression

$$[\underline{F}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$\det \underline{F} = J$

invariant.

$= \lambda_1 \lambda_2 \lambda_3$

Rubber is almost incompressible, so $J \simeq 1$.

In general, $\det \underline{F} = \frac{dV}{dV_0}$ the deform of the volume over the reference (original) volume.

always true.

$= \frac{V}{V_0}$ ← new

reference volume

in a tension bar: not uniform.

$\underline{E} = E_{ij} \underline{e}_i \underline{e}_j$

$$E_{ij} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)}_{\varepsilon_{ij}} + \frac{1}{2}\left[\frac{\partial u_k}{\partial X_i} \cdot \frac{\partial u_k}{\partial X_j}\right]$$

small strain tensor (1% ~ 2%).

effect of large defor. quadratic term.

→ (10^{-2}% ~ $4 \cdot 10^{-2}$%)

$10^{-2} \cdot 10^{-2} = 10^{-4}$.

Week 3.
Sep. 15th (Wed.)

Review. $\underset{\sim}{F}$ ($\underset{\sim}{F}$) Deformation Gradient Tensor

↑ completely characterize the local deformation at a point $\underline{X}$.

$\underset{\sim}{F}(\underline{X}, t)$, $\underline{X}$, t, independent variables.

↳ Material description.

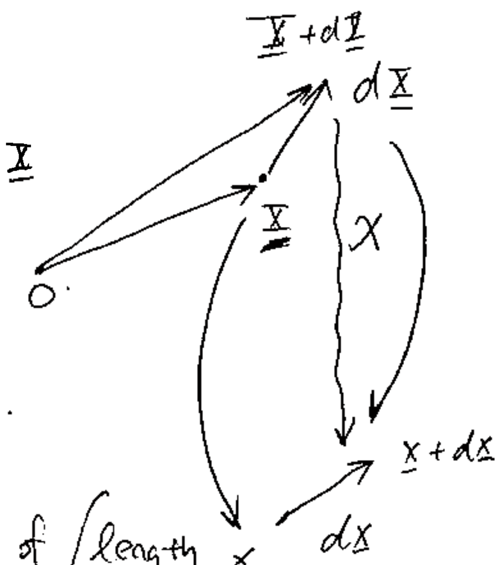
$\underset{\approx}{C} = \underset{\approx}{F}^T \underset{\approx}{F}$ Right Caughty - Green Tensor

$\underset{\approx}{E} = \underset{\approx}{F}^T \underset{\approx}{F} - \underset{\approx}{I}$.

$d\underline{x} \cdot d\underline{x} = d\underline{X} \cdot \underset{\approx}{C} \cdot d\underline{X}$

$\underline{N}$ is a unit vector.

$$\frac{\|d\underline{x}\|^2}{\|d\underline{X}\|^2} = \overbrace{\frac{d\underline{X}}{\|d\underline{X}\|}}^{\underline{N}} \cdot \underset{\approx}{C} \cdot \overbrace{\frac{d\underline{X}}{\|d\underline{X}\|}}^{\underline{N}}.$$



$\frac{\|d\underline{x}\|}{\|d\underline{X}\|} \equiv \lambda$. → Ratio of length. ↳ of material line element.

Stretch Ratio = $\dfrac{\text{length of mat. line ele. in the current configuration.}}{\text{length of mat. line ele. in the Ref. configuration.}}$

$\lambda_n^2 = \underline{N} \cdot \underset{\approx}{C} \cdot \underline{N}$.

$\underline{N} \cdot \underset{\approx}{E} \cdot \underline{N} = \lambda_n^2 - 1$. measure the deformation.

↓ Lagrangian strain tensor.

$$E_{ij} = \underbrace{\frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right]}_{\varepsilon_{ij}} + \frac{1}{2}\left[\frac{\partial u_k}{\partial X_i} \frac{\partial u_k}{\partial X_j}\right].$$

Remember:
$\underline{x} = \underline{X} + \underline{u}(\underline{X}, t)$.

Both $\underset{\approx}{C}$ & $\underset{\approx}{E}$ are symmetric Tensors.

$\underset{\approx}{C}^T = \underset{\approx}{C}$. Recall $\underset{\approx}{F}$ is revertible.

$\underset{\approx}{F}^T \underset{\approx}{F} = \underset{\approx}{C}$.
↓ revertible → $\underset{\approx}{C}$ is positive definite

$d\underline{X} \cdot \underset{\approx}{C} \cdot d\underline{X} \geq 0$. only when $d\underline{X} = 0$.

identical (exactly) $\|\underset{\approx}{F} \cdot d\underline{X}\|^2 > 0$. $d\underline{X} \neq \underline{0}$.

$\underset{\approx}{C}$ symmetric implies that $\underset{\approx}{C}$ has Real eigen values $\lambda_1^2, \lambda_2^2, \lambda_3^2$.

$\lambda_1^2 > \lambda_2^2 > \lambda_3^2 \leftarrow$

$\underline{\underline{C}}$ positive definite implies $\lambda_i^2 > 0$.
$i = 1, 2, 3$.

$\underline{\underline{C}}$ Can be diagonalized

that is, $\underline{\underline{C}}$ can be written as

$$\underline{\underline{C}} = \lambda_1^2 \underline{n}_1 \underline{n}_1 + \lambda_2^2 \underline{n}_2 \underline{n}_2 + \lambda_3^2 \underline{n}_3 \underline{n}_3$$

$\underline{n}$ are orthonormal eigen vectors of $\underline{\underline{C}}$

that is $\underline{n}_i \cdot \underline{n}_j = \delta_{ij}$

THIS $\lambda_1, \lambda_3, \lambda_2$ are called principal stretches.

~~N_i~~ $\underline{n}_i$'s are the principal direction.

Polar Decomposition Theorem

$$\boxed{\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}} = \underline{\underline{V}}\,\underline{\underline{R}}.$$

$\underline{\underline{R}}$ is a rigid body rotation tensor, $\underline{\underline{R}}^T = \underline{\underline{R}}^{-1}$

$$\underline{\underline{R}}^T \underline{\underline{R}} = \underline{\underline{R}}\,\underline{\underline{R}}^T = \underline{\underline{I}}.$$

$\underline{\underline{U}}$ is symmetric, positive definite.

and $\underline{\underline{U}}^2 = \underline{\underline{C}}$ & $\underline{\underline{U}} = \sqrt{\underline{\underline{C}}}$.

$$\underline{\underline{U}} = \lambda_1 \underline{N}_1 \underline{N}_1 + \lambda_2 \underline{N}_2 \underline{N}_2 + \lambda_3 \underline{N}_3 \underline{N}_3$$

check $\underline{\underline{U}} \cdot \underline{\underline{U}} = \underline{\underline{U}}^2 = \underline{\underline{C}}$.

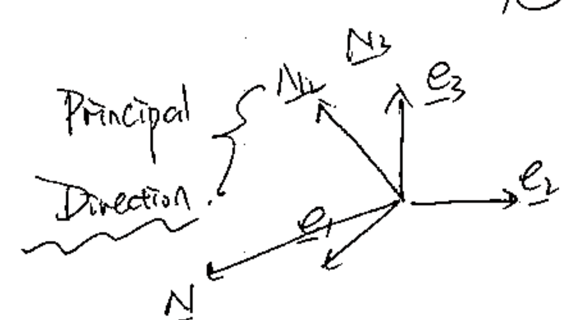
$\underline{\underline{F}}$ can be decompose into two simple tensor,

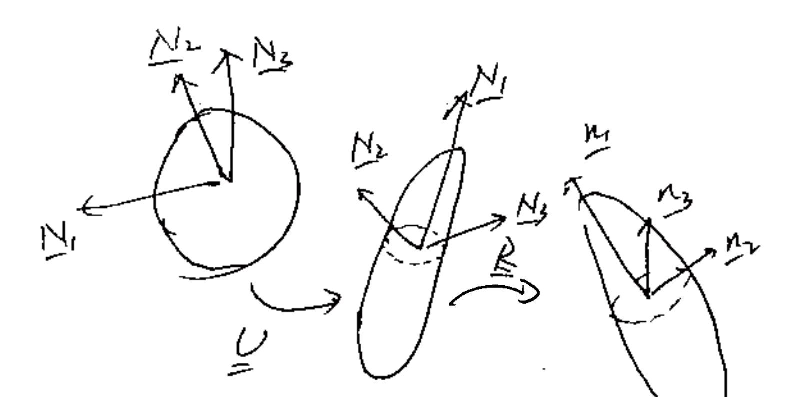
where first tensor, $\underline{\underline{U}} \rightarrow$ stretch tensor
↓
it stretch the material point in principal directions

the it rotate with $\underline{\underline{R}}$.

$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} \cdot \underline{X}$ → this a local theorem

$\underline{\underline{U}}$ comes first, and $\underline{\underline{R}}$ comes second.





$\underline{n}_i = \underline{\underline{R}}(\underline{N}_i)$

★ Only need to prove

$$\underline{\underline{R}} \equiv \underline{\underline{F}}\,\underline{\underline{U}}^{-1} \quad \text{is a rotation.}$$

$$\underline{\underline{R}}^T\underline{\underline{R}} = \underline{\underline{I}}.$$

$$\underline{\underline{R}}^T\underline{\underline{R}} = (\underline{\underline{F}}\,\underline{\underline{U}}^{-1})^T(\underline{\underline{F}}\,\underline{\underline{U}}^{-1}). \qquad \underline{\underline{C}} = \underline{\underline{U}}^2$$

$$= (\underline{\underline{U}}^{-T}\underline{\underline{F}}^T)(\underline{\underline{F}}\,\underline{\underline{U}}^{-1}) = \underline{\underline{U}}^{-1}\underbrace{\underline{\underline{F}}^T\underline{\underline{F}}}_{\underline{\underline{C}}=\underline{\underline{U}}^2}\underline{\underline{U}}^{-1}$$

symmetric ↓ U^{-1}

$$= \underbrace{\underline{\underline{U}}^{-1}\underline{\underline{U}}}_{\underline{\underline{I}}}\,\underbrace{\underline{\underline{U}}\,\underline{\underline{U}}^{-1}}_{\underline{\underline{I}}} = \underline{\underline{I}}$$

Therefore we prove: $\underline{\underline{R}}\,\underline{\underline{R}}^T = \underline{\underline{R}}^T\underline{\underline{R}} = \underline{\underline{I}}$.

$$\underline{n}_i = \underline{\underline{R}}(\underline{N}_i).$$

$$\underline{\underline{R}} = \underline{n}_i\underline{N}_i = \underline{n}_1\underline{N}_1 + \underline{n}_2\underline{N}_2 + \underline{n}_3\underline{N}_3.$$

*** Decomposition is unique.

if we define: $\underline{\underline{V}} = \lambda_1\underline{n}_1\underline{n}_1 + \lambda_2\underline{n}_2\underline{n}_2 + \lambda_3\underline{n}_3\underline{n}_3$.

$$\underline{\underline{V}}\,\underline{\underline{R}} = (\lambda_1\underline{n}_1\underline{n}_1 + \lambda_2\underline{n}_2\underline{n}_2 + \lambda_3\underline{n}_3\underline{n}_3)\cdot(\underline{n}_1\underline{N}_1 + \underline{n}_2\underline{N}_2 + \underline{n}_3\underline{N}_3).$$

$$= \lambda_1\underline{n}_1\underline{N}_1 + \lambda_2\underline{n}_2\underline{N}_2 + \lambda_3\underline{n}_3\underline{N}_3$$

⇓ then we can check:

$$\underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}.$$

[IN CASE] in mechanics theorem paper,

Someone writes:

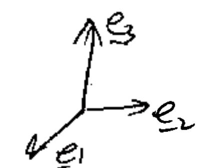
$$\underline{\underline{F}} = \cancel{F_{iA}\,e}$$

$$F_{iA}\,e_i\,E_A \to \text{two point tensor}$$

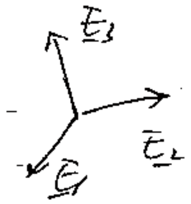
$$e_i\cdot e_j = \delta_{ij}$$

$$E_A\cdot E_B = \delta_{AB}$$

in the ref. config. e_1, e_2, e_3



current config. E_1, E_2, E_3



Simple Shear Deformation.

$$\begin{cases} x_1 = X_1 + X_2\tan\gamma \\ x_2 = X_2 \\ x_3 = X_3 \end{cases}$$

← fixed number $(0, \pi/2)$.

$\underline{x} = x_i e_i$

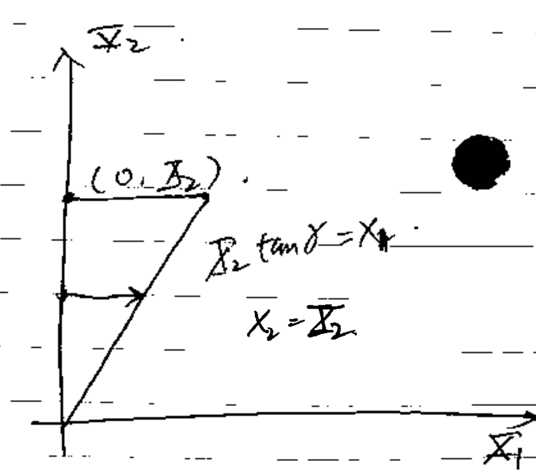
$$[\underline{\underline{F}}] = \begin{bmatrix} 1 & \tan\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

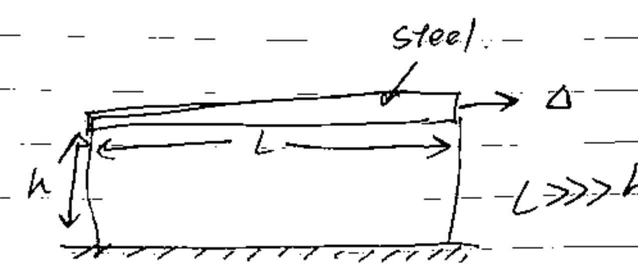
$\underline{\underline{F}} = \underline{e}_1 \underline{e}_1 + \tan\gamma\, \underline{e}_1 \underline{e}_2 + \underline{e}_2 \underline{e}_2 + \underline{e}_3 \underline{e}_3$.

$\det \underline{\underline{F}} = 1$.

$\hookrightarrow \det(\underline{\underline{R}}\,\underline{\underline{U}}) = \underbrace{\det \underline{\underline{R}}}_{=1} + \det \underline{\underline{U}} \rightarrow \lambda_1 \lambda_2 \lambda_3 \rightarrow \frac{V}{V_0}$

$\underline{\underline{F}} =$





Office Hour:

$\underline{\underline{F}} = (\delta_{ij} + u_{ij})\, \underline{e}_i \underline{e}_j$

$\underline{\underline{E}} = \frac{1}{2}\left[u_{i,j} + u_{j,i} + u_{k,i} u_{k,j}\right]$.

$_{,j} = \frac{\partial}{\partial X_j}$

Week 4 (5). Mon.

Linear Theory. (small deformation).

↳ perturbation theory.

geometry change small.

(gradients of displacements small, << 1).

$$\underset{\approx}{E} = \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \underbrace{\frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}}_{\text{higher order terms}}\right].$$

$\underline{u} = u_k e_k$.

$$\approx \underbrace{\frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right]}_{\text{leading order terms}}.$$

$\underline{\underline{C}} - \underline{\underline{I}} \equiv 2\underline{\underline{E}}$.

$\underline{\underline{F}} = \underline{\underline{I}} + \frac{\partial u_i}{\partial X_j} e_i e_j = \underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}}$. (small rotation tensor ω)

$$\frac{\partial u_i}{\partial X_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)}_{\text{sym}\ \to\ \varepsilon_{ij}} + \underbrace{\frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right]}_{\text{Anti-sym}\ \to\ \omega_{ij}}$$

$C_{ij} = (\delta_{ij} + \varepsilon_{ij} + \omega_{ij})(\delta_{ji} + \varepsilon_{ji} + \omega_{ji})$
$= \delta_{ij}\delta_{ji} + \delta_{ij}\varepsilon_{ji} + \delta_{ij}\omega_{ji} + \varepsilon_{ij}\delta_{ji} + \varepsilon_{ij}\varepsilon_{ji} + \varepsilon_{ij}\omega_{ji} + \omega_{ij}\delta_{ji} + \omega_{ij}\varepsilon_{ji} + \omega_{ij}\omega_{ji}$

$\underline{\underline{C}} = (\underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}})^T(\underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}})$

$= (\underline{\underline{I}} + \underline{\underline{\varepsilon}} - \underline{\underline{\omega}})(\underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}})$.

$= \underline{\underline{I}} + 2\underline{\underline{\varepsilon}} + H.O.T.$

$\underline{\underline{C}} = \underline{\underline{U}}^2$. (???)

$\underline{\underline{C}} \approx \underline{\underline{I}} + 2\underline{\underline{\varepsilon}}$.

$\underline{\underline{U}} \simeq \underline{\underline{I}} + \underline{\underline{\varepsilon}}$. $\quad \underline{\underline{U}}^2 = \underline{\underline{I}} + 2\underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}}\,\underline{\underline{\varepsilon}}$
$\approx \underline{\underline{I}} + 2\underline{\underline{\varepsilon}}$.

$\underline{\underline{U}}^{-1} \simeq \underline{\underline{I}} - \underline{\underline{\varepsilon}}$.

$\underline{\underline{F}} \cancel{A} = \underline{\underline{R}}\,\underline{\underline{U}}$

$\underline{\underline{R}} = \underline{\underline{F}}\,\underline{\underline{U}}^{-1}$. $\quad \underline{\underline{R}} \approx (\underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}})(\underline{\underline{I}} - \underline{\underline{\varepsilon}})$
$= \underline{\underline{I}} \cancel{-\underline{\underline{\varepsilon}}} + \underline{\underline{\omega}}$.

$\underline{V} = \frac{\partial \underline{X}}{\partial t}\Big|_{\underline{X}\ \text{fix}} = \frac{\partial u}{\partial t}\Big|_{\underline{X}}$.

↳ (at a fixed material point).

Sin

$\underline{X} = \underline{X} + u(\underline{X}, t)$.

$\underline{A} = \frac{\partial^2 \underline{X}}{\partial t^2}\Big|_{\underline{X}} = \frac{\partial^2 u}{\partial t^2}\Big|_{\underline{X}} \cancel{\frac{\partial u}{\partial t}\Big|_{\underline{X}}}$

$V(X,t)$.

mechanics: quantities in spatial descrip.

density

$\rho(X,t)$.

the material derivative:

$$f(x,t) = f(\chi(X,t),t).$$

$$\frac{D}{Dt}f \equiv \dot{f} \qquad \text{Fixed } X.$$

$$= \frac{\partial f}{\partial x_i}\underbrace{\left.\frac{\partial \chi_i}{\partial t}\right|_{X=\chi^{-1}(x,t)}}_{\text{velocity: } V_i(x,t)} + \left.\frac{\partial f}{\partial t}\right|_x$$

$$= \frac{\partial f}{\partial x_i}V_i(x,t) + \left.\frac{\partial f}{\partial t}\right|_x = \nabla_x f \cdot V + \left.\frac{\partial f}{\partial t}\right|_x$$

$$\nabla_x f = \frac{\partial f}{\partial x_j}e_j.$$

$$\nabla_X g = \frac{\partial g}{\partial X_j}e_j.$$

$$V = V(X,t). \equiv \left.\frac{\partial \chi}{\partial t}\right|_X.$$

$$v = V(\chi^{-1}(x,t),t) \quad X = \chi^{-1}(x,t).$$

In the spatial configuration.

$$a = A(\chi^{-1}(x,t),t).$$

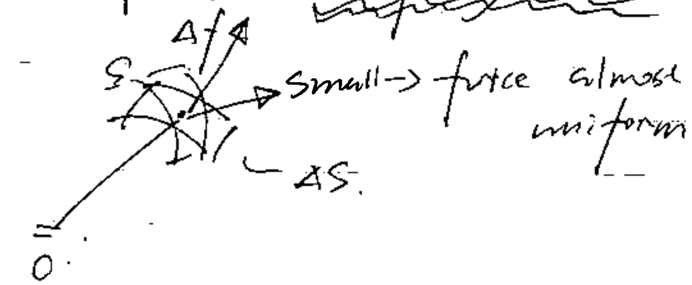
$$a = v\cdot\nabla_x V + \left.\frac{\partial V}{\partial t}\right|_x.$$

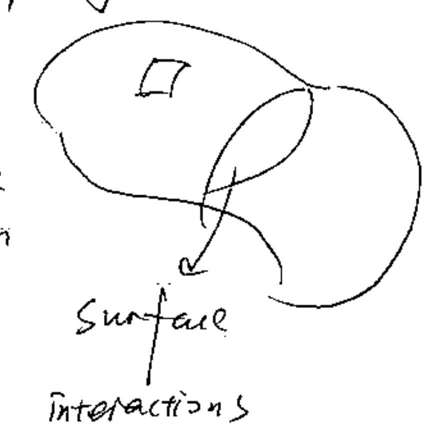
Concept of Stress

if given displacement field, & ref config.

↳ then we can calculate everything.

Assumption. Cauchy's hypothesis.





Important: orientation → X shape.

$$t = \frac{\Delta f}{\Delta S} \qquad \Delta S \to 0$$

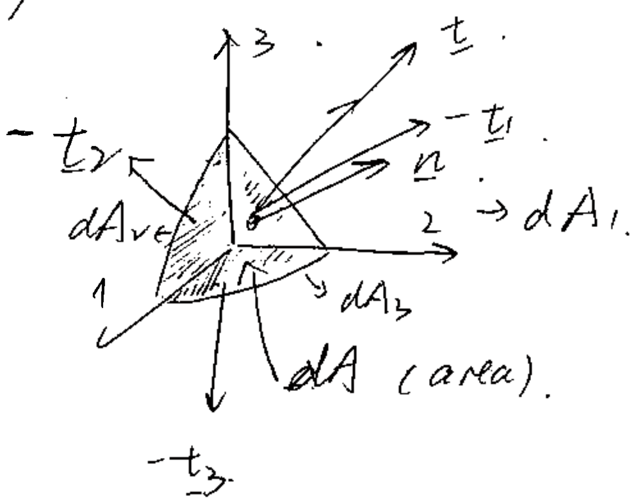
$$= t(n, \cancel{X}, t).$$

$\underline{n}$ → outward unit normal vector in the current configuration

$\underline{t}$ → traction vector (stress).

※ if we know traction in 3D, then we know the stress at this point.

Cauchy's theorem



Force on pyramid?

Body forces: → depends on volume of element.

ρ = mass per unit volume in current configuration.

Body forces per unit volume in

linear momentum Balance.

$$\underline{t}\,dA - \underline{t}_1 dA_1 - t_2 dA_2 - t_3 dA_3 + \rho\, b\, dV$$

$$= \cancel{m\frac{dV}{dt}}\ (\rho\, dV)\,\underline{a}.$$

$\cancel{dV}$ is the volume of small pyramid

dv

$$dA \gg dV, \quad \frac{dV}{dA} \to 0.$$

(in small pyramid).

$$\underline{t} = \underline{t}_1 \frac{dA_1}{dV} + \underline{t}_2 \frac{dA_2}{dV} + \underline{t}_3 \frac{dA_3}{dV}$$

$$\underline{t} = \cancel{t_1 n_1 + t_2 n_2 + t_3 n_3}$$

$$\underline{t}_1 \underline{e}_1 \cdot \underline{n} + \underline{t}_2 \underline{e}_2 \cdot \underline{n} + \underline{t}_3 \underline{e}_3 \cdot \underline{n}.$$

$$\underline{t}_1 = \sigma_1 \underline{e}_1$$

$$\underline{t}_k = \sigma_k \underline{e}_k, \quad k = 1, 2, 3.$$

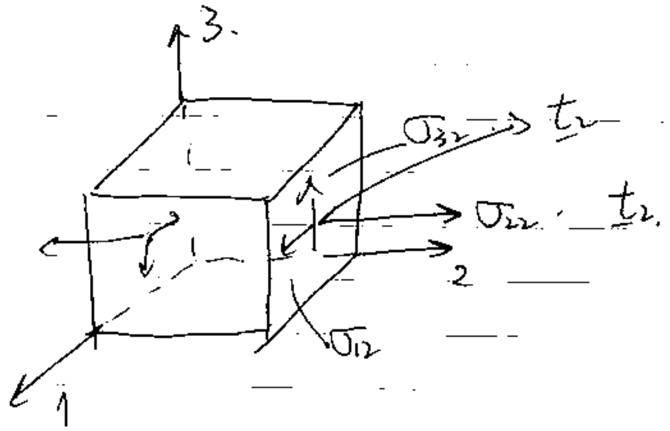
$$\underline{t}_j = \sigma_{ij}\, \underline{e}_i$$

$$= \sigma_{ij}\, \underline{e}_i\, \underline{e}_j \cdot \underline{n}.$$

(other text book: $\underline{t} = \underline{\sigma}^T \cdot \underline{n}$).

$\underline{\sigma}$ = Cauchy or True stress tensor

$$t_j = \sigma_{ji}\, n_i. \qquad \underline{t} = \underline{\sigma} \cdot \underline{n}.$$



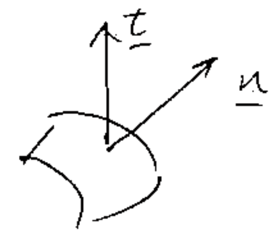
$$\underline{t}_2 = \sigma_{12}\,\underline{e}_1 + \sigma_{22}\,\underline{e}_2 + \sigma_{32}\,\underline{e}_3$$

Week 4, Wed.

$\underline{\underline{\sigma}}$ True stress tensor in current config.

Cauchy

$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{t} \rightarrow$ traction vector.



$$[\underline{t}] = \frac{\text{Force}}{\text{Unit area}}$$

* Equilibrium equation – Deformed configuration
(LMB)
↓
Linear momentum Balance.

Key Results.

$$\underbrace{\nabla_x \cdot \underline{\underline{\sigma}}}_{\text{Divergence}} + \rho \underline{b} = \rho \cdot \underline{a}$$

body force; acceleration; spatial.

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho a_i \quad \dots (\mathcal{N}1)$$

Current config.
!!
In real-world applications, cuz you ~~didn't~~ don't know current config.

in elastic prob., we use reference config.

In Ref config.

Eq. (N1) become :

$$\nabla_X \cdot \underset{=}{P} + \rho_0 \underline{B} = \rho_0 \underline{A}.$$

$\underline{A} = \underline{A}(X, t)$

$$\frac{\partial P_{ij}}{\partial X_j} + \rho_0 B_i = \rho_0 A_i.$$

First Piola Tensor $\Rightarrow$ Nominal Stress tensor

$\boxed{AMB}$ Angular momentum balance.

$\sigma_{ij} = \sigma_{ji}.$

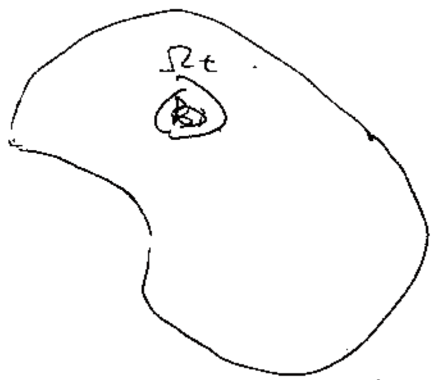
$\underset{=}{\sigma} = \underset{=}{\sigma}^T$

AMB.

$$\underset{=}{P}\,\underset{=}{F}^T = \underset{=}{F}\,\underset{=}{P}^T.$$

Derivation

METHOD I : Forces acting on Ω_t.



(Force balance) integral of $\int_{\Omega_t} \rho\,\underline{b}\,dV$

$$+ \int_{\partial\Omega_t} \underset{=}{\sigma} \cdot \underline{n}\,dS \quad \text{(traction)} \qquad (F.1)$$

LMB

(Newton's law) $\quad + \frac{D}{Dt}\int_{\Omega_t} \rho \underline{v}\,dV$ cannot take inside.

$\rho\,dV = \rho_0\,dV_0$ conservation of mass

$$\frac{dV}{dV_0} = \frac{\rho_0}{\rho} = \det \underset{=}{F}.$$

Jacobian.

Eq. (F.1) becomes.

$$\frac{D}{Dt}\int_{\Omega_t} \rho\,\underline{v}\,dV = \frac{D}{Dt}\int_{\Omega_0} \rho_0\,\underline{V}\,d V_0$$

fixed

So can take $\frac{D}{Dt}$ inside

$$= \int_{\Omega_0} \rho_0\,\underline{A}\,dV_0.$$

$$= \int_{\Omega_t} \rho\,\underline{a}\,dV$$

$$\underbrace{\int_{\partial\Omega_t} \underset{=}{\sigma} \cdot \underline{n}\,ds}_{\text{Divergence theorem}} + \int_{\Omega_t} (\rho\,\underline{b} - \rho\,\underline{a})\,dV.$$

$$\Downarrow$$

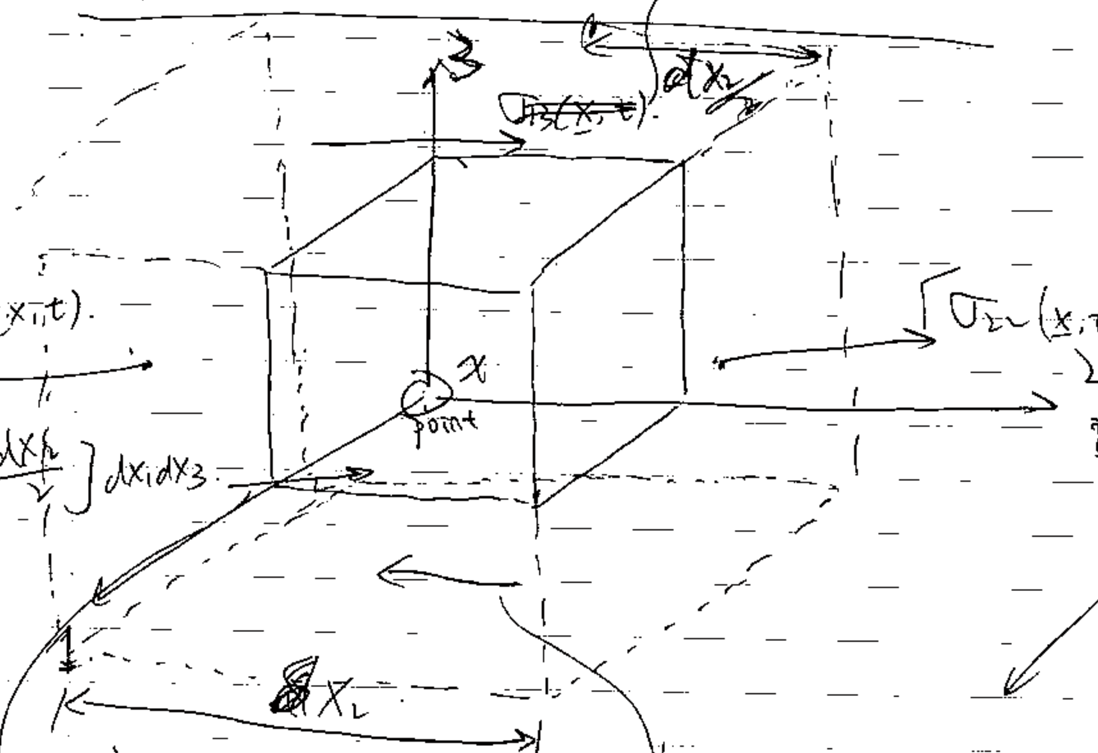
$$\int_{\Omega_t} \nabla_x \cdot \underset{=}{\sigma}\,dV.$$

$$\int_{\Omega_t} [\nabla_x \cdot \underset{=}{\sigma} + \rho\,\underline{b} - \rho\,\underline{a}]\,dV = 0$$

This is true for any $\Omega_t \Rightarrow$

$$\Rightarrow \boxed{\nabla_{\underline{x}} \cdot \underline{\underline{\sigma}} + \rho \underline{b} = \rho \underline{a}}$$

FINISHED !!!



$$\left[\sigma_{23}(\underline{x},t) + \frac{\partial \sigma_{23}}{\partial x_3}\frac{dx_3}{2}\right]$$

$\sigma_{23}(\underline{x},t)$, $\frac{dx_2}{2}$

$$\left[\sigma_{22}(\underline{x},t) - \frac{\partial \sigma_{22}}{\partial x_2}\frac{dx_2}{2}\right] dx_1 dx_3$$

$$\left[\sigma_{22}(\underline{x},t) + \frac{\partial \sigma_{22}}{\partial x_2}\frac{dx_2}{2}\right] dx_1 dx_3$$

x point

dx_2

$$\left[\sigma_{23}(\underline{x},t) - \frac{\partial \sigma_{23}}{\partial x_3}\frac{dx_3}{2}\right] dx_1 dx_2$$

LMB in 2 direction

σ_{21} ...

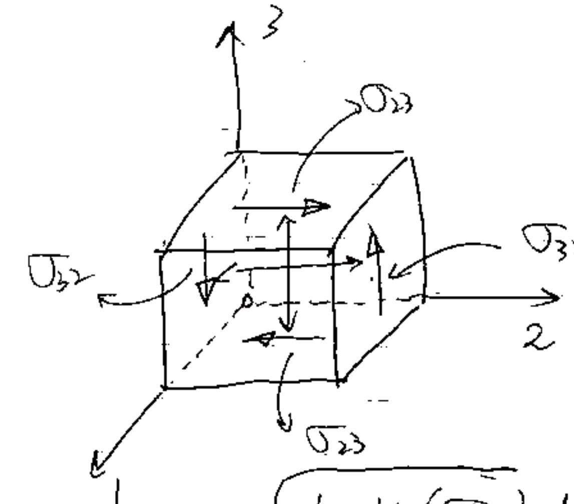
Net force in 1D,

$$\left[\frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \frac{\partial \sigma_{21}}{\partial x_1}\right] dx_1 \cancel{dx_2} dx_3$$

$$+ \rho \underline{b}\, dx_1 \cancel{dx_2} dx_3 = \rho \underline{a}\, dx_1 \cancel{dx_2} dx_3$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho a_i$$

Δ Δ $\Delta \rightarrow$ total force



$$\boxed{dx_1 dx_2 (\sigma_{23})} dx_3 = (\sigma_{32} \cdot dx_2)\, dx_1 dx_3$$

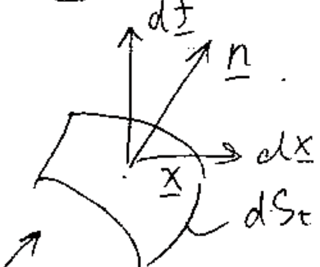
just times

force

$$\sigma_{23} = \sigma_{32}$$

$$\sigma_{ij} = \sigma_{ji}$$

$$\int_{\Omega_0} \rho_0 \underline{B}\, dV_0 + \int_{\partial R_t} \underline{\underline{\sigma}} \cdot \underline{n}\, dS = \int_{\Omega_0} \rho_0 \cdot \underline{A}\, dV_0$$

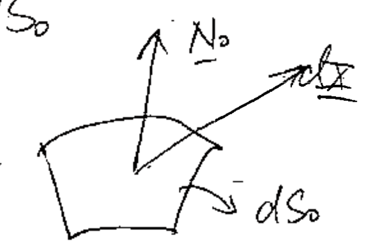


$\underline{n}\, dS_t$

$\underline{n}\, dS_t$

$$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n} = \frac{d\underline{f}}{dS_t} \Rightarrow \underline{t}\, dS_t = d\underline{f}$$

$\underline{N}_0 dS_0$



$$dV = \underline{n}\, dS_t \cdot d\underline{x} := \underline{n} \cdot dS_t \cdot \underline{\underline{F}} \cdot d\underline{X}.$$

$$dV_0 = \underline{N}\, dS_0 \cdot d\underline{X}.$$

$$\frac{dV}{dV_0} = \det \underline{\underline{F}} \equiv J.$$

$$\underline{n}\, dS_t \cdot \underline{\underline{F}} \cdot d\underline{X} = J \underline{N}\, dS_0 \cdot d\underline{X}.$$

$$d\underline{X} \cdot dS_t\, \underline{\underline{F}}^T \cdot \underline{n} = d\underline{X} \cdot J \underline{N}\, dS_0.$$

$d\underline{X}$ is solitary!

$$dS_t \cdot \underline{\underline{F}}^T \cdot \underline{n} = J \underline{N}\, dS_0.$$

$$\underline{\underline{F}}^T \cdot \underline{n}\, dS_0 = J \underline{N}\, dS_0$$

$$\boxed{\underline{n}\, dS_0 = J \underline{\underline{F}}^T \cdot \underline{N}\, dS_0}$$

Substitute $\int_{\partial\Omega_t} \underline{\underline{\sigma}} \cdot \underline{n}\, dS$

~~Naensen's~~ Formula

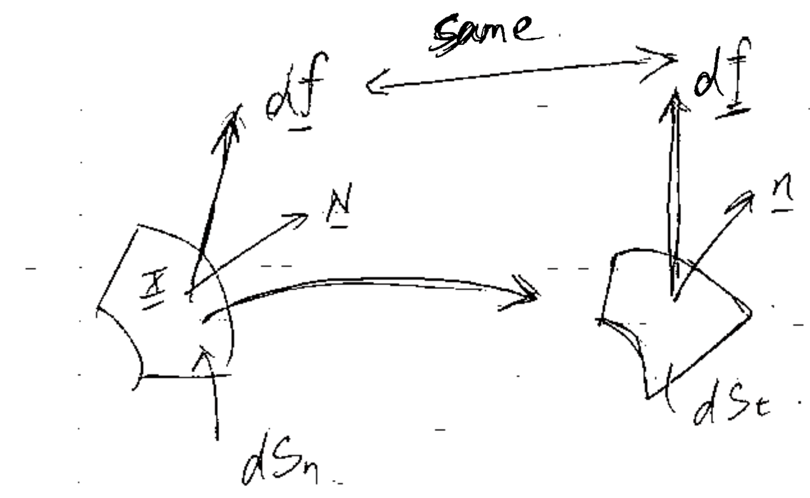
Nansen's.

$$\int_{\partial\Omega_0} \underbrace{J \underline{\underline{\sigma}} \cdot \underline{\underline{F}}^{-T}}_{\underline{\underline{P}}} \cdot \underline{N}\, dS_0.$$

By definition $\quad \underline{\underline{P}} = J \underline{\underline{\sigma}}\, \underline{\underline{F}}^{-T}.$

Divergence

$$\int_{\Omega_0} \nabla_X \cdot \underline{\underline{P}}\, dV_0 \rightarrow \nabla_X \cdot \underline{\underline{P}} + \rho_0 \underline{B} = \rho_0 \underline{A}.$$



$$\underline{t}_0\, dS_0 = d\underline{f} = \underline{t}\, dS_t.$$

~~$\underline{\underline{P}} \cdot \underline{n}$~~ $\quad \underline{\underline{P}} \cdot \underline{n} = \underline{t}$

↓

not a symmetric tensor

$$\underline{\underline{P}} = J \underline{\underline{\sigma}}\, \underline{\underline{F}}^{-T}$$

↓ symmetric $\quad$ ↳ not symmetric

AMB

$$\frac{(\underline{\underline{P}}\,\underline{\underline{F}}^T)^T}{J} = \frac{\underline{\underline{P}}\,\underline{\underline{F}}^T}{J} \Rightarrow \boxed{\underline{\underline{F}}\,\underline{\underline{P}}^T = \underline{\underline{P}}\,\underline{\underline{F}}^T}$$

In fluid mech, use current config as variables.

☆☆☆ Basic balance laws of continuum mechanics.

(Derived in current configuration)

Office hour Fri. 3:30pm

HW #2

$\underline{\underline{A}} = a_{ij}\, \underline{e}_i\, \underline{e}_j$

$\{e_i\}$ original basis.

$\{E_j\}$ New

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & & \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\underbrace{\qquad\qquad}_{\text{Symmetry}}$

whole num

λ_i : eigenvalues

$\underline{\underline{A}}$ in new basis: $\underline{\underline{A}} = \lambda_1 \underline{E}_1 \underline{E}_1 + \lambda_2 \underline{E}_2 \underline{E}_2 + \lambda_3 \underline{E}_3 \underline{E}_3$

$$\rightarrow \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

eigenvalues: $\lambda_1 = 8,\ \lambda_2 = 6,\ \lambda_3 = 3$

Original one: $\underline{\underline{A}} = 6\underline{e}_1\underline{e}_1 - 2\underline{e}_1\underline{e}_2 - 1\underline{e}_1\underline{e}_3 \ldots\ldots$

$\hookrightarrow$ So, $\underline{\underline{A}} = 8\underline{E}_1\underline{E}_1 + 6\underline{E}_2\underline{E}_2 + 3\underline{E}_3\underline{E}_3$

what is $\underline{E}_1$?

Original tensor $\underline{\underline{A}} = a_{ij}\underline{e}_i\underline{e}_j$

$= \lambda_1 \underline{E}_1\underline{E}_1 + \lambda_2 \underline{E}_2\underline{E}_2 + \lambda_3 \underline{E}_3\underline{E}_3$

$\underline{E}_1 \cdot \underline{E}_1 = 1,\ \underline{E}_2 \cdot \underline{E}_1 = 0,\ \underline{E}_3 \cdot \underline{E}_1 = 0$

need to normalize to 1

$(a_{ij}\underline{e}_i\underline{e}_j) \cdot \underline{E}_1 = \lambda_1 \underline{E}_1$

$a_{ij}\,\underline{e}_i \underbrace{(\underline{e}_j \cdot \underline{E}_1)}_{P_{1j}} = \lambda_1 \underline{E}_1$

$\underline{E}_1 \cdot \underline{e}_j = P_{1j}$

$\Downarrow$

$\underline{E}_1 = P_{1i}\,\underline{e}_i$

$a_{ij} P_{1j}\, \underline{e}_i = \lambda_1 \underline{E}_1 = \lambda_1 P_{1i}\, \underline{e}_i$

or $[a_{ij} P_{1j} - \lambda_1 P_{1i}]\, \underline{e}_j = \underline{0}$

$\Rightarrow a_{ij} P_{ij} - \lambda P_{1i} = 0 \longrightarrow i = 1, 2, 3$

$A = [a_{ij}] \quad A\vec{P}_1 - \lambda \vec{P}_1 = 0$

eigenvector of $\underline{A}$ for $\underline{E}$ $\longleftrightarrow$ $\vec{P}_1 = \begin{bmatrix} P_{11} \\ P_{12} \\ P_{13} \end{bmatrix}$

or $(A - \lambda I)\vec{P}_1 = \vec{0}$

$\lambda_1 \rightarrow$ eigenvalue of $\underline{A}$

$\vec{P}_1$ is a eigenvector for A $[\lambda_1]$.

$\underline{E}_1 = P_{1i} \underline{e}_i$

$\underline{E}_1 \cdot \underline{E}_1 = 1$. $\quad P_{1i} P_{1i} = 1$

Same idea applies to E_2

$\underline{E}_2 = P_{2i} \underline{e}_i \longleftarrow \lambda_2$

E_3 ....

$\underline{A} \cdot (\underline{e}_1 + \underline{e}_2) = \cdots$

☆ get the same no matter which basis ...

$\underline{A} = (\underline{e}_1 \underline{e}_2)$. Matrix.

linear transformation

$\underline{A} \cdot \underline{e}_1 =$ first column of $\underline{A}$.

$(\underline{e}_1 \underline{e}_2) \cdot \underline{e}_1 = \underline{0}$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\underline{A} \cdot \underline{e}_2 = (\underline{e}_1 \underline{e}_2) \cdot \underline{e}_2 = \underline{e}_1 (\underline{e}_2 \cdot \underline{e}_2) = \underline{e}_1$

$= 1\underline{e}_1 + 0\underline{e}_2 + 0\underline{e}_3$

$\underline{A} \cdot \underline{e}_3 = \underline{0}$ $\longrightarrow$ A respect to basis $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

$\underline{A} = a_{ij} \underline{e}_i \underline{e}_j$

$$[\underline{e}_1 \underline{e}_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[A] = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} +$$

$$= a_{11} \underline{e}_1 \underline{e}_1 + \cdots\cdots$$

$$\underline{V} = V_1 \underline{e}_1 + V_2 \underline{e}_2 + V_3 \underline{e}_3 .$$

$$\underline{E}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\underline{V} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$\to V_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + V_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \cdots$$

$$\underline{V} = \underline{e}_1$$

$$\underline{e}_i \underline{e}_j$$

$$\underline{\underline{A}} = a_{11} \underline{e}_1 \underline{e}_1 + a_{12} \underline{e}_1 \underline{e}_2 + a_{13} \underline{e}_1 \underline{e}_3 + \cdots\cdots$$

$$[\underline{\underline{A}}] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

break it down into simple linear trans.

$$\underline{V} = V_1 \underline{e}_1 + V_2 \underline{e}_2 + V_3 \underline{e}_3$$

☆ eigenvalues → invariants

eigenvectors → be care of the basis!!!

$\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ → $\underline{E}_i = P_{ij} \underline{e}_j$

$\{\underline{E}_1, \underline{E}_2, \underline{E}_3\}$.

with respect to this basis.

$$\underline{P}_1 = \begin{bmatrix} P_{11} \\ P_{12} \\ P_{13} \end{bmatrix}$$

associated with P_{ij}.

$\underline{\underline{W}}$ = skew symmetric.

$\underline{\underline{W}} \cdot \underline{v} = \underline{\omega} \times \underline{v}$

*** need to tell what $\underline{\omega}$ is

expand:

$\underline{\underline{W}} = W_{12}\, \underline{e}_1 \underline{e}_2 + W_{21}\, \underline{e}_2 \underline{e}_1 + \cdots$

$W_{11}, W_{22}, W_{33}, = 0.$

$\underline{\underline{W}} \cdot \underline{v} = \underline{P} \times \underline{v}.$

$W_{21} = -W_{12}$

$= W_{12}\, \underline{e}_1 \underline{e}_2 - W_{12}\, \underline{e}_2 \underline{e}_1 + \cdots$

$\underline{\underline{W}} \cdot \underline{v} \rightarrow \underline{v} = v_i \underline{e}_i$

$\underline{P} \times \underline{v} = \cdots \quad v_1 \cdots v_2 \cdots v_3$

$P_1 = W_{32}$

$P_2 = W_{13}.$

$P_3 = W_{21}.$

$\therefore \underline{P} = W_{32}\, \underline{e}_1 + W_{13}\, \underline{e}_2 + W_{21}\, \underline{e}_3.$

Sep 27. Week 5.

PERSONAL REVIEW - SO FAR:

- Cartesian Tensors.
  - Review on Notation
- Summation Convention (Indical notation)
- Permutation symbol
- Transformation Rule for vectors
- Second Order tensors
- Transpose of tensor
- Symmetric & Skew-symmetric tensor
- Tensor transformation (basis, ...)
- Operation of tensors: (products, ...)
- Symmetric tensors: Diagonalization.
- High order tensor
- Trace of second order tensor

~~High order tensor~~

Tensor fields.

Kinematics

Sep 27, Week 5. Mon.

—Review: Last lecture: Balance laws.

True stress tensor ↳ current cooridanate.

$$\rightarrow \quad \nabla_{\underline{x}} \cdot \underline{\underline{\sigma}} + \rho \underline{b} = \rho \underline{a} \qquad \text{invariante form}$$

(× config. influence).

$$\underbrace{\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho a_i.}_{3\text{ PDEs.}} \quad i=1,2,3.$$

↳ in the current cooridinate

independent spatial variable are x_i.

AMB $\sigma_{ij} = \sigma_{ji}$, $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$

Balance law in reference configuration, independent variables are X_i.

$\underline{\underline{P}}$ (Nominal or 1st Piola stress tensor).

$$\underset{(\underline{X})}{\underline{\underline{P}}} = J \, \underset{(\underline{x})}{\underline{\underline{\sigma}}} \, \underline{\underline{F}}^{-T} \longrightarrow J = \det \underline{\underline{F}}.$$

true stress.

$$\underline{\underline{\sigma}} = \frac{1}{J} \underline{\underline{P}} \, \underline{\underline{F}}^T \qquad \underline{\underline{\sigma}}(\underline{x} = \underline{\chi}(\underline{X},t), t).$$

$$\nabla_{\underline{X}} \underline{\underline{P}} + \rho_0 \underline{B}_0 = \rho_0 \underline{A}.$$

$$\frac{\partial P_{ij}}{\partial X_j} + \rho_0 B_{0i} = \rho_0 A_i.$$

$\underline{\underline{P}}$ is not always symmetric

So. an.. (AMB) $\quad \underline{\underline{P}}\,\underline{\underline{F}}^T = \underline{\underline{F}}\,\underline{\underline{P}}^T$

Proof.

$$\det \underline{\underline{F}} = J = \frac{dV}{dV_0}, \qquad \rho_0 \, dV_0 = \rho \, dV$$

$$\frac{\rho_0}{\rho} = \frac{dV}{dV_0} = J = \det \underline{\underline{F}}$$

Material is called immcompressible is $J=1$, $\forall \underline{X}$

Review Part

$$\underline{\underline{P}} \cdot \underline{N} \, dS_0 = \underline{\underline{\sigma}} \cdot \underline{n} \cdot dS_0$$

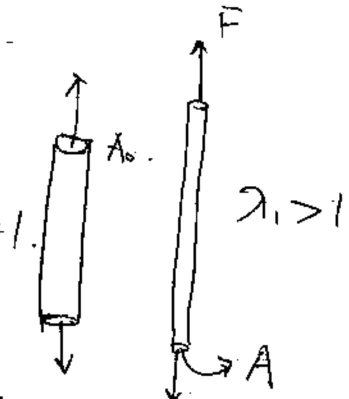
Simple example. $\underline{\underline{\sigma}} = \sigma_{11} \underline{e}_1 \underline{e}_1$

***

$$\sigma_{11} = \frac{F}{A}$$

?

$$P_0 = \frac{F}{A_0}$$



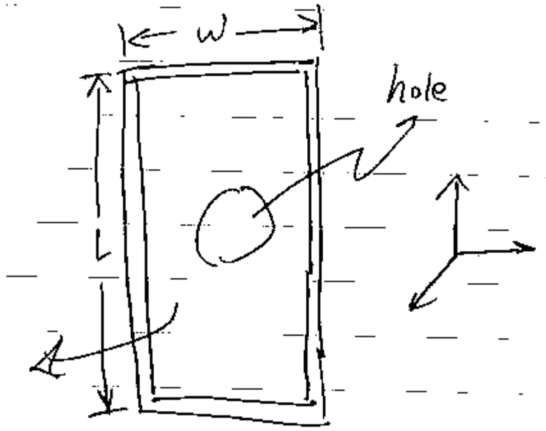
Incompressible solid. $J = \lambda_1 \lambda_2 \lambda_3 = 1$.

$$\lambda = \lambda_2 = \lambda_3. \qquad \lambda = \frac{1}{\sqrt{\lambda_1}}.$$

Plane stress.

$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0, \quad \forall X_3.$

Sym $\mapsto \sigma_{31} = \sigma_{32} = 0.$



Pressure

None-vanishing Stress state. hydrostatic pressure (atmosphere)

$L, w >> t.$

$\sigma_{11}, \sigma_{12} = \sigma_{21}, \sigma_{22}$ $\sigma_{\alpha,\beta}.\ \alpha, \beta = 1, 2.$

Assumption: $\sigma_{\alpha\beta}(X_1, X_2, t)$ independent of X_3

Equilibrium Eq: $\frac{\partial \sigma_{\alpha\beta}}{\partial X_\beta} + \rho b_\alpha = \rho \cdot 0$

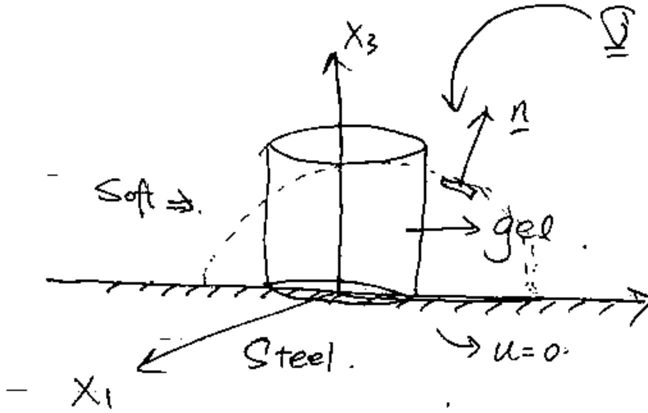
Quasi-Static problem $A = 0$

↳ Reduce to a 2D problem

Pure shear: $\underline{\underline{\sigma}} = \sigma_{12} \underline{e}_1 \underline{e}_2 + \sigma_{21} \underline{e}_2 \underline{e}_1$

$= \tau (\underline{e}_1 \underline{e}_2 + \underline{e}_2 \underline{e}_1).$

Eg.



$\underline{\underline{\sigma}} \cdot \underline{n} = 0$ (no external traction).

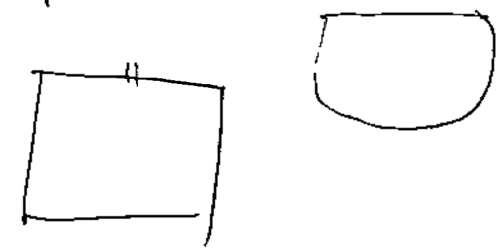
(no disp.)

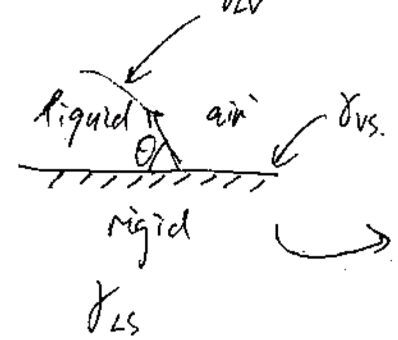
$\hat{u}_3 (X_1, X_2, X_3 = 0) = 0.$

(no shear)

$\tau_{31} = \tau_{32} = 0$, on $X_3 = 0$

→ Solve the shape of gel $|X_1^2 + X_2^2| = a.$





↳ Laplace-Young Equation.

Equilibrium.

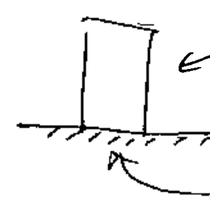
$\gamma_{LV} \cos\theta - \gamma_{LS} = \gamma_{VS}$

$\frac{\gamma}{G a}$ = elastopillary no.

G: Shear modulus. a: typical length scale

water leak out → Poroelasticity

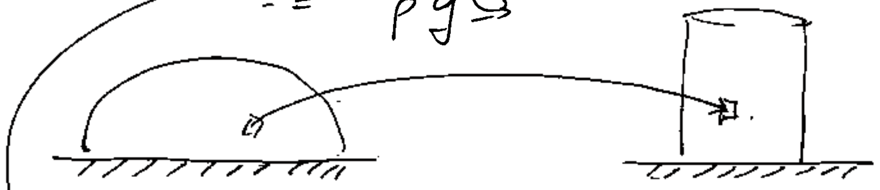
EASY WAY TO DO THIS:



$\leftarrow \underline{P} \cdot \underline{N} = 0$ $\quad \nabla_X \underline{P} = -\rho_0 \underline{B}_0$

glued. $\to u_3 = 0$ $\quad = -\rho_0 g \underline{e}_3$

$$\nabla_x \cdot \underline{\underline{\sigma}} = \rho g \underline{e}_3$$



3 equations → 6 unknown. ($\sigma_{\alpha\beta}$)

3 unknowns → $\hat{\underline{u}}$

9 unknowns.

Nonlinear Elasticity

Continuum Mechanics.

Sep 29, Wed, Week 5.

Constitutive law.

$$\underline{\underline{\sigma}} = \underline{\underline{\Psi}}\left(\underline{\underline{F}}(t'),\ -\infty < t' \le t\right).$$

$\downarrow$

$\underline{\underline{\sigma}}(t)$ $\Updownarrow$ Follow the whole deformation history.

$\updownarrow$

how to obtain the function → Ψ.

▽ Hyperelasticity (Green's elasticity).

Elasticity $\underline{\underline{\sigma}}$ depends only on $\underline{\underline{F}}(t)$.

$$\underline{\underline{\sigma}} = \underline{\underline{\Psi}}\left(\underline{\underline{F}}(t)\right).$$

↳ response function.

Mathematically, → the strain energy density.

$$\underline{\underline{P}} = \frac{\partial W(\underline{\underline{F}})}{\partial \underline{\underline{F}}}$$ ← energy per unit volume

$\Uparrow$

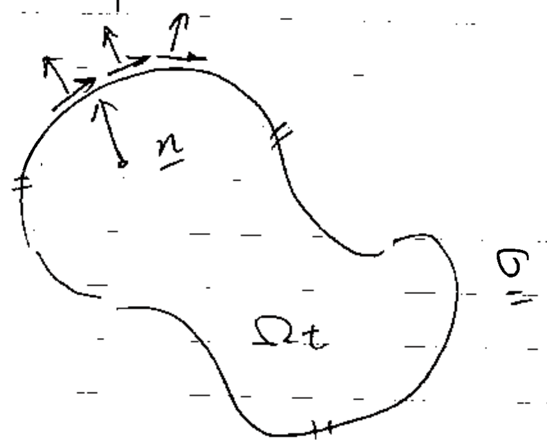
* the definition of hyperelastic materials.

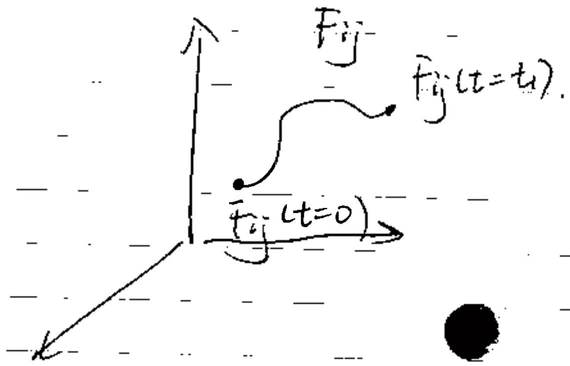
$$P_{ij} = \frac{\partial W}{\partial F_{ij}}$$

$$dW = \frac{\partial W}{\partial F_{ij}} dF_{ij} = \frac{\partial W}{\partial \underline{\underline{F}}} : d\underline{\underline{F}}$$

$$= P_{ij}\, dF_{ij} = \underline{\underline{P}} : d\underline{\underline{F}}$$ — work

$$d\phi = -\underline{E} \cdot d\underline{x}$$ Potential





external work rate

$$= \int_{\partial \Omega_t} (\underline{\sigma} \cdot \underline{n}) \cdot \underline{v}\, ds + \int_{\Omega_t} \rho\, \underline{b} \cdot \underline{v}\, dV_t$$

in the reference configuration

(Piola stress) $$\int_{\partial \Omega_0} (\underline{\underline{P}} \cdot \underline{N}) \cdot \underline{V}\, dS_0 + \int_{\Omega_0} \rho_0\, b_0 \cdot V\, dV_0$$

$$\int_{\partial \Omega_0} (\underline{\underline{P}} \cdot N)\, dS_0 = \int_{\partial \Omega_0} P_{ij} N_j V_i\, dS_0$$

div. Theo.

$$= \int_{\partial \Omega_0} (P_{ij} V_i)_{,j}\, dV_0 \qquad {}_{,j} = \frac{\partial}{\partial X_j}$$

$$= \int_{\partial \Omega_0} \underbrace{P_{ij,j}}_{\text{LMB}} V_i\, dV_0 + \int_{\partial \Omega_0} P_{ij} V_{i,j}\, dV_0$$

$$P_{ij,j} = -\rho_0 B + \rho_0 A_i$$

$$EWR = \int_{\Omega_0} \rho_0\, \underline{A} \cdot \underline{V}\, dV_0 + \int_{\Omega_0} P_{ij} V_{i,j}\, dV_0$$

$$\int_{\Omega_0} P_{ij} \dot{F}_{ij}\, dV_0$$

$$F_{ij} = \delta_{ij} + u_{i,j}$$

$$V_{i,j} = \frac{\partial}{\partial X_j}\left[\frac{\partial u_i}{\partial t}\bigg|_{X_i}\right]$$

$$= \frac{\partial F_{ij}}{\partial t}\bigg|_{\underline{X}}$$

$$\frac{DF_{ij}}{Dt} = \dot{F}_{ij}$$

move the body

kinetic energy (KE)

$$= \frac{1}{2}\frac{D}{Dt}\left(\int_{\Omega_0} \rho_0 \, (\underline{V}\cdot\underline{V})\, dV_0\right)$$

in the reference configuration

$$+ \underbrace{\int_{\Omega_0} \underline{\underline{P}} : \dot{\underline{\underline{F}}} \, dV_0}_{\text{work rate}}$$

completely general

→ deform the body

there could be dissipation in the process.

☆ hyperelastic

$$\frac{D}{Dt}\int_{\Omega_0} W(\underline{\underline{F}})\, dV_0$$

$$\frac{\partial W}{\partial \underline{\underline{F}}} = \underline{\underline{P}}$$

assume $P_{ij} = \dfrac{\partial W}{\partial F_{ij}}$.

$$dW = \frac{\partial W}{\partial F_{ij}} dF_{ij}.$$

$$\cancel{(\dot{dw}) = \frac{\partial w}{\partial F_{ij}} \dot{dF_{ij}}}$$

$$\frac{DW}{Dt} = \underbrace{\frac{\partial W}{\partial F_{ij}}}_{P_{ij}} \frac{DF_{ij}}{Dt} \rightarrow \dot{F}_{ij}$$

$t_1 \to t_2$; have to integrate the rate to time.

Total work from $t_1 \Rightarrow t_2$

$$\int_{t_1}^{t_2} EWR \, dt = \int_{t_1}^{t_2} \frac{D}{Dt} \int_{\Omega_0} \rho \, (\underline{V})^2 \, dV_0 \, dt$$

$$+ \int_{t_1}^{t_2} \frac{D}{Dt}\left(\int_{\Omega_0} W(\underline{\underline{F}}) \cdot dV_0\right) dt$$

t_1: $\underline{\underline{F}}_1$ $\underline{V}_1$

t_2: $\underline{\underline{F}}_2$ $\underline{V}_2$

$\underline{\underline{F}}$ $\underline{V}$

$$\int_{\Omega_0} W(\underline{\underline{F}}_2)\, dV_0$$

$$- \int_{\Omega_0} W(\underline{\underline{F}}_1)\, dV_0 = 0$$

Motivation for Hyperelasticity

↳ No energy loss during loading

Objectivity

• $\underset{\approx}{W}(\underline{F}) = W(\underline{Q}\,\underline{F})$

↓

rigid body rotation.

$\underline{Q}$: rotation

↑ (any)

$\underline{F} = \underline{Q}\,\underline{R}\,\underline{U}$

only depends on stretch tensor

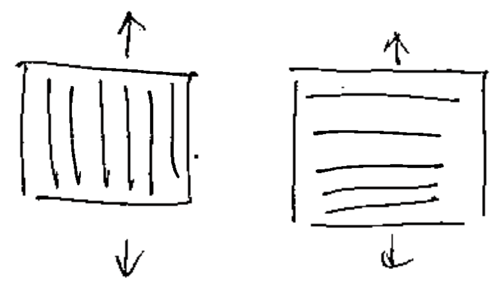
$$W(\underline{F}) = W(\underline{R}^T\underline{R}\,\underline{U}) = \bar{W}(\underline{U})$$

☆ only the stretching parts make a diff

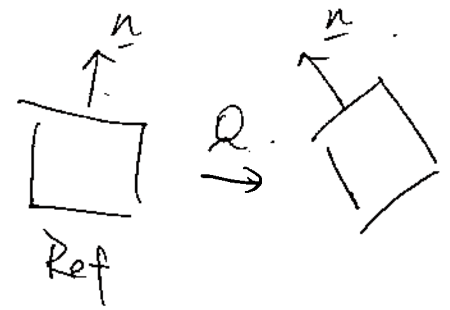
$$= \hat{W}(\underline{C}). \quad (\because \underline{U}^2 = \underline{C}).$$

$$\underline{P} \Rightarrow P_{ij} = \frac{\partial W}{\partial F_{ij}} = \frac{\partial \hat{W}}{\partial C_{kl}} \frac{\partial C_{kl}}{\partial F_{ij}}$$

$$= 2\underline{F}\,\frac{\partial \hat{W}}{\partial C_{ij}} \quad \text{(HW)}.$$



Not isotropic.



$$W(\underline{F}) = W(\underline{F}\,\underline{Q})$$

~~only for isotropic~~

Isotropic : true for all $\underline{Q}$

depends on the symmetry of the materials.

$\underline{x} = \underline{X} + \underline{u}(\underline{X}, t)$

$\underline{X} = \underline{\chi}^{-1}(\underline{x}, t)$

$\underline{u}(\underline{\chi}^{-1}(\underline{x}, t)) = \hat{\underline{u}}(\underline{x}, t)$

$\underline{x} - \underline{X} = \underline{u}$ $\nearrow$ $\frac{\partial u_i}{\partial X_j} = \frac{\partial u_i}{\partial x_k} \cdot \frac{\partial x_k}{\partial X_j}$

$\frac{\partial u_i}{\partial X_j} = \frac{\partial \hat{u}_i}{\partial x_k} \cdot \frac{\partial x_k}{\partial X_j}$.

$\underline{x} - \underline{X} = \underline{u}$.

$\underline{X} - d\underline{x} = \underline{u}$

Oct. 5, Mon, Week 6.

$\det(\underline{\underline{I}} + d\underline{\underline{B}}) = I + \frac{1}{\mathbb{R}}(d\underline{\underline{B}}) + O(d\underline{\underline{B}})^2$

$d(\det \underline{\underline{A}}) = \det A \cdot (1 + \cancel{\det}\ \mathrm{tr}(\underline{\underline{A}}^{-1} d\underline{\underline{A}})) - \det \underline{\underline{A}}$

$= (\det A)\, \mathrm{tr}(\underline{\underline{A}}^{-1} d\underline{\underline{A}})$.

$= (\det A)(\underline{\underline{A}}^{-T} : d\underline{\underline{A}})$

$= \frac{\partial \det \underline{\underline{A}}}{\partial \underline{\underline{A}}} : d\underline{\underline{A}}$.

$\hookrightarrow$ hold true for all $d\underline{\underline{A}}$.

$\Rightarrow \frac{\partial \det \underline{\underline{A}}}{\partial \underline{\underline{A}}} = (\det \underline{\underline{A}})\underline{\underline{A}}^{-T}$.

$\frac{\partial (\det \underline{\underline{A}})}{\partial A_{ij}} = (\det \underline{\underline{A}}) A^{-1}_{jk}$

$I_1 \underline{\underline{I}} = \underline{\underline{C}}$

$\hookrightarrow \underline{\underline{P}} = \frac{\partial W}{\partial \underline{\underline{F}}} = 2\underline{\underline{F}} \frac{\partial \hat{W}}{\partial \underline{\underline{C}}} = 2\underline{\underline{F}}\left[\frac{\partial \Phi}{\partial I_1}\underbrace{\frac{\partial I_1}{\partial \underline{\underline{C}}}}_{\underline{\underline{I}}} + \frac{\partial \Phi}{\partial I_2}\frac{\partial I_2}{\partial \underline{\underline{C}}} + \frac{\partial \Phi}{\partial I_3}\frac{\partial I_3}{\partial \underline{\underline{C}}}\right]$

1st Piola stress.

$$\underline{\underline{P}} = 2\left[\left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2}\right)\underline{\underline{F}} + \frac{\partial \Phi}{\partial I_2}\underline{\underline{F}}\,\underline{\underline{C}} + I_3 \frac{\partial \Phi}{\partial I_3}\underline{\underline{F}}^{-T}\right.$$

$$\underline{\underline{F}}\,\underline{\underline{C}}^{-1} = \underline{\underline{F}}\,(\underline{\underline{F}}^T\underline{\underline{F}})^{-1}$$
$$= \underline{\underline{F}}\,(\underline{\underline{F}}^{-1}\underline{\underline{F}}^{-T})$$
$$= \underline{\underline{F}}^{-T}$$

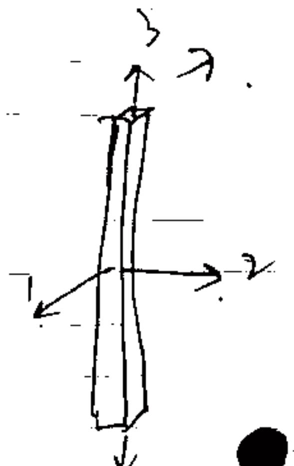
Recall

$$\underline{\underline{\sigma}} = \frac{1}{J}\underline{\underline{P}}\,\underline{\underline{F}}^T.$$

implies $\Longrightarrow$ $J = \det \underline{\underline{F}}$

$$\hookrightarrow \frac{1}{J}\left[\left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2}\right)\underline{\underline{F}}\,\underline{\underline{F}}^T - \frac{\partial \Phi}{\partial I_2}(\underline{\underline{F}}\,\underline{\underline{F}}^T)^2 + I_3 \frac{\partial \Phi}{\partial I_3}\underline{\underline{I}}\right] = \underline{\underline{\sigma}}.$$

Tension Test



$$\underline{\underline{F}} \rightarrow \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}.$$

$$\underline{\underline{F}} = \lambda\, e_1 e_1 + \lambda_2\, \underline{e}_2\underline{e}_2 + \lambda_3\, e_3 e_3$$

$I_1 \neq$

$I_2 \neq$

$$\underline{\underline{F}}^T\underline{\underline{F}} \Rightarrow \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_1^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$$

$$\mathrm{tr}(\underline{\underline{C}}) = \lambda_1^2 + \lambda_1^2 + \lambda_3^2.$$

$\lambda_1 = \lambda_2 = \lambda.$

$\det \underline{\underline{C}} = \lambda^4 \lambda_3^2.$

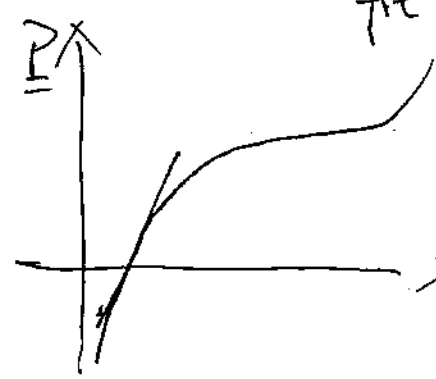
$\underline{\underline{P}} = \underline{\underline{\sigma}}/\lambda.$

$P_{33} = \dfrac{\sigma_{33}}{\lambda}.$ $\quad \sigma_{ij} = 0,\ j, i \neq 3.$

$\lambda \rightarrow I_2, I_1, I_3$

loading $\rightarrow$ λ $\rightarrow$ curve.

fit: $\surd$ constitutive modeling



$= 1 - \varepsilon \rightarrow \dfrac{\delta L}{L_0}$

strain ener. dens. function

det $\underline{F}$

$$\Phi = C_1\left(I_1 - 3 - 2\log(J)\right) + C_2(\ln J)^2$$

C_1, C_2 are material constants.

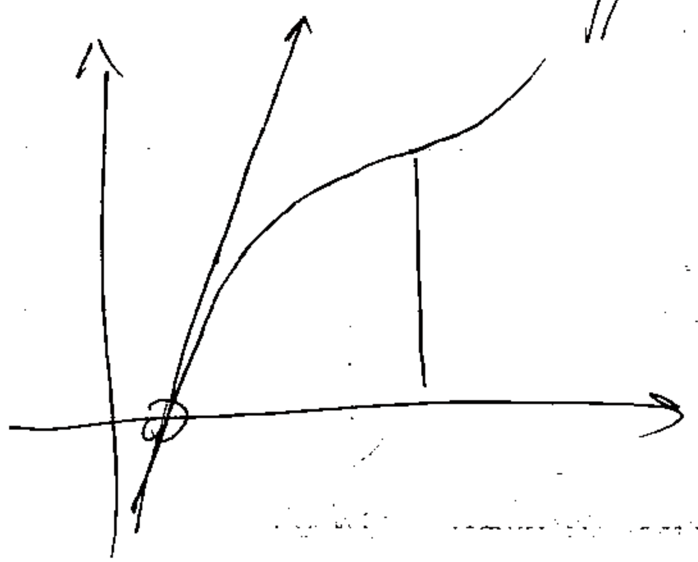
common model

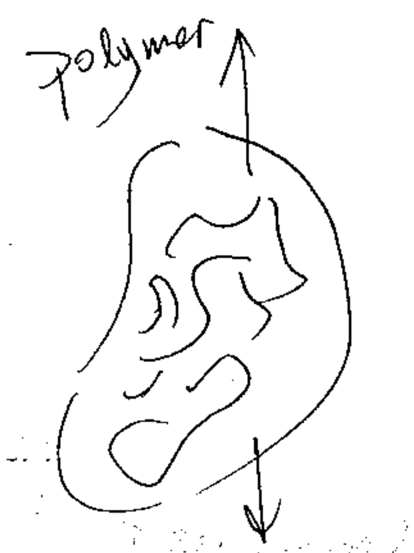
locally linear model.

$$\underline{\sigma} = \lambda\, \underline{\varepsilon}.$$

simple test & simple shear

determine different parameters.





Temperature is ~~major~~ importance

↳ entropy.

Incompressible.

$$\frac{dV}{dV_0} = \overbrace{\det \underline{F}}^{J} = \underline{\underline{I}}.$$

Isotropic deformation.

Define new energy function.

$$W_{new} = W(\underline{F}) - P(J-1)$$

Lagrangian multiplier

(impose constraints)

$$\underline{P} = \frac{\partial W_{new}}{\partial \underline{F}} = \frac{\partial W}{\partial \underline{F}} - p\cdot\frac{\partial \det(\underline{F})}{\partial \underline{F}}$$

$$= \frac{\partial W}{\partial \underline{F}} - p\,\underbrace{(\det \underline{F})}_{1}\,\underline{F}^{-T}.$$

$J=1$

$$\underline{\sigma} = \underline{P}\,\underline{F}^{T}.$$

$$\boxed{\underline{\sigma} = \frac{\partial W}{\partial \underline{F}}\,\underline{F}^{T} - p\,\underline{\underline{I}}} \longrightarrow \text{model.}$$

$$I_3 = \det \underline{C} = \det(\underline{F}^T \underline{F}) = \det(\underline{F}^T)\det \underline{F} = 1$$

Isotropic incompressible solid

$$\underbrace{W}_{} = \Phi(I_1, I_2).$$

$$\underline{\underline{\sigma}} = \left[\left(\frac{\partial \Phi}{\partial I_1} + I_2 \frac{\partial \Phi}{\partial I_2}\right)\underline{\underline{b}} - \frac{\partial \Phi}{\partial I_2} \underline{\underline{b}}^2\right] - p\underline{\underline{I}}$$

Corrected Note: (Based on Wiley books).

Review.

Hyperelasticity $\rightarrow$ $\underline{\underline{P}} = \dfrac{\partial W(\underline{\underline{F}})}{\partial \underline{\underline{F}}}$

$\rightarrow$ gradient of strain energy density with respect to $\underline{\underline{F}}$.

$\rightarrow$ end exactly where u start.

Objectivity $\sim$ $W(\underline{\underline{F}}) = \hat{W}(\underline{\underline{C}}) = \bar{W}(\underline{\underline{E}})$.

$\underline{\underline{C}} = \underline{\underline{F}}^T\underline{\underline{F}}$.

Isotropic Material $\sim$ $W(\underline{\underline{F}}) = W(\underline{\underline{F}}\,\underline{\underline{Q}})$.

$\forall$ orthogonal tensor $\underline{\underline{Q}}$

Define $\bar{\underline{\underline{F}}} = \underline{\underline{F}}\,\underline{\underline{Q}}$ $\sim$ if isotropic.

$W(\bar{\underline{\underline{F}}}) = W(\underline{\underline{F}}) \quad \forall Q.$

Objectivity $\sim$ $W(\underline{\underline{F}}) = \hat{W}(\underline{\underline{C}}) = W(\underline{\underline{F}}^T\underline{\underline{F}})$.

$= \hat{W}(\underline{\underline{F}}\,\underline{\underline{Q}})^T(\underline{\underline{F}}\,\underline{\underline{Q}}) = \hat{W}(\underline{\underline{Q}}^T\underline{\underline{C}}\,\underline{\underline{Q}})$.

$\hat{W}$ is a scalar invariant of Tensor $\underline{\underline{C}}$.

$$\underbrace{\det[\underline{\underline{C}} - \lambda\underline{\underline{I}}]}_{\text{independent of } Q} = (-\lambda)^3 + I_1\lambda^2 - I_2\lambda + I_3.$$

$$\rightarrow \begin{cases} I_1 = \operatorname{tr}\underline{\underline{C}} \\ I_2 = \frac{1}{2}\left[(\operatorname{tr}\underline{\underline{C}})^2 - \operatorname{tr}\underline{\underline{C}}^2\right]. \\ I_3 = \det\underline{\underline{C}}. \end{cases}$$

$\rightarrow$ For isotropic material.

$\hat{W} = \Phi(I_1, I_2, I_3)$.

isotropic.

$$\rightarrow \underline{\underline{P}} = \frac{\partial W}{\partial \underline{\underline{F}}} = 2\underline{\underline{F}}\frac{\partial \hat{W}}{\partial \underline{\underline{C}}}.$$

$$= 2\underline{\underline{F}}\left[\frac{\partial \Phi}{\partial I_1}\frac{\partial I_1}{\partial \underline{\underline{C}}} + \frac{\partial \Phi}{\partial I_2}\frac{\partial I_2}{\partial \underline{\underline{C}}} + \frac{\partial \Phi}{\partial I_3}\frac{\partial I_3}{\partial \underline{\underline{C}}}\right]$$

$\frac{\partial I_1}{\partial \underline{\underline{C}}} = \underline{\underline{I}}$, $\quad \frac{\partial I_2}{\partial \underline{\underline{C}}} = I_1\underline{\underline{I}} - \underline{\underline{C}}$, $\quad \frac{\partial I_3}{\partial \underline{\underline{C}}} = I_3\underline{\underline{C}}^{-T} = I_3\underline{\underline{C}}^{-1}$

$I_3 = \det\underline{\underline{C}}$.

<u>most general</u>: how to find $\dfrac{\partial \det\underline{\underline{A}}}{\partial \underline{\underline{A}}}$.

$$d(\det\underline{\underline{A}}) = \frac{\partial \det\underline{\underline{A}}}{\partial \underline{\underline{A}}} : d\underline{\underline{A}}$$

$$= \frac{\partial(\det\underline{\underline{A}})}{\partial A_{ij}}\, dA_{ij}.$$

$$d(\det\underline{\underline{A}}) = \det(\underline{\underline{A}} + d\underline{\underline{A}}) - \det(\underline{\underline{A}})$$

$$= \det(\underline{\underline{A}}(\underline{\underline{I}} + \underbrace{\underline{\underline{A}}^{-1}d\underline{\underline{A}}}_{d\underline{\underline{B}}})) - \det\underline{\underline{A}}$$

$$\underline{I} + d\underline{B} \to \begin{bmatrix} 1+dB_{11} & dB_{12} & \cdots \\ \cdots & 1+dB_{22} & \cdots \\ \cdots & \cdots & 1+dB_{33}\end{bmatrix}$$

$$\det(\underline{I} + d\underline{B}) = 1 + tr(d\underline{B}) + o(d\underline{B})^2.$$

$$d(\det(d\underline{B})) = \det \underline{A}\,(1 + tr(\underline{A}^{-1} d\underline{A})) - \det \underline{A}$$

$$= (\det \underline{A})\, tr(\underline{A}^{-1} d\underline{A}).$$

$$= (\det \underline{A}).(\underline{A}^{-T} : d\underline{A}) = \frac{\partial \det \underline{A}}{\partial \underline{A}} : d\underline{A}$$

$$\Rightarrow \frac{\partial \det \underline{A}}{\partial \underline{A}} = (\det \underline{A})\, \underline{A}^{-T}.$$

$$\cdots + \frac{\partial \Phi}{\partial I_3} \frac{\partial I_3}{\partial \underline{C}} \xrightarrow{2} I_3 \underline{C}^{-T} = I_3 \underline{C}^{-1}$$

$$I_3 = \det \underline{C}.$$

$$\underline{P} = 2\left[\left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2}\right)\underline{F} - \frac{\partial \Phi}{\partial I_2} \underline{F}\underline{C} + I_3 \frac{\partial \Phi}{\partial I_3} \underline{F}^{-T}\right]$$

Recall: $\underline{\sigma} = \frac{1}{J} \cdot \underline{P}\, \underline{F}^T \to$ True Stress.

Allison Carter

808 278-5794.

Detective Todd.

— Mike Hughes → detective

Tension Test

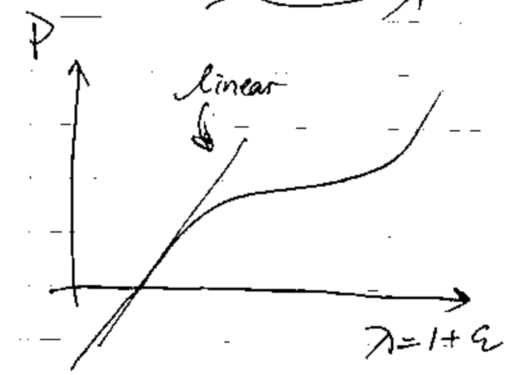
$$\underline{F} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & \lambda_3 \end{bmatrix}.$$

$$\underline{F} = \lambda_1 \underline{e}_1 \underline{e}_1 + \lambda_1 \underline{e}_2 \underline{e}_2 + \lambda_3 \underline{e}_3 \underline{e}_3.$$

$$I_1 = tr\, \underline{C} = tr(\underline{F}^T \underline{F}) = 2\lambda_1^2 + \lambda_3^2$$

$$I_3 = \det \underline{C} = \lambda_1^4 \lambda_3^2$$

$$I_2 = \cancel{P_{33} = \frac{\sigma_{33}}{\lambda}} \qquad P_{33} = \frac{\sigma_{33}}{\lambda}.$$



common model.

$$\to \Phi = C_1 (I_1 - 3 - 2\ln J) + C_2 (\ln J)^2$$

$J = \det \underline{F}$. ~ C_1, C_2 material constants.

Compressible isotropic materials.

Incompressible Solid.

$$\frac{dV}{dV_0} = \det \underline{F} = 1. \quad \text{"Isotropic Deformation"}$$

$$W_N = W(\underline{F}) - p(J-1).$$

Lagrangian multiplier

$$\underline{P} = \frac{\partial W_n}{\partial \underline{F}} = \frac{\partial W}{\partial \underline{F}} - p(J-1) = \frac{\partial W}{\partial \underline{F}} - p(\det \underline{F})(\underline{F}^{-T})$$

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$$J=1 \sim \underline{\underline{\sigma}} = \underline{\underline{P}}\,\underline{\underline{F}}^{T} = \frac{\partial W}{\partial \underline{\underline{F}}}\underline{\underline{F}}^{T} - p\,\underline{\underline{I}}$$

$$d(\det(\underline{\underline{A}})) = \det \underline{\underline{A}}(1 + tr(\underline{\underline{A}}^{-1} d\underline{\underline{A}})) - \det \underline{\underline{A}}$$

$$= (\det \underline{\underline{A}})\, tr(\underline{\underline{A}}^{-1} d\underline{\underline{A}})$$

$$= (\det \underline{\underline{A}})(\underline{\underline{A}}^{-T} : d\underline{\underline{A}})$$

$$= \frac{\partial \det \underline{\underline{A}}}{\partial \underline{\underline{A}}} : d\underline{\underline{A}}$$

$$\rightarrow \frac{\partial \det \underline{\underline{A}}}{\partial \underline{\underline{A}}} = (\det \underline{\underline{A}})\,\underline{\underline{A}}^{-T}$$

$$\cdots + \frac{\partial \Phi}{\partial I_3}\frac{\partial I_3}{\partial \underline{\underline{C}}} \rightarrow I_3\, \underline{\underline{C}}^{-T} = I_3\, \underline{\underline{C}}^{-1}$$

$$I_3 = \det \underline{\underline{C}}$$

$$\underline{\underline{P}} = 2\left[\left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2}\right)\underline{\underline{F}} - \frac{\partial \Phi}{\partial I_2}\underline{\underline{F}}\,\underline{\underline{C}} + I_3 \frac{\partial \Phi}{\partial I_3}\underline{\underline{F}}^{-T}\right]$$

$$\underline{\underline{\sigma}} = \frac{1}{J}\underline{\underline{P}}\,\underline{\underline{F}}^{T} \rightarrow \text{true stress.}$$

Tension Test

For isotropic.

$$\underline{\underline{\sigma}} = 2\left[\left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2}\right)b - \frac{\partial \Phi}{\partial I_2} b^2\right] - p\underline{\underline{I}}$$

7. Wed, Week

Incompressible hyperelasticity.

- Kinematics — quantities deformation.

  $\underline{\underline{C}}$, $\underline{\underline{E}}$, $\underline{\underline{U}}$ strain measures.

- Balance laws. — stresses.

  $\underline{\underline{P}}$, $\underline{\underline{\sigma}}$, ... other stresses measures.

  e.g. 2nd Piola stress.

  Biot stress.

- Constitutive Model.

  Relationship stress – strain.

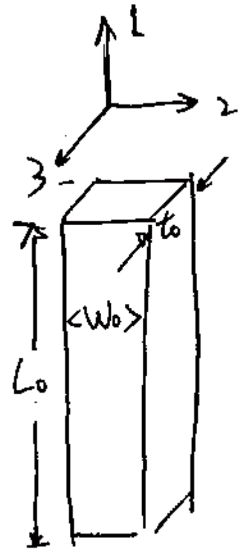
$$\underline{\underline{P}} = -p\,\underline{\underline{F}}^{-T} + 2\left[\left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2}\right)\underline{\underline{F}} - \frac{\partial \Phi}{\partial I_2}\underline{\underline{F}}\,\underline{\underline{C}}\right] \quad (1)$$

Recall I_1, I_2 are invariants of $\underline{\underline{C}}$.

$$I_1 = tr\,\underline{\underline{C}}, \quad I_2 = \frac{1}{2}\left[(tr\,\underline{\underline{C}})^2 - tr(\underline{\underline{C}})^2\right].$$

Lagrange multiplier enforce $\det \underline{\underline{F}} = J = 1$.

E.g. Uniaxial Tension or Compression test.



Ref. config.

$L_0 \gg W_0$ and t_0, Tension test.

$\underline{P}\cdot\underline{N}$ on all lateral surface is 0.

$\underline{N} = \underline{e}_2$ or $\underline{e}_3$

Undeformed lateral surfaces. ← $\boxed{P_{13} = P_{23} = P_{33} = 0,\ P_{12} = P_{22} = P_{32} = 0}$

$\nabla_X \cdot \underline{P} = \underline{0}$. ↔ Balance law.

Simplest model: $\Phi = \frac{\mu}{2}(I_1 - 3)$

Ideal rubber → Neo-Hookean solid.

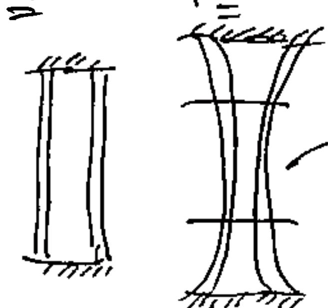
$= \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$

Principal coordinate.

Principal stretches.

Eqn (1): →

$\underline{P} = -p\underline{F}^{-T} + \mu\underline{F}$. ← material model. (constitutive).



$u_1 = (\lambda_1 - 1)X_1$.

$u_2 = u_3 = (\lambda_2 - 1)X_2$

↪ $(\lambda_2 - 1)X_3$

$\lambda_2 = \lambda_3$

$$\underline{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$\det \underline{F} = 1$. $\lambda_1\lambda_2^2 = 1$. → $\lambda_2 = \frac{1}{\sqrt{\lambda_1}}$

incompressibility.

Subs. into constitutive model.

make sure you satisfy boundary conditions.

* Not satisfy balance law → nonequilibrium states.

$$[\underline{F}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda_1}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\lambda_1}} \end{bmatrix}$$

$$[\underline{F}^{-T}] = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_1} & 0 \\ 0 & 0 & \sqrt{\lambda_1} \end{bmatrix}$$

$P_{11} = -p/\lambda_1 + \mu\lambda_1$

$P_{22} = -p\sqrt{\lambda_1} + \mu/\sqrt{\lambda_1} = P_{33}$.

$P_{12} = P_{21} = P_{23} = P_{32} = P_{13} = P_{31} = 0$.

$\lambda_1 = \text{const.}$ ⇒ equilibrium equation automatically satisfied.

B.C. are automatically satisfied.

*. Now, determine P.

$\underline{\underline{BC}} = P_{22} = P_{33} = 0 \Rightarrow P\sqrt{\lambda_1} = \mu / \sqrt{\lambda_1}$

$\Rightarrow P = \dfrac{\mu}{\lambda_1}$

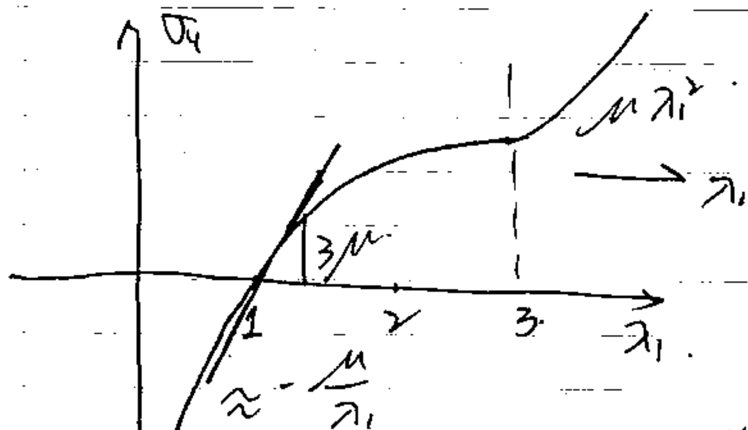
Substitute back to P_{11}.

$P_{11} = -\dfrac{\mu}{\lambda_1^2} + \mu\lambda_1$.

$\underline{\underline{\sigma}} = \underline{\underline{P}}\,\underline{\underline{F}}^T$

$\sigma_{11} = P_{11}\lambda_1$

$\sigma_{11} = -\dfrac{\mu}{\lambda_1} + \mu\lambda_1^2$.



$\lambda_1 \approx 1+\varepsilon$. $\quad \sigma_{11} = -\dfrac{\mu}{1+\varepsilon} + \mu(1+\varepsilon)^2$ feasible modules

$\varepsilon << 1$ $\quad \cong -\mu(1-\varepsilon) + \mu(1+2\varepsilon) = 3\mu\varepsilon$

$\mu = E/3$.

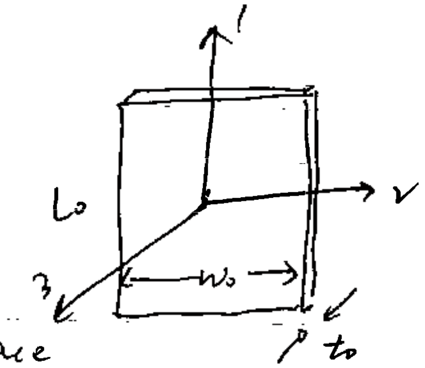
$\dfrac{E}{2(1+\nu)} = \mu$

↳ $\nu = 1/2$.

Poisson ratio.

plane stress deformation

(Surface : traction free).



B.C.: $\underline{\underline{P}} \cdot \underline{e}_3 = 0$, on surface

$t_0 << L_0$

$t_0 << w_0$.

$u_1(X_1, X_2)$. $u_2(X_1, X_2)$.

$u_3(X_1, X_2)$. (Assumption).

$P_{13} = P_{23} = P_{33} \equiv 0$, in the region (every where).

↳ plane stress assumption.

$$\underline{\underline{F}} = \begin{bmatrix} 1+\frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & 0 \\ \frac{\partial u_2}{\partial X_1} & 1+\frac{\partial u_2}{\partial X_2} & 0 \\ 0 & 0 & \cancel{\frac{\partial u_3}{\partial X_3}} \rightarrow \text{wrong} \end{bmatrix}$$

$\dfrac{\partial u_3}{\partial X_1} = 0$ $\quad \dfrac{\partial u_3}{\partial X_1} = 0$ $\quad \lambda_3 \rightarrow \lambda_3(X_1, X_2)$. $\boxed{1+\frac{\partial u_3}{\partial X_3}}$

$$\underline{\underline{P}} = -p\,\underline{\underline{F}}^{-T} + \mu \underline{\underline{F}}.$$

$$[\underline{\underline{P}}] = -p \underbrace{\begin{bmatrix} 1+\frac{\partial u_2}{\partial X_2} & -\frac{\partial u_2}{\partial X_1} & 0 \\ -\frac{\partial u_1}{\partial X_2} & 1+\frac{\partial u_1}{\partial X_1} & 0 \\ 0 & 0 & \frac{1}{\lambda_3} \end{bmatrix}}_{[\underline{\underline{F}}^{-T}]}.$$

$$+ \mu \begin{bmatrix} 1+\frac{\partial u_1}{\partial X_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad \Phi = \frac{\mu}{2}(I_1 - 3)$$

$$P_{33} = -\frac{p}{\lambda_3(X_1, X_2)} + \mu \lambda_3(X_1, X_2) = 0.$$

equilibrium

$$\boxed{\begin{aligned} &\frac{\partial P_{11}}{\partial X_1} + \frac{\partial P_{12}}{\partial X_2} + \cancel{\frac{\partial P_{13}}{\partial X_3}} = 0 \\ &\frac{\partial P_{21}}{\partial X_1} + \frac{\partial P_{22}}{\partial X_2} = 0. \end{aligned}}$$

$$P_{11} = \mu \lambda_3^3 \cdot \left(1+\frac{\partial u_2}{\partial X_2}\right) + \mu\left(1+\frac{\partial u_1}{\partial X_1}\right).$$

Oct, 13, 2021. Wed.

Review: plane stress : incompressible neoHookean Solid

$\underline{\underline{F}}$ for plane stress.

$$[\underline{\underline{F}}] = \begin{bmatrix} \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \lambda_3 \end{bmatrix} \quad [\underline{\underline{F}}_{in}] \qquad ,\alpha = \frac{\partial}{\partial X_\alpha}$$

$$x_\alpha = X_\alpha + u_\alpha(X_1, X_2) \qquad \alpha = 1, 2.$$

Independent of X_3.

λ_3 is the out-of-plane stretch ratio, $\lambda_3(X_1, X_2)$.

$$\underline{\underline{F}}_{in} = x_{\alpha,\beta}\, \underline{e}_\alpha \underline{e}_\beta$$

<u>neo-Hookean</u>

$$\underline{\underline{P}} = -p\underline{\underline{F}}^{-T} + \mu \underline{\underline{F}}$$

<u>~~for~~ ~~imp~~ incompressibility</u>

$$J = \det \underline{\underline{F}} = 1 = (\det \underline{\underline{F}}_{in})\lambda_3 = 1 \qquad (x_{1,1}x_{2,2} - x_{1,2}x_{2,1})$$

$$\Rightarrow \det \underline{\underline{F}}_{in} = \frac{1}{\lambda_3}$$

$$[\underline{\underline{F}}^{-T}] = \begin{bmatrix} x_{2,2}\lambda_3 & -x_{2,1}\lambda_3 & 0 \\ -x_{1,2}\lambda_3 & x_{1,1}\lambda_3 & 0 \\ 0 & 0 & 1/\lambda_3 \end{bmatrix}$$

P-11 Next step:

$$\begin{cases} P_{11} = -p\, x_{2,2}\lambda_3 + \mu x_{1,1} \\ P_{12} = +p\lambda_3 x_{2,1} + \mu x_{1,2} \\ P_{21} = p\lambda_3 x_{1,2} + \mu x_{2,1} \\ P_{22} = -p\lambda_3 x_{1,1} + \mu x_{2,2} \end{cases}$$

$P_{13} = P_{23} = P_{31} = P_{32} = 0$, consistent with the plane stress assump.

$\underline{\underline{P_{33} = 0}} = -p\frac{1}{\lambda_3} + \mu\lambda_3 = 0$

$\hookrightarrow p = \mu\lambda_3^2$

Substitute

use LMB: (ignore body forces.) & acceleration.

$P_{11,1} + P_{12,2} = 0$

$(-(\mu\lambda_3^2 x_{2,2})_{,1} + \mu x_{1,11} + (\mu\lambda_3^2 x_{1,2})_{,2}$

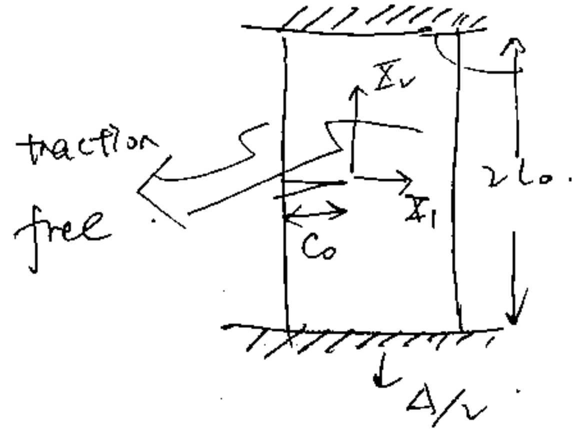
$+ \mu x_{1,22} = 0$

$0 = P_{21,1} + P_{22,2} = (\mu\lambda_3^3 + x_{1,2})_{,1} + \mu x_{2,11}$

$-(\mu\lambda_3^3 x_{1,1})_{,2} + \mu x_{2,22} = 0$

$$\begin{cases} \mu\nabla_X^2 x_1 + \mu\left[(\lambda_3^3 x_{1,2})_{,2} - (\lambda_3^3 x_{2,2})_{,1}\right] = 0 \quad (1) \\ \mu\nabla_X^2 x_2 + \mu\left[\qquad\qquad\right] = 0 \\ \lambda_3 = \dfrac{1}{x_{1,1}x_{2,2} - x_{2,1}x_{1,2}} \end{cases}$$

coupled PDEs for unknowns x_1, x_2.



BCs. $X_2 = \pm L_0$,
$u_1 = 0$. $u_2 = \frac{\pm\Delta}{2}$

on lateral sides:
$X_1 = -C_0 \quad P_{21} = P_{11} = 0$
$X_1 = W_0 - C_0 \quad P_{21} = P_{11} = 0$

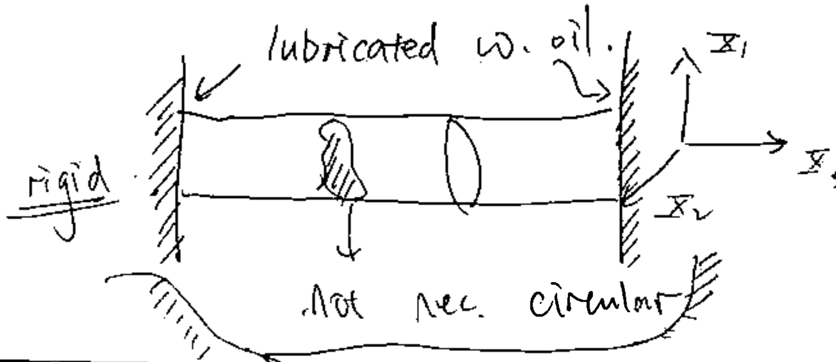
on Crack forces. $-C_0 < X_1 < 0,\ X_2 = 0^\pm$
$P_{12} = P_{22} = 0$

Plane Strain

Assumption.

$$\left\{\begin{array}{l} u_1 = u_1(X_1, X_2) \\ u_2 = u_2(X_1, X_2) \\ u_3 = 0 \end{array}\right. \xRightarrow{\text{imply}} \underline{F} = \begin{bmatrix} x_{1,1} & x_{1,2} & 0 \\ x_{2,1} & x_{2,2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\det \underline{F} = 1. \longrightarrow$ to determine $\underline{P}$



$\nearrow P_{33}$ is not zero.
σ_{33}

Linear Elasticity

$$\underline{E} = \frac{u_{i,j} + u_{j,i}}{2} \rightarrow \text{kinematics} \qquad ,i = \frac{\partial}{\partial X_i}$$

(one simple strain measure, only).

$\rightarrow$ small strain tensor.

$$\frac{\partial \sigma_{ij}}{\partial X_j} = -\rho_0 B_i \qquad \text{Equilibrium}$$

All you need, is constitutive model.

large deformation $\xrightarrow{\text{linearize}}$ constitutive model.

$$\frac{\partial W(\underline{E})}{\partial \underline{E}} = \underline{\sigma} \quad \rightarrow \text{Small for all in linear stage} \quad (\hat{W} = \bar{W} = W).$$

$$\boxed{\sigma_{ij} = K_{ijkl}\ \varepsilon_{kl}} \qquad 3^4 \qquad (\underline{\text{Expect}})$$

K_{ijkl}: independent of strain

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \rightarrow \text{quadratic function of strain tensor}$$

(try) $W = \frac{1}{2} K_{ijkl} \varepsilon_{ij} \varepsilon_{kl}$

$$\sigma_{rs} \leftarrow \frac{\partial W}{\partial \varepsilon_{rs}} = \frac{1}{2}\left[\underbrace{K_{ijkl}\delta_{ir}\delta_{js}\varepsilon_{kl}}_{K_{rskl}} + \underbrace{K_{ijkl}\varepsilon_{ij}\delta_{kr}\delta_{ls}}_{K_{ijrs}}\right]$$

$$\sigma_{rs} = \frac{1}{2}\left[K_{rskl}\varepsilon_{kl} + K_{ijrs}\varepsilon_{ij}\right]$$

$$= \frac{1}{2}\left[K_{rskl}\varepsilon_{kl} + K_{klrs}\varepsilon_{kl}\right]$$

$$\sigma_{ij} = \frac{1}{2}\left[K_{ijkl}\varepsilon_{kl} + K_{klij}\varepsilon_{kl}\right]$$

$$\sigma_{ij} = \frac{1}{2}\underbrace{\left[K_{ijkl} + K_{klij}\right]}_{\hat{K}}\varepsilon_{kl}$$

$\hat{K}_{ijkl} = \hat{K}_{klij}$ symmetric in kl, ij

81 component

Symmetry of $\sigma_{ij} \Rightarrow K_{ijkl} = K_{jikl}$

Symmetry of $\varepsilon_{kl} \Rightarrow K_{ijkl} = K_{ijlk}$.

$9 \times 9 \rightarrow 6 \times 6$

36 independent components.

The existence of W.

$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}$

implies $K_{ijkl} = K_{klij}$

~~18~~ 21 ind. comp.

Most common model for elastic.

Oct. 15, office hours.

C_1 C_2, $\propto$ shear modulus }
Poisson's ratio. }

letting $\lambda \to 1$

at very small → agrees with Hooke's law.

Plot the curve.

normalize the stress for shear modulus

other terms $\propto$ ratio of C_1, C_2
↓
function only of the Poisson's ratio

reasonable choice $\nu = 0.45$

0.5 (~~in~~ incompressible).

a lot of curve with different Poisson's ratio.

* normalized shear modulus G.

Piola & Cauchy normalized by

You can normal the stress by G.

λ_1, λ_3

$$\cancel{\lambda_3^{-2} - 1 + \frac{C_2}{C_1} \ln(\lambda_1^2 \lambda_3) = 0}$$

$$\lambda^2 - 1 + \frac{C_2}{C_1} \ln(\lambda_3^3 \lambda^2) = 0$$

$f(\lambda)$ incompressible. $\frac{C_2}{C_1} \to$ huge

↳ G.

$\lambda_3 \lambda^2 = 1$. ← Lambert function

1st order expansion John Hutchinston

Oct. 18 th, 2021, Mon.

Review.

Linear Elasticity.

$$W = \frac{1}{2} K_{ijkl} \varepsilon_{ij} \varepsilon_{kl}.$$

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}$$

K_{ijkl} has 21 independent constants.

$$K_{ijkl} = K_{jikl} = K_{ijlk} = K_{klij}$$

Anisotropic

$$\sigma_{ij} = K_{ijkl} \varepsilon_{kl}.$$

$$W = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} = \frac{1}{2} \underline{\sigma} \cdot \underline{\varepsilon}$$

Isotropy solids

$$K_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

λ, μ are constants.

General form of isotropic 4th order Tensor

"Introduction to Cartesian Tensors"

'Jim ~~Knowles~~ Knowles.

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + \mu [\varepsilon_{ij} + \varepsilon_{ji}].$$

$$= 2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk} \delta_{ij}.$$

Generalized Hooke's LAW

μ, λ are called Lame constants.

$$\sigma_{kk} = 2\mu \varepsilon_{kk} + 3\lambda \varepsilon_{kk}.$$

$$\sigma_{kk} = (2\mu + 3\lambda) \varepsilon_{kk}.$$

$$2\mu \varepsilon_{ij} + \lambda \frac{\sigma_{kk}}{(2\mu + 3\lambda)} \delta_{ij} = \sigma_{ij}.$$

$$\varepsilon_{ij} = \frac{\sigma_{ij}}{2\mu} - \frac{\lambda \sigma_{kk}}{(2\mu + 3\lambda)} \cdot \frac{1}{2\mu} \delta_{ij}$$

$$\varepsilon_{ij} = \frac{(1+\mu)}{E} \sigma_{ij} - \frac{\nu \sigma_{kk}}{E} \delta_{ij}.$$

~~μ~~ ν — Poisson's ratio.

E — Young's Modulus

$$\frac{1}{2\mu} = \frac{1+\nu}{E} \Rightarrow \mu = \frac{E}{2(1+\nu)}$$

Shear Modulus.

$$\frac{\nu}{E} = \frac{\lambda}{(2\mu + 3\lambda) 2\mu}.$$

Tension test

$\sigma_{11} = \sigma$. $\sigma_{ij} = 0$. $(i,j \neq 1)$

$\varepsilon_{11} = \frac{\sigma_{11}}{E}$.

↳ tension modulus.

$\varepsilon_{22} = \varepsilon_{33} = -\frac{\mu}{E}\sigma_{11}$.

$-\frac{\varepsilon_{22}}{\varepsilon_{11}} = \mu$,

Poisson's ratio ≥ 0.

There are negative Poisson's ratio material but anisotropic.

Apply a pure hydrostatic tension,

$\varepsilon_{ij} \leftarrow \varepsilon_{kk} = -\frac{(1+\nu)}{E} p\,\delta_{ij} + \frac{3p\nu}{E}\delta_{ij}$.

if $\sigma_{ij} = -p\delta_{ij}$.

$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = -\underbrace{\frac{(1+2\nu)}{E}}P$

Bulk Modulus $= -\frac{1}{K}P \to -P/K$.

$K = \cancel{\frac{E}{1-\nu}}\ \frac{E}{1-2\nu}$.

$\nu \to \frac{1}{2}$, $K \to \infty$.

$\Downarrow$

$\mathrm{tr}\,\underline{\underline{\varepsilon}} = 0$. incompressible solid

$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$.

$\sigma_{ij} = \cancel{2\mu\varepsilon_{ij}} \to \sigma_{ij} = 2\mu\varepsilon_{ij} - P\delta_{ij}$.

~~AAA~~. Office hour.

$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$

General form relation for linear Elasticity

① $\varepsilon_{ij} = \frac{u_{i,j}+u_{j,i}}{2}$

② $\sigma_{ij,j} = -\rho_0 B_i$.

② $\sigma_{ij} = \sigma_{ji}$

③ $\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$

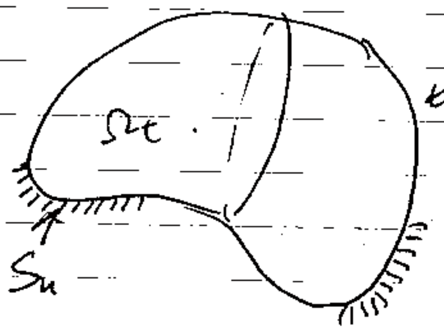
- Substitute ① into ③ to express strains in terms of displacements.
- Substitute stress into ② to obtain.

$G\nabla^2 \underline{u} + (\lambda + G)\nabla(\nabla\cdot\underline{u}) = -\rho_0\underline{B}$

↳ Navier's equation (3 PDEs)

Subject Navier's Eq. to BCs

Typical e.g.

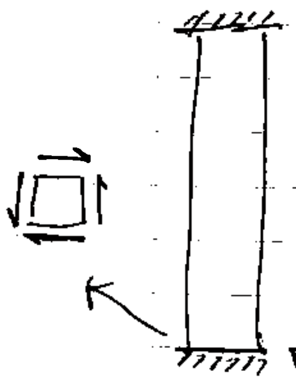


On. traction is prescribed.

mixed BCs.

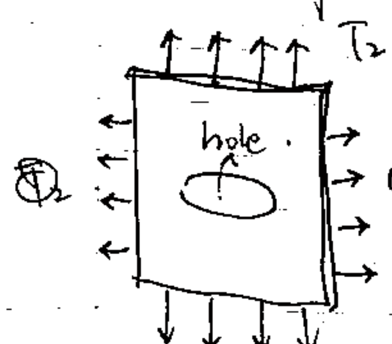
Displacement prescribed.

$$\begin{cases} \sigma_{ij} n_j = T_n(\underline{X}), & \underline{X} \in S_T \\ S_u:\ u_i = f(\underline{X}), & \underline{X} \in S_U \end{cases}$$



$$\left[2\mu\left[\frac{u_{i,j}+u_{j,i}}{2}\right] + \lambda u_{k,k}\delta_{ij}\right] n_j = T_i(\underline{X})$$

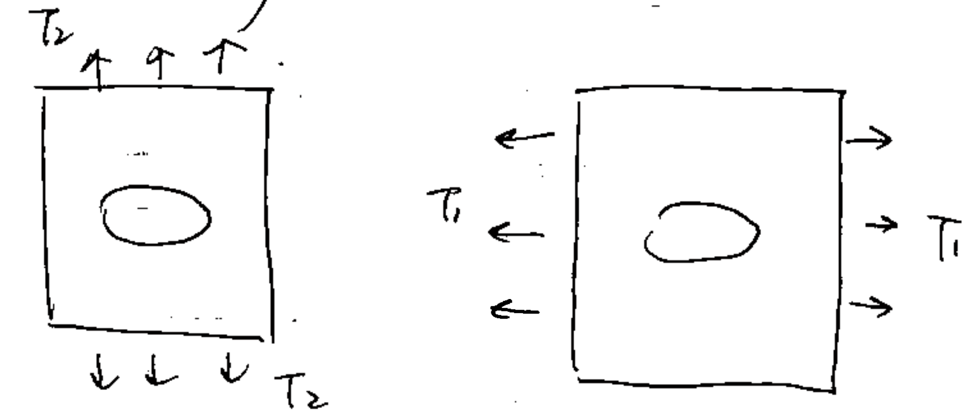
fixed boundary



(thin sheet)

In Linear Elasticity.

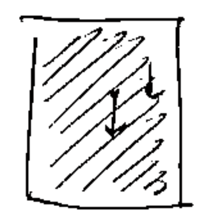
(⟺) equiv



Build up complex solutions from simple ones.

$$\sigma_{ij,j} = -\rho_0 B_i$$

Suppose we guess a solution for ⊛ that also satisfies the Boundary Conditions (BC) (traction BC).



$\underline{B} = g\,\underline{e}_3$

If we guess: σ^*_{ij}

compute displacement.

$$\varepsilon^*_{ij} = \frac{(1+\nu)}{E}\sigma^*_{ij} - \frac{\nu}{E}\sigma^*_{kk}\delta_{ij}$$

↓

Integrate strain $\underline{\varepsilon}^*$ to get displacement field

There are three unknown disp.

$u_1, u_2, u_3 \rightarrow$ (position).

I guess $\sigma_{ij}^* \rightarrow \varepsilon_{ij}^*$

$$\varepsilon_{ij}^* = \frac{u_{i,j} + u_{j,i}}{2}$$

6 equations here

6 equations, & 3 unknowns.

the solutions may not exists, if e. not unique.

Plane strain.

$\hookrightarrow$ linear elas.

$$u_3 = 0, \quad \varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = u_{1,1}$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = u_{2,2}.$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{u_{1,2} + u_{2,1}}{2}$$

$$\varepsilon_{ij} \text{ (rest) } = 0.$$

$u_3 = 0$, u_1, u_2 depends on X_1, X_2 only.

You can show that

$$-2\varepsilon_{12,12} + \varepsilon_{11,22} + \varepsilon_{22,11} = 0$$

$$\frac{\partial^2(\ )}{\partial X_1 \partial X_2} = (\ )_{,12}$$

$\hookrightarrow$ compatibility equation for plane strain.

: puts a constraint on the strain.

$$\varepsilon_{ij} = \frac{\sigma_{ij}(1+\nu)}{E} - \frac{\nu \sigma_{kk} \delta_{ij}}{E}$$

$\hookrightarrow$ you will find this:

$$\nabla^2(\sigma_{11} + \sigma_{22}) = \frac{1}{(1-\nu)} \nabla \cdot (\rho_0 \underline{b}).$$

compatibility equation for stress.

Wed., Oct. 20, 2021   Week 9 (?)

Linear Elasticity

6 Eqns: $\varepsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2}$ — kinematics.

3 Eqns: $\sigma_{ij,j} = -\rho_0 B_i$ — Balance laws.

6 Eqns: $\varepsilon_{ij} = \frac{(1+\nu)\sigma_{ij}}{E} - \frac{\nu \sigma_{kk} \delta_{ij}}{E}$

Constitutive model.

15 Eqns.

unknowns: ε_{ij}, u_i, σ_{ij}. 15 unknowns.

Navier Eqns. (Displacement formulation).

$$G \nabla^2 \underline{u} + (\lambda + G)\nabla(\nabla \cdot \underline{u}) = -\rho_0 \underline{B}$$

3 Eqns, & 3 unknowns. u_1, u_2, u_3.

independent variables, X_i.
positions.

dependent is u_i.

Most useful when body is subject to BCs

Antiplane shear deformation.

$u_\alpha \equiv 0$. $\alpha = 1, 2$. No in-plane disp.

$u_3 = u(X_1, X_2)$. independent of X_3.

$\Downarrow$

$\varepsilon_{\alpha\beta} = 0$, $\alpha = 1, 2$.

$\varepsilon_{33} = \frac{\partial u_3}{\partial X_3} = 0$.

Only non-vanishing strain are.

engineering strain $\left\langle \begin{array}{l} \varepsilon_{13} = \varepsilon_{31} = \frac{1}{2}\frac{\partial u}{\partial X_1} = \frac{1}{2}\gamma_1 \\ \varepsilon_{23} = \varepsilon_{32} = \frac{1}{2}\frac{\partial u}{\partial X_2} = \frac{1}{2}\gamma_2 \end{array} \right.$

Constitutive model.

$\sigma_{\alpha\beta} = 0$. in-plane stess

$\sigma_{33} = 0$

$$\begin{cases} \sigma_{13} = \sigma_{31} = G\gamma_1 \\ \sigma_{23} = \sigma_{32} = G\gamma_2 \end{cases}$$

Equilibrium Eqns are identically satisfied in 1 & 2 directions ($B_1 = B_2 = 0$)

$$\frac{\partial \sigma_{31}}{\partial X_1} + \frac{\partial \sigma_{32}}{\partial X_2} = 0$$

$$\Updownarrow$$

no body force.

$$\frac{\partial \tau_1}{\partial X_1} + \frac{\partial \tau_2}{\partial X_2} = 0 \iff \nabla_X \cdot \vec{\tau} = 0. \quad (1)$$

$$\underline{\tau} = \tau_1 \underline{e}_1 + \tau_2 \underline{e}_2 .$$

γ_1 & γ_2 must satisfy the fact that.

$$\gamma_1 = \frac{\partial w}{\partial X_1}, \quad \gamma_2 = \frac{\partial w}{\partial X_2} \Rightarrow \frac{\partial \gamma_1}{\partial X_2} = \frac{\partial \gamma_2}{\partial X_1}.$$

Stress compatibility

the eqn. int. $\Longrightarrow$ $\dfrac{\partial \tau_1}{\partial X_2} = \dfrac{\partial \tau_2}{\partial X_1}.$ (2)

Introduce a stress function ϕ.

$$\tau_1 \equiv \frac{\partial \phi}{\partial X_2} \qquad \tau_2 \equiv -\frac{\partial \phi}{\partial X_1}. \qquad (3)$$

Subs. (2) $\downarrow$ (3) into (1), we see that

(1) is satisfied automatically.

$$\frac{\partial \tau_1}{\partial X_1} = \frac{\partial^2 \phi}{\partial X_1 \partial X_2} \qquad \frac{\partial \tau_2}{\partial X_2} = \frac{\partial^2 \phi}{\partial X_2 \partial X_1}$$

Substitute (3) into (2).

$$\frac{\partial^2 \phi}{\partial X_2^2} + \frac{\partial^2 \phi}{\partial X_1^2} = 0 \quad \text{or} \quad \nabla_X^2 \phi = 0.$$

Laplace Eqn in 2D.

<u>Stress function approach</u>

Disp. Formulation. $\tau_1 = G \dfrac{\partial w}{\partial X_1}$

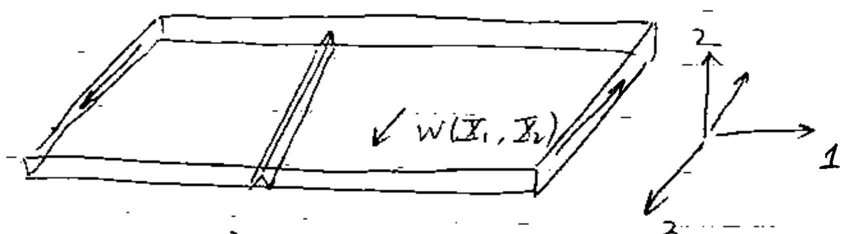
$\tau_2 = G \dfrac{\partial w}{\partial X_2}$ $\Big\rangle$ $\nabla_X^2 w = 0.$

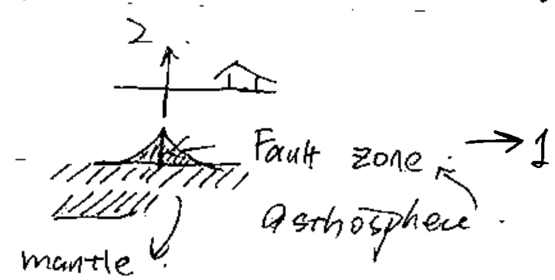
Substitute into (1). Simpliest form of Navier's Equation.

$$\phi + i w = f(z).$$

$\phi + iGw$

☆ Anti-plane shear.





Reminder: Plane strain

$u_\alpha(X_1, X_2)$, $\alpha = 1, 2$

$u_3 \equiv 0$

$\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}$ (all others strain components = 0).

Compatibility

$$\boxed{\frac{\partial^2 \varepsilon_{11}}{\partial X_2^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial X_1 \partial X_2} + \frac{\partial^2 \varepsilon_{22}}{\partial X_1^2} = 0}$$

Note

$\frac{\partial \sigma_{3i}}{\partial X_i} = 0$, is automatically satisfied

$\sigma_{31} = \sigma_{32} = 0$, σ_{33} is independent of X_3.

$\varepsilon_{33} = 0 \Rightarrow \frac{\sigma_{33}}{E} - \frac{\nu(\sigma_{11} + \sigma_{22})}{E} = 0$

$$\boxed{\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})}$$

Equilibrium equation is identically satisfied in 3 direction.

Therefore: $\frac{\partial \sigma_{11}}{\partial X_1} + \frac{\partial \sigma_{12}}{\partial X_2} = \rho_0 B_1$

$$\frac{\partial \sigma_{21}}{\partial X_1} + \frac{\partial \sigma_{22}}{\partial X_2} = \rho_0 B_2$$

Equilibrium Eqns (LMB) $\rightarrow$ (4a, b)

Assuming $\underline{B} = \underline{0}$

Airy stress function, ϕ.

$\sigma_{11} = \frac{\partial^2 \phi}{\partial X_2^2}$ $\quad \sigma_{12} = -\frac{\partial^2 \phi}{\partial X_1 \partial X_2}$

$\sigma_{22} = \frac{\partial^2 \phi}{\partial X_1^2}$ (5)

Substitute (5) into (4a, b).

$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \frac{\nu(\sigma_{22} + \sigma_{33})}{E}$

$\varepsilon_{22} = \frac{\sigma_{22}}{E} - \frac{\nu(\sigma_{11} + \sigma_{33})}{E}$ $\quad \sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$

$\varepsilon_{33} = \cancel{\frac{\sigma_{33}}{E}} - \cancel{\frac{\nu(\sigma_{11} + \sigma_{22})}{E}}$ $\quad \frac{\sigma_{12}}{2G}$ $\sigma_{12} \rightarrow \tau$. $\Downarrow$

Simplified plane strain constitutive model

$\varepsilon_{11} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{11} - \nu\sigma_{22}\right]$.

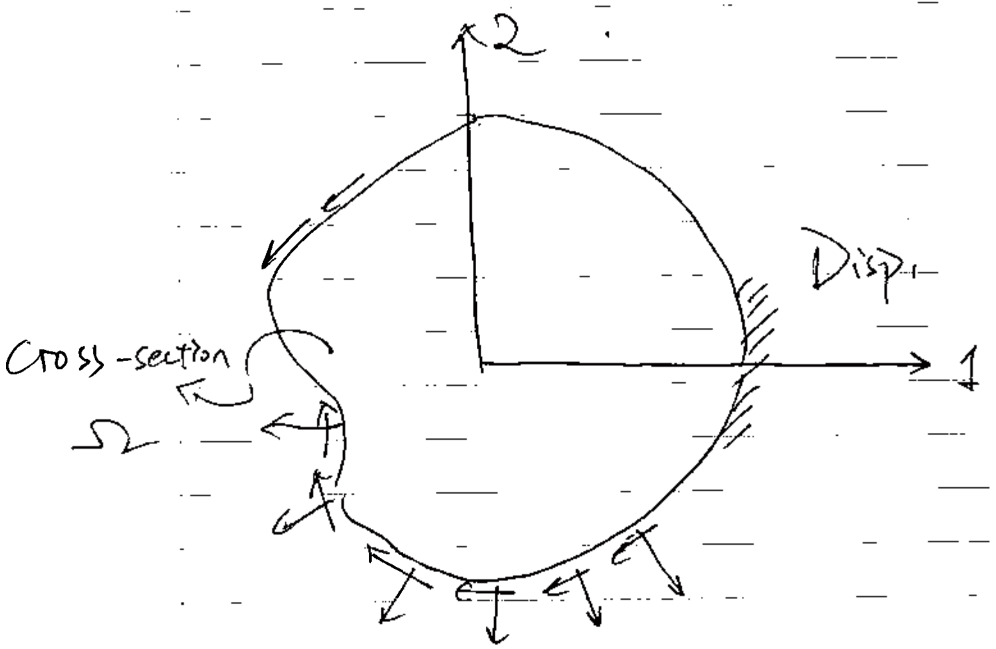
$\gamma = \frac{\tau}{G}$ $\quad \varepsilon_{22} = \frac{(1+\nu)}{E}\left[(1-\nu)\sigma_{22} - \nu\sigma_{11}\right]$

Non-zero Inplane stress fields are

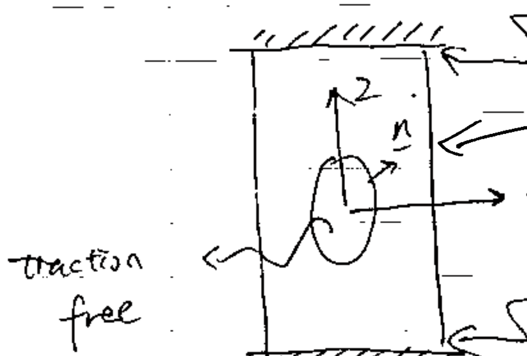
σ_{11}, σ_{12}, σ_{22}

Non-zero Out-of-plane stress.

$\sigma_{33} = -\nu(\sigma_{11} + \sigma_{22})$.



S_T : Traction Boundary Conditions.

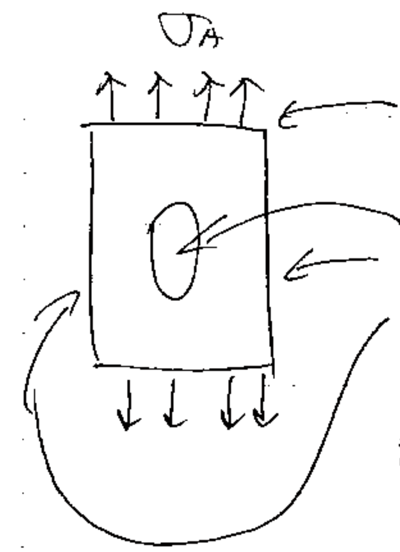


$u_1 = u_2 = 0$

$\sigma_{21} = \sigma_{11} = 0$. $(\underline{\sigma} \cdot \underline{n} = 0)$

$\underline{\sigma} \cdot \underline{n} = 0$

$u_1 = u_2 = 0$



no shear stress.

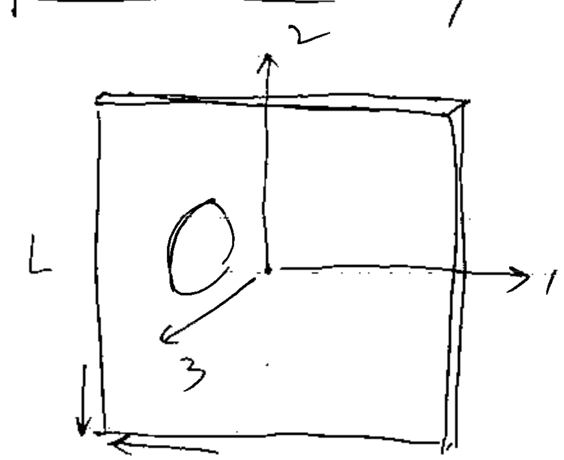
free boundary.

Solution independent of material properties.

Stresses $\propto$ stress function.

$$\begin{cases} \sigma_{11} = \dfrac{\partial^2 \phi}{\partial x_1^2} \\ \sigma_{22} = \dfrac{\partial^2 \phi}{\partial x_2^2} \\ \sigma_{12} = -\dfrac{\partial^2 \phi}{\partial x_1 \partial x_2} \end{cases}$$

<u>plane stress</u> (finite deformation).



$t \ll L$. and other in-plane dimensions.

$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$. $\rightarrow$ three non zero stresses

σ_{11}, σ_{22}, σ_{12}.

$$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{32} = \varepsilon_{31} \cong 0.$$

$\varepsilon_{\alpha\beta}$ is approx. independent of x_3.

$$\frac{\Delta V}{V} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 0$$

$\downarrow$ incompressible

Constitutive model

Plane stress $\quad \sigma_{13} = \sigma_{23} = \sigma_{33} = 0.$

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \frac{\nu(\sigma_{22} + \overset{0}{\sigma_{33}})}{E} = \frac{\sigma_{11}}{E} - \frac{\nu\sigma_{22}}{E}$$

$$\varepsilon_{12} = \frac{\sigma_{12}}{2G}.$$

$$\varepsilon_{22} = \frac{\sigma_{22}}{E} - \frac{\nu(\sigma_{11} + \overset{0}{\sigma_{33}})}{E} = \frac{\sigma_{22}}{E} - \frac{\nu\sigma_{11}}{E}$$

Plane strain

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \frac{\nu(\sigma_{22} + \sigma_{33})}{E} \qquad \sigma_{33}: \cancel{0} \to \nu(\sigma_{11}+\sigma_{22})$$

$$= \frac{\sigma_{11}}{E} - \frac{\nu(\sigma_{22} + \nu(\sigma_{11}+\sigma_{22}))}{E}$$

$$= \frac{(1-\nu^2)\sigma_{11}}{E} - \frac{\nu(1+\nu)\sigma_{22}}{E}$$

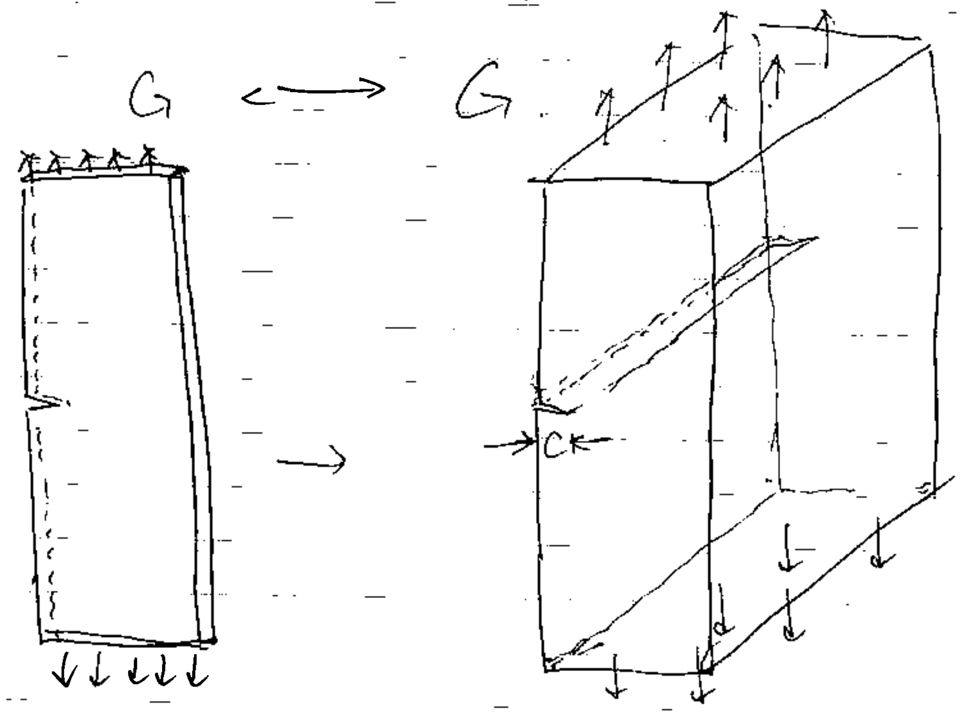
Compatibility & Equilibrium.

$\nabla^4 \phi = 0.$ (No Body force).

$\nu \longrightarrow \frac{\nu}{1-\nu}$

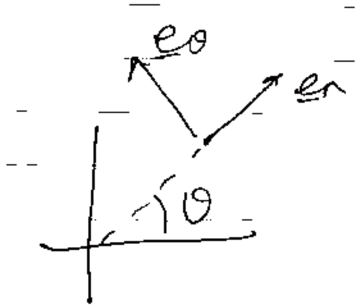
plane stress $\quad$ plane strain

$G \longleftrightarrow G$



$$\underset{\sim}{\varepsilon} = \frac{\nabla \underline{u} + (\nabla \underline{u})^T}{2}$$

$$\varepsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2}$$



$$\underline{u} = u_r \underline{e_r} + u_\theta \underline{e_\theta}$$

$$\begin{cases} \underline{e_r} = \cos\theta\, \underline{e_1} + \sin\theta\, \underline{e_2} \\ \underline{e_\theta} = -\sin\theta\, \underline{e_1} + \cos\theta\, \underline{e_2}. \end{cases}$$

In Cartesian coordinates

$$\nabla \underline{u} = \frac{\partial u_i}{\partial x_j} \underline{e}_i \underline{e}_j$$

$$\overleftarrow{\nabla} \underline{u} = \frac{\partial u_i}{\partial x_j} \underline{e}_i \underline{e}_j$$

$$\overrightarrow{\nabla} \underline{u} = \frac{\partial u_i}{\partial x_j} \underline{e}_j \underline{e}_i = (\overleftarrow{\nabla} \underline{u})^T$$

$$\nabla = \cancel{e_\theta \frac{\partial}{\partial r} + e_r \frac{\partial}{\partial \theta}}$$

$$e_r \frac{\partial}{\partial r} + \frac{e_\theta}{r}\frac{\partial}{\partial \theta}$$

$$\nabla \cdot \underline{u} = \left(e_r \frac{\partial}{\partial r} + \frac{e_\theta}{r}\frac{\partial}{\partial \theta}\right)(u_r \underline{e}_r + u_\theta \underline{e}_\theta)$$

$$= e_r \cdot \frac{\partial (u_r \underline{e}_r)}{\partial r} + e_r \frac{\partial (u_\theta \underline{e}_\theta)}{\partial r}$$

$$= e_r \cdot \frac{\partial u_r}{\partial r} \cdot e_r + e_r \left(\frac{\partial u_\theta}{\partial r} e_\theta \right)$$

$$+ \frac{e_\theta}{r}\left[\frac{\partial u_r}{\partial \theta} e_r + u_r \underbrace{\frac{\partial e_r}{\partial \theta}}_{e_\theta} \right]$$

$$+ \frac{e_\theta}{r} \frac{\partial u_\theta}{\partial \theta} e_\theta + \frac{e_\theta}{r} u_\theta (e_r)$$

$$\frac{\partial e_\theta}{\partial \theta} = -\cos\theta \, \underline{e}_1 - \sin\theta \, \underline{e}_2 = -\underline{e}_r$$

$$\left(\quad \sim \quad -\frac{u_\theta}{r} \cdot \right) e_\theta e_\theta$$

$$\nabla \underline{u} = \frac{\partial u_r}{\partial r} e_r \underline{e}_r + \frac{\partial u_\theta}{\partial r} e_\theta \underline{e}_\theta + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} e_\theta e_r$$

$$+ \left[\underbrace{\frac{u_r}{r} \cdot e_\theta e_\theta}_{\varepsilon_{rr}} + \frac{1}{r} \cdot \frac{\partial u_\theta}{\partial \theta} \right] e_\theta e_\theta$$

$$\underline{\underline{\varepsilon}} = \frac{\partial u_r}{\partial r} e_r e_r + \left[\frac{u_\theta}{r} + \frac{1}{r} \cdot \frac{\partial u_\theta}{\partial \theta} \right] e_\theta e_\theta \quad \varepsilon_{\theta\theta}$$

$$+ \frac{1}{2}\underbrace{\left[\frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} - \frac{u_\theta}{r} \right]}_{} e_\theta e_r + \frac{1}{2}[\quad] e_r e_\theta$$

$\varepsilon_{\theta r}$.

$\sigma_{\alpha\beta}$ (Cartesian) $\{ \underline{e}_1, \underline{e}_2 \}$.

↓

σ_{rr}, $\sigma_{r\theta}$, $\sigma_{\theta\theta}$ polar coordinates.

$\{ e_r, e_\theta \}$.

Oct. 27. 2021. Wed

☆☆☆ { Anti-plane shear. / plane strain / plane stress } Stress functions.

⇓

$\nabla^2 \phi = 0$ harmonic

$\nabla^4 \phi = 0$ biharmonic

$\underline{u} = u_r \underline{e}_r + u_\theta \underline{e}_\theta$

$\underline{e}_r = \cos\theta \, \underline{e}_1 + \sin\theta \, \underline{e}_2$

$\underline{e}_\theta = -\sin\theta \, \underline{e}_1 + \cos\theta \, \underline{e}_2$.

$\underline{\underline{\sigma}} = \sigma_{ij} \, \underline{e}_i \underline{e}_j = \sigma_{rr} \, \underline{e}_r \underline{e}_r + \cdots$

$\nabla \cdot \underline{\underline{\sigma}}$ → Easy in Cartesian Coordinate

plane strain or plane stress

Derive THIS ONE !!!!

$$\begin{cases} \dfrac{\partial \sigma_{rr}}{\partial r} + \dfrac{1}{r} \cdot \dfrac{\partial \sigma_{r\theta}}{\partial \theta} + \dfrac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 . \\ \dfrac{1}{r} \cdot \dfrac{\partial \sigma_{\theta\theta}}{\partial \theta} + \dfrac{\partial \sigma_{r\theta}}{\partial r} + \dfrac{2 \sigma_{r\theta}}{\partial r} = 0 \end{cases}$$

Third eqn. automatically satisfied.

$$\sigma_{rr} = \frac{\phi_{,r}}{r} + \frac{\phi_{,\theta\theta}}{r} .$$

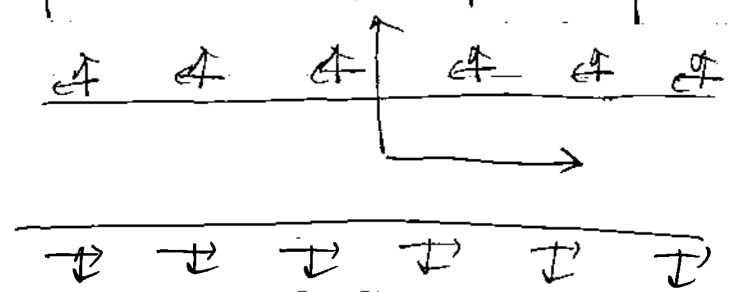
$$\sigma_{\theta\theta} = \phi_{,rr} \qquad \sigma_{r\theta} = -(\phi_{,\theta} / r)_{,r}$$

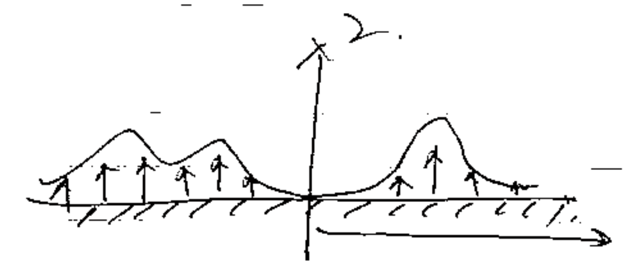
$$\nabla^4 \phi = 0 = \overbrace{\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2} \cdot \frac{\partial^2}{\partial r^2}\right]}^{\nabla^2} \underbrace{\left[\frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2} \cdot \frac{\partial^2}{\partial \theta^2}\right]}_{\nabla^2} \phi = 0$$

Technique of solution

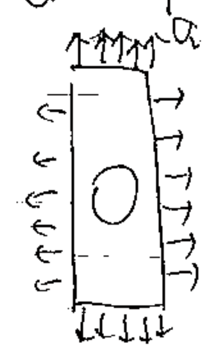
① Fourier transform.

[Strip or half space problem]





② superposition. (simple idea × technique)



> Sum of two solutions.

⇒ ~~xxx~~ work for nonlinear elasticity

↳ traction free.

③ separation of variables.

↳ works for simple geometry.

Complex variable method

function theory.
(Antiplane shear).

$\nabla^2\phi = 0$ and $\nabla^2 w = 0$.

$$\boxed{\Phi_{,x} = \sigma_{23}, \quad \Phi_{,y} = \sigma_{32}} \quad (1)$$

$X = X_1$, $y = X_2$ (stress function)

$$\boxed{\begin{aligned}\sigma_{13} &= G\frac{\partial w}{\partial x} = G w_{,x} \\ \sigma_{23} &= G\frac{\partial w}{\partial y} = G w_{,y}\end{aligned}} \quad (2)$$

(1), (2).

$$\Rightarrow \phi_{,x} = G w_{,y}$$
$$\Phi_{,y} = -G w_{,x} \quad (3)$$

Define a complex function.

$f(z) \equiv \phi + iGw$

ϕ ← real part of f.

iGw ↳ imaginary part of f

(3) is a rotation between real part of f and its Imaginary parts

(3) is called the $\underbrace{\text{Cauchy - Riemann Eqns.}}_{CR}$

$h(z) = u + iv$.

$$\begin{bmatrix} u_{,x} = v_{,y} \\ u_{,y} = -v_{,x} \end{bmatrix}$$

Any fct with Real & Imaginary parts that satisfies the CR Eqns is called an analytic fct in D
↳ in a Domain D.

$$\begin{cases} \Phi_{,xx} = G w_{,xy}. \\ \Phi_{,xy} = -G w_{,xy}. \\ \nabla^2\phi = 0 \end{cases}$$

$\cos x$ ← Replace x by z. $= \cos z$.

$\cos x = \dfrac{e^x + e^{-x}}{2}$

$$\cos z = \frac{e^z + e^{-z}}{2}$$

$$e^z = e^{x+iy} = e^x e^{iy}$$
$$= e^x[\cos y + i\sin y]$$
$$= e^x\cos y + i e^x \sin y.$$

$$\frac{(e^x\cos y + ie^x\sin y) + e^{-x}(\cos y - i\sin y)}{2}$$

$$= \frac{(e^x + e^{-x})\cos y + i(e^x - e^{-x})\sin y}{2}$$

$$= \underbrace{\cosh x\cos y}_{u} + i\underbrace{\sinh x\sin y}_{v} = \cos z.$$

$$\begin{cases} u_{,x} = v_{,y} \\ u_{,y} = -v_{,x}. \end{cases} \quad \checkmark \to CR$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \to a_n = \frac{1}{n!} f^{n}(z_0)$$

$$= \left.\frac{\partial^n f}{\partial z^n}\right|_{z_0}$$

↖ analytic solution.

$$f'(z) = \sum_{i=0}^{\infty} a_n n (z - z_0)^{n+1}$$

$$f'(z) = \phi_{,x} + i G\omega_{,x}$$

$$= \frac{\phi_{,y}}{i} + \frac{iG\omega_{,y}}{i}$$

$$= -i\phi_{,y} + G\omega_{,y}$$

$$\boxed{\phi = Re[\bar{z}\varphi(z) + \chi(z)] \qquad \bar{z} \equiv x - iy.}$$

Analytic

⟸ take this methods.

displacements :

$$2G(u_1 + iu_2) = \chi\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}$$

$$\psi(z) = \chi'(z) = \frac{d\chi}{dz}.$$

$\chi = 3 - 4\nu$. plane strain .

$= \frac{3-\nu}{1+\nu}$ stress

$$\sigma_{11} + \sigma_{22} = 2[\varphi'(z) + \overline{\varphi'(z)}].$$

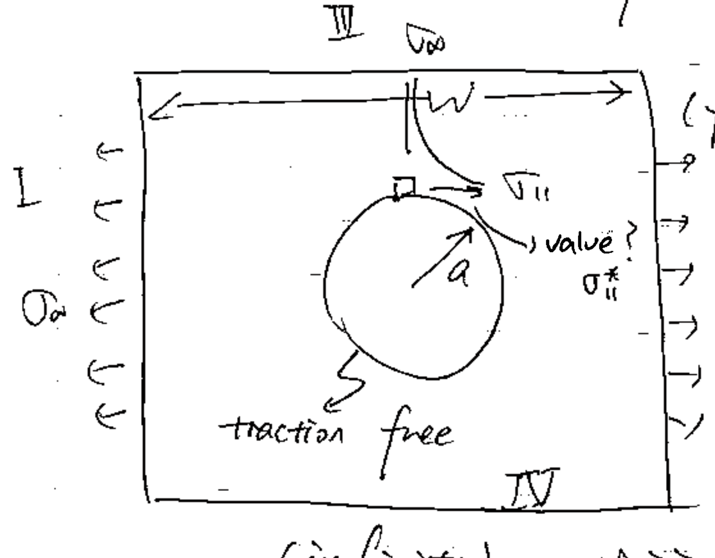
$$\sigma_{22} + i\sigma_{12} = \varphi'(z) + \overline{\varphi'(z)} + \bar{z}\varphi''(z) + \psi'(z)$$

$$\frac{dz^n}{dz} = n z^{n-1}$$

* Any rule with differentiations can be applied to complex variable theory.

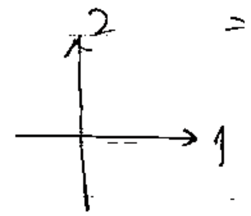
(doesn't depend on the hole size).

$\frac{\sigma_{11}^*}{\sigma_\infty}$ = const. independent of σ_∞



$\sigma_{\alpha\beta} = \sigma_\infty f(\frac{r}{a}, \theta)$

$= \sigma_\infty f(\frac{x}{a}, \frac{y}{a})$



$\frac{\sigma_{11}^*}{\sigma_\infty} = f(0, 1) = 3$

(infinite). → $w \gg a$.

BC : $\sigma_{11} = \sigma_\infty$, as $|x| \to \infty$

I & II. $\sigma_{21} = 0$, as $|x| \to \infty$

III & IV : $\sigma_{22} = 0$. as $|y| \to \infty$

$\sigma_{12} = 0$, as $|y| \to \infty$.

BC on hole : $\underline{\sigma} \cdot \underline{n} = 0$ on hole.

$\underline{n} = \underline{e_r} = \cos\theta\, \underline{e_1} + \sin\theta\, \underline{e_2}$

$\sigma_{\alpha\beta} n_\beta = 0$ → $r = a = \sqrt{x^2 + y^2}$

$\sigma_{11}\cos\theta + \sigma_{12}\sin\theta = 0.$

$\sigma_{21}\cos\theta + \sigma_{22}\sin\theta = 0$ $\quad |\theta| \le \pi$.

∴ linear problem

1: σ_{ij} ($\sigma_{\alpha\beta}$) independent of G, ν.

→ $\sigma_{\alpha\beta}$. proportional to σ_∞.

Mon. ~~Oct.~~ Nov. 1st, Week 11

Theory of Torsion

Semi-inverse Method

① Guess a form for the disp. field

⇓

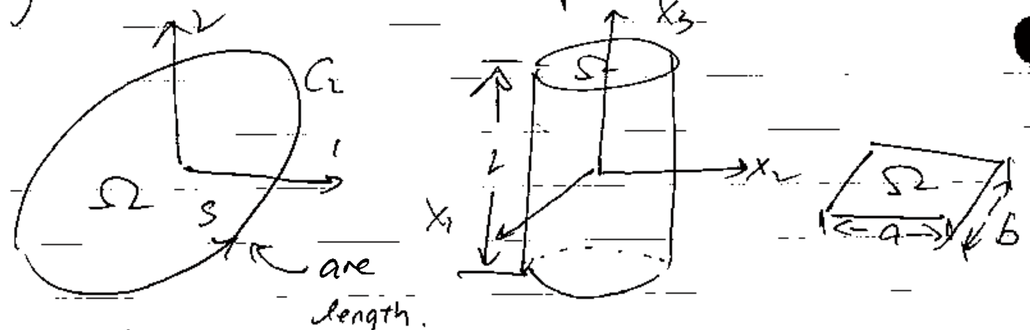
strain

⇓

stress → satisfy equilibrium.

Check the BCs are satisfied.

* Cylinder with a uniform cross-section.



$X_1(s),\ X_2(s)$ parameterize by

$\underline{u}$ is the displacement field.

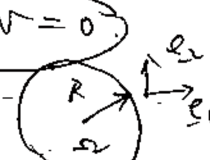
$u_1 = -\alpha X_2 X_3$ $\quad u_2 = \alpha X_1 X_3$ $\quad u_3 = W(X_1, X_2)$

walking function.

α: a constant.

To motivate this, look at a special case

Assume $W = 0$, Bar is circular



$\underline{u} = u_1 e_1 + u_2 e_2 = u_r e_r + u_\theta e_\theta$

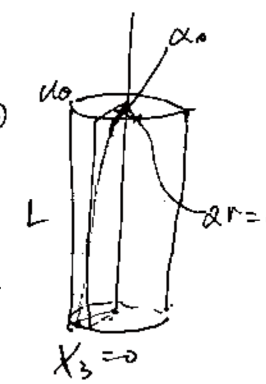
$e_r = e_1 \cos\theta + e_2 \sin\theta$ ①

$e_\theta = -e_1 \sin\theta + \cos\theta\, e_2$

$u_r = \underline{u} \cdot e_r \quad u_\theta = \underline{u} \cdot e_\theta$ ②

$u_1 = -\alpha \sin\theta X_3 \quad u_2 = +\alpha r \cos\theta X_3$

$u_r = 0, \quad u_\theta = \alpha r X_3$ ③



on surface: $u_\theta = \alpha R X_3$ $(r = R)$

$u_\theta = \alpha r L = r\theta_0$

$rL = \theta\theta_0$. $\quad \alpha = \frac{\theta_0}{L}$ > the unit of twist per unit length

Strain tensor in cylindrical coordinate

the strain due to < $\varepsilon_{rr} = \varepsilon_{\theta\theta} = \varepsilon_{r\theta} = \varepsilon_{r3} = \varepsilon_{33} = 0$

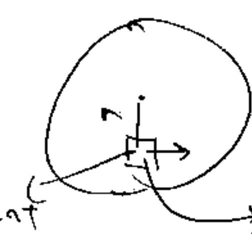
$\varepsilon_{\theta 3} = \frac{1}{2}\alpha r$

$L\gamma = R\theta_0 \quad \gamma = \frac{R\theta_0}{L} = R\alpha$

$\tau_{\theta 3} = \frac{G}{2}\alpha r$ stress

↳ only non-trivial stress component in polar coordinate

• traction free BCs automatically satisfied on surface of cylinder



$\iint \tau_{\theta 3}\, r\, dA = M$

(Recall the walking function of z)

$\varepsilon_{11} = 0, \quad \varepsilon_{22} = 0, \quad \varepsilon_{12} = 0, \quad \varepsilon_{33} = 0.$

$$\varepsilon_{13} = \frac{1}{2}\left[-\alpha x_2 + \frac{\partial w}{\partial x_1}\right]$$

$$\varepsilon_{23} = \frac{1}{2}\left[\alpha x_1 + \frac{\partial w}{\partial x_2}\right]$$

$$\Rightarrow \begin{cases} \tau_{\alpha\beta} = 0, & \alpha = 1, 2. \\ \tau_{33} = 0 \end{cases}$$

$$\tau_{13} = G[-\alpha x_2 + w_{,1}] \qquad \tau_{23} = G[\alpha x_1 + w_{,2}]$$

Enforce equilibrium,

Equilibrium in 1, 2 directions are satisfied automatically. → No Body Force.

in 3 direction. $\boxed{w \text{ is harmonic.}}$

↳ equilibrium is satisfied.

$$\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} = 0 \Rightarrow \text{Traction free BCs on the side of the Bar}$$

$$\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} = 0 \leftarrow \text{in some sense, anti-plane shear.}$$

$$\nabla^2 w = 0.$$

$\tau_{ij} n_j = 0$ (traction free).

$$\underline{t} = \frac{dx_1}{ds} e_1 + \frac{dx_2}{ds} e_2$$

$$\underline{n} = \frac{dx_2}{ds} e_1 - \frac{dx_1}{ds} e_2$$

$$\tau_{ij} n_j = 0$$

$$\Downarrow$$

$$\boxed{\tau_{31} \underline{n}_1 + \tau_{32} \underline{n}_2 = 0} \quad (\text{BCs}).$$

$$\Downarrow$$

$$\boxed{\tau_{31} \cdot \frac{dx_2}{ds} - \tau_{32} \frac{dx_1}{ds} = 0}$$

$$G[-\alpha x_2 + w_{,1}] \frac{dx_2}{ds} - G[\alpha x_1 + w_{,2}] \frac{dx_1}{ds} = 0.$$

(we can cancel the G).

$$\underbrace{w_{,1} \frac{dx_2}{ds} - w_{,2} \frac{dx_1}{ds}}_{\text{grad: } \nabla w \cdot \underline{n}} = \alpha\left[x_1 \frac{dx_1}{ds} + x_2 \frac{dx_2}{ds}\right]$$

$$\text{grad: } \underbrace{\nabla w \cdot \underline{n}}_{\frac{dw}{dn}} = \frac{\alpha}{2}\left[\frac{d(x_1^2 + x_2^2)}{ds}\right]$$

Traction free BCs

$$\frac{dw}{dn} = \frac{\alpha}{2} \frac{d(x_1^2 + x_2^2)}{ds} \quad > \text{BCs for Laplace}$$

$\nabla^2 w = 0.$ ↖ Neumann BCs.

guarantee this *

$$\oint_C \frac{dw}{dn} ds = 0 \quad (\text{existence of solution}) \quad \text{condition to be satisfied}$$

||

automatically satisfied. → $\frac{dw}{dn}$

$$M = \alpha G \underbrace{\iint_A [x_\alpha x_\alpha + x_1 w_{,2} - x_2 w_{,1}]\, dx_1 dx_2}_{K:\ \text{torsional stiffness}}$$

$\boxed{M = k\alpha}$

w is harmonic

$\Updownarrow$

w is the real / Im part of an analytical function.

$f(z) = w + i\phi$ $i = \sqrt{-1}$

$z = x_1 + i x_2$

w, ϕ are related by CR Eqns.

$$\begin{cases} w_{,1} = \phi_{,2} \\ w_{,2} = -\phi_{,1} \end{cases}$$

$\sigma_{13} = G[-\alpha x_2 + w_{,1}] = G[-\alpha x_2 + \phi_{,2}]$

$\sigma_{23} = G[\alpha x_1 + w_{,2}] = G[\alpha x_1 - \phi_{,1}]$

Note: w, ϕ, is harmonic, so ϕ also satisfy $\nabla^2\phi = 0$.

Check. Equilibrium Eqn is automatically satisfied

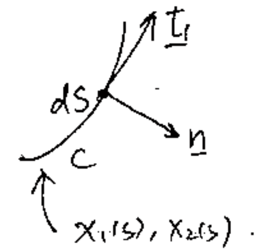
$\sigma_{13} n_1 + \sigma_{23} n_2 = 0 \Rightarrow$ BCs

$\sigma_{13}\frac{dx_2}{ds} - \sigma_{23}\frac{dx_1}{ds} = 0$.

Wed, Nov. 3rd, Week 12

REVIEW. Displacement. $\rightarrow \nabla^2 w = 0$

TRACTION FREE BCs: $\frac{dw}{dn} = x_2 n_1 - x_1 n_2$



$= x_2\frac{dx_2}{ds} + x_1\frac{dx_1}{ds} = \frac{1}{2}\frac{d(x_1^2+x_2^2)}{ds}$

$\underline{t} = \frac{dx_1}{ds}\underline{e}_1 + \frac{dx_2}{ds}\underline{e}_2$

$\underline{n} = \frac{dx_2}{ds}\underline{e}_1 - \frac{dx_1}{ds}\underline{e}_2$.

$$\begin{cases} \sigma_{13} = \frac{G}{2}\alpha\left[-x_2 + w_{,1}\right] \\ \sigma_{23} = \frac{G}{2}\alpha\left[x_1 + w_{,2}\right] \end{cases} \qquad \begin{cases} u_1 = -\alpha x_2 x_3 \\ u_2 = \alpha x_1 x_3 \\ u_3 = \alpha w(x_1, \cancel{x_2}) \end{cases}$$

A different approach.

$f(z) = w + i\phi$ ← conjugate harmonic function to w.

$w_{,1} = \phi_{,2}$ & $w_{,2} = -\phi_{,1}$ CR

$\nabla^2\phi = 0$

$\sigma_{13} = \frac{G}{2}\alpha\left[-x_2 + \phi_{,2}\right]$, $\sigma_{23} = \frac{G}{2}\alpha\left[x_1 - \phi_{,1}\right]$

T.F. BCs $\sigma_{31} n_1 + \sigma_{32} n_2 = 0 \Rightarrow [-x_2 + \phi_{,2}]n_1 + [x_1 - \phi_{,1}]n_2 = 0$

$\underbrace{-x_2\frac{dx_2}{ds} + x_1\left(-\frac{dx_1}{ds}\right)}_{\frac{d(x_1^2+x_2^2)}{ds}}$ $-\phi_{,1} n_2 + \phi_{,2} n_1 = 0$.

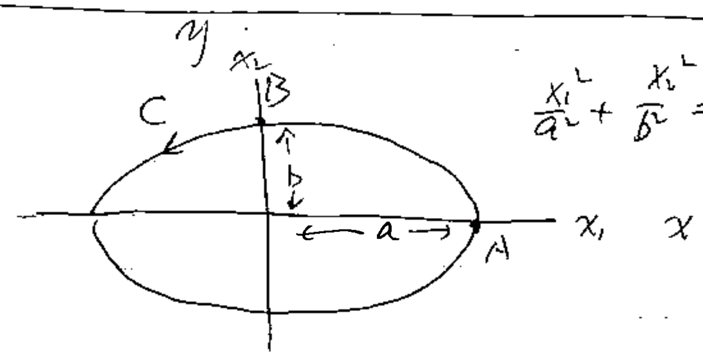
$\underbrace{\phi_{,1} t_1 \quad \phi_{,2} t_2}_{\nabla\phi\cdot\underline{t} = \frac{d\phi}{ds}}$

$$\Rightarrow \frac{d\phi}{ds} = \frac{d(x_1^2+x_2^2)}{ds} \Rightarrow \phi = \frac{x_1^2+x_2^2}{2} + \text{const} \Rightarrow \text{Curve } C$$

Set const $=0$.

BC: $$\boxed{\begin{array}{l} \phi = \dfrac{x_1^2+x_2^2}{2} \quad \text{on } C \\ \nabla^2\phi = 0 \end{array}}$$

Example



$$\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=1$$

$f(z) = w + i\phi$.

$f(z) = i(\underbrace{C^2}_{\text{const}})z^2$, $z = x+iy$ $\Downarrow$ $z^2 = x^2-y^2+2ixy$.

$\vee$

$iC^2z^2 + ik^2 \rightarrow$ some other const.

C k k are real numbers.

hopefully satisfy the boundary conditions.

$$= \underbrace{iC^2(x^2-y^2) + ik^2}_{\text{imaginary part}} \underbrace{-2C^2xy}_{\text{real part}}$$

BC on C: $C^2(x^2-y^2) - k^2 = \dfrac{x^2+y^2}{2}$

$$k^2 = \underbrace{x^2\left[\tfrac{1}{2} - C^2\right]}_{\to \frac{1}{a^2}} + \underbrace{y^2\left[\tfrac{1}{2}+C^2\right]}_{\to \frac{1}{b^2}}$$

$$C^2 = \frac{1}{2}\,\frac{a^2-b^2}{a^2+b^2} \qquad k^2 = \frac{a^2b^2}{a^2+b^2}$$

that means $\rightarrow \phi$ $\Downarrow$ ∇

$$\sigma_{13} = G\alpha[-x_2 + \phi_{,2}]$$
$$\sigma_{23} = G\alpha[x_1 - \phi_{,1}]$$

$$\begin{cases} \sigma_{13} = \dfrac{-2G\alpha a^2 y}{a^2+b^2} \\ \sigma_{23} = \dfrac{2G\alpha b^2 x}{a^2+b^2} \end{cases}$$

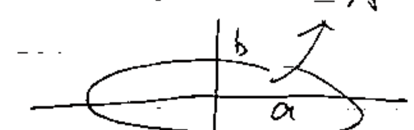
$w = -2C^2xy$ Also know waping function.

Note — $C=0$, $a=b$, circle

$$M = \iint_A (\sigma_{23}x - \sigma_{13}y)\,dA$$
$$= \frac{G\pi\alpha b^3a^3}{a^2+b^2} = k\alpha$$

$\hookrightarrow k = G(\pi ab)\dfrac{a^2b^2}{(a^2+b^2)}$

$A = \pi ab$



how to calculate waping function / stresses in torsion

Prandtl stress function approach:

$$\nabla^2\phi = 0$$
$$\phi = \frac{1}{2}(x^2+y^2), \text{ on } C$$

Define a function: $\Phi = \phi - \frac{1}{2}(x^2+y^2)$

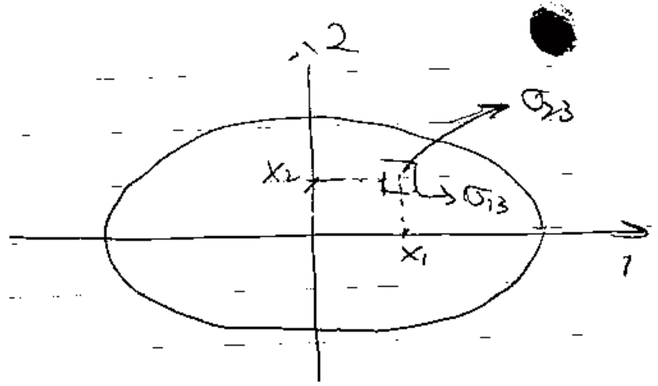
on the BCs, $\Phi = 0$ on C

$\nabla^2\Phi = -2 \rightarrow$ Poisson's equation

$$\phi = \Phi + \frac{1}{2}(x^2+y^2) \quad > \quad \begin{cases} \sigma_{13} = G\alpha\,\Phi_{,2} \\ \sigma_{23} = -G\alpha\,\Phi_{,1} \end{cases}$$

Calculate the moment:

$M = \alpha 2G \iint_A \Phi \, dA$



Constant Φ curve

$\Phi = c$

the gradient of Φ, normal to surface

$\nabla\Phi = \Phi_{,1} e_1 + \Phi_{,2} e_2$

$\underline{\tau} = G\alpha$

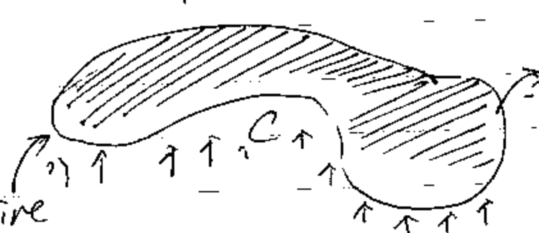
$[\Phi_{,2} e_1 - \Phi_{,1} e_2] \Rightarrow \underline{\tau} \cdot \nabla\Phi = 0$

The constant line are the direction of shear vector

lines of shear stresses

$\nabla \cdot \Phi = \Phi_{,1}^2 + \Phi_{,2}^2 \propto [|\tau|/G\alpha]$

Prandtl soap film analogy

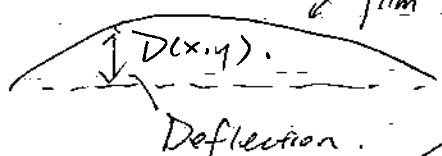


$T \Rightarrow$ surface tension.

$(\approx$ const$)$

$\hookrightarrow \frac{\text{Force}}{\text{length}}$

film deformed.



$\nabla^2 D = -\frac{P}{T}$ pressure

Small membrane deflection formula

define new const.

$\frac{1}{2}d = \frac{DT}{P}$

$D = \frac{dP}{2T}$

$\nabla^2 \phi = -2$

Similar !!

also $D = 0$ on C

$\nabla^2 d = -2 \Rightarrow d = 0$ on C

Solid Mechanics

Nov. 8, Mon. Week 12.

Linear viscoelasticity

ideal model:

Deformation $\not\propto$ stress history.

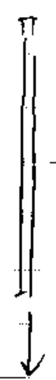
real material:

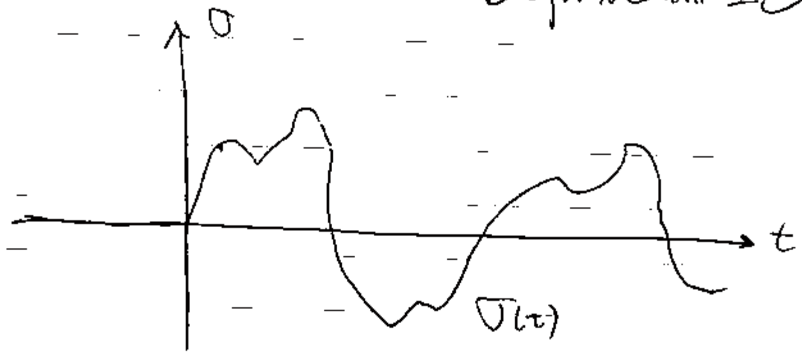
Deformation $= D$ (history).

Cartoon Models

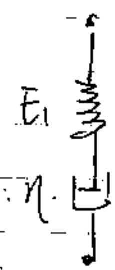
- uniaxial tension test.

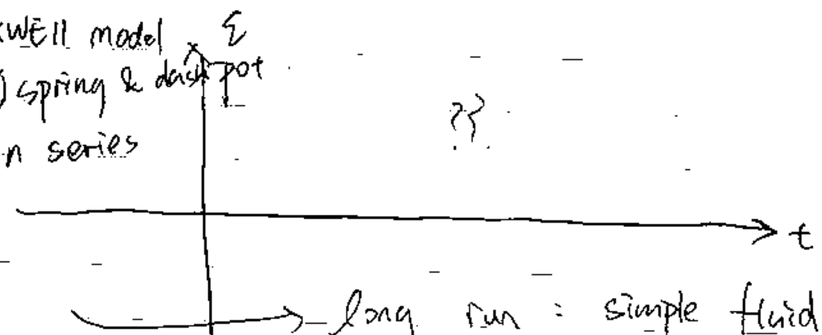
$\uparrow \sigma(t)$



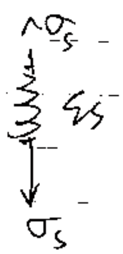


- MAXWELL model: spring & dashpot in series

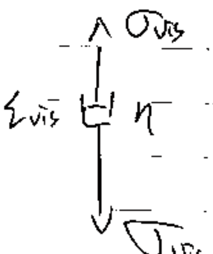




long run : simple fluid.
(suddenly apply a stress)



$\sigma_s = E\,\varepsilon_s$



$\dot{\varepsilon}_{vis} = \dfrac{\sigma_{vis}}{\eta}$.

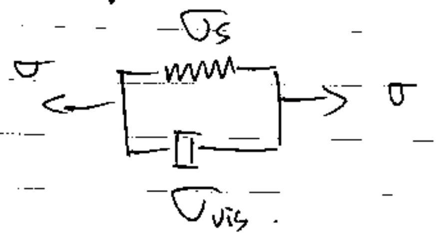
$\varepsilon = \dfrac{\sigma}{E}$

$\dot{\varepsilon}_{vis} = \dfrac{\sigma}{\eta}$

Same since they are in series.

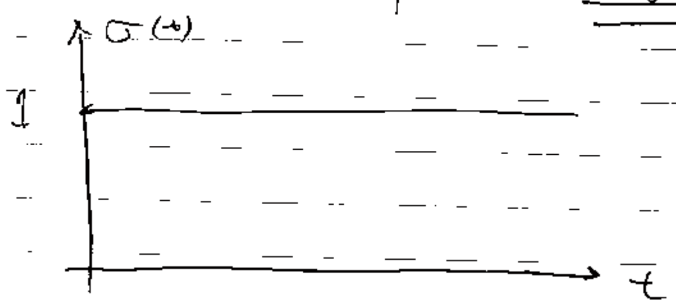
$$\boxed{\underbrace{\dot{\varepsilon}}_{\dot{\varepsilon}_s + \dot{\varepsilon}_{vis}} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\varepsilon}} \Rightarrow \text{ODE in time}$$

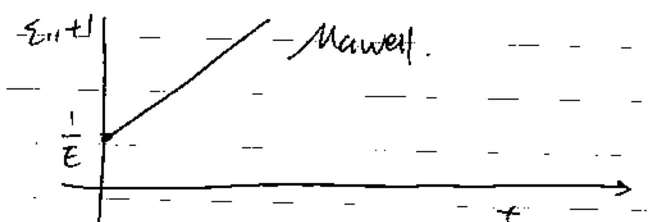
Voigt model



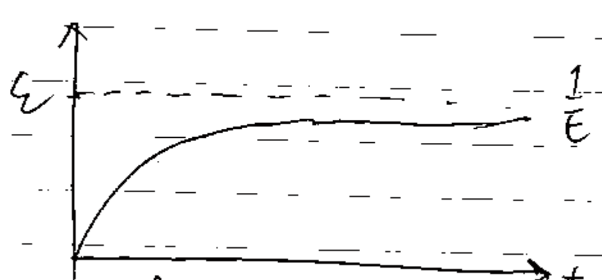
$\sigma = \sigma_s + \sigma_{vis}$

- $\sigma = E\varepsilon + \eta\dot{\varepsilon}$ => Voigt model





Solve a ODE to get this curve.



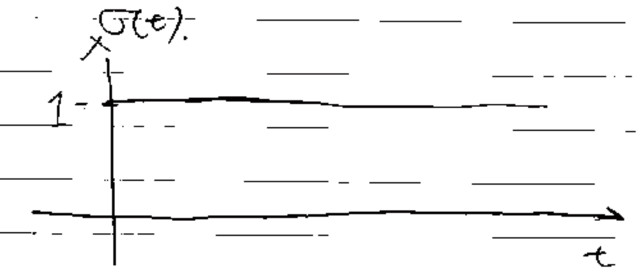
$t > 0$, $\Rightarrow \eta\dot{\varepsilon} + E\varepsilon = 1$

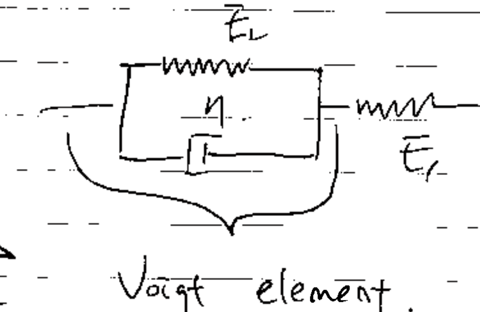
$\varepsilon = A e^{-\frac{E}{\eta}t} + \frac{1}{E}$

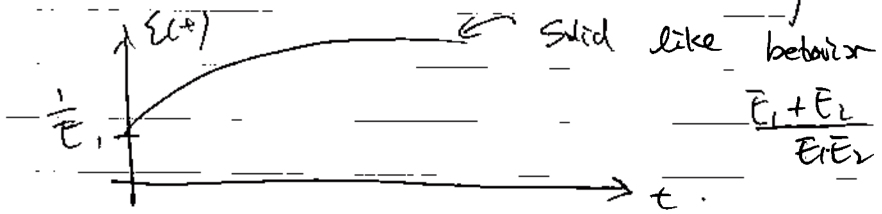
$= \frac{1}{E}\left[1 - e^{-\frac{E}{\eta}t}\right]$

Short time : fluid ; & long time solid

- Standard model







$\frac{E_1 + E_2}{E_1 E_2} = \frac{1}{E_2} + \frac{1}{E_1}$

Solid behavior for both long & short time

$$\underbrace{\dot{\sigma} + \frac{E_1 + E_2}{\eta}\sigma = E_1\dot{\varepsilon} + \frac{E_2 E_1}{\eta}\varepsilon}_{\text{3 parameters to determine}}$$ Linear ODE

- Concept of Creep function.

Creep function $C(t)$ strain history due to a unit stress $\sigma = H(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$

$C(t) = \left(\frac{1}{E} + \frac{t}{\eta}\right)H(t)$. <=> Maxwell

$C(t) = \frac{1}{E}\left(1 - e^{-\frac{Et}{\eta}}\right)H(t)$. <=> Voigt.

$C(t) = \underbrace{\frac{E_1 + E_2}{E_1 E_2}}_{1/E_\infty} - \frac{1}{E_2} e^{-E_2 t/\eta}$

$\frac{E_2}{\eta}$ units : $[\frac{1}{t}]$

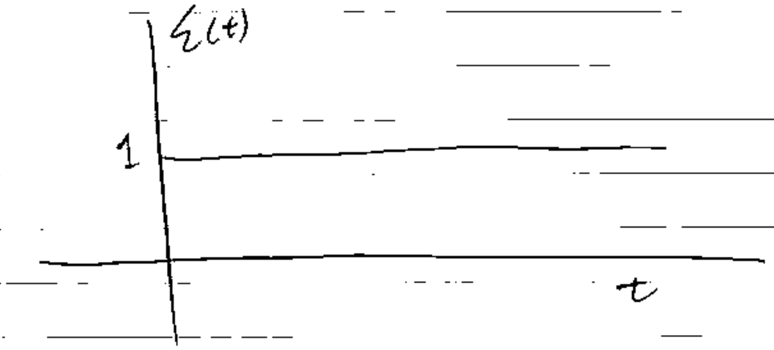
t_c <=> Creep relaxation time

$C(t=0) = \frac{1}{E_1}$ ← short time modulus.

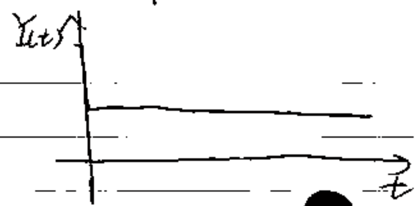
$C(t=\infty) = \frac{1}{E_2} + \frac{1}{E_1} = \frac{1}{E_\infty}$

↑ long time modulus

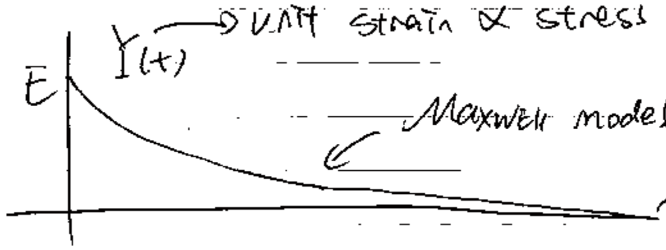
$\frac{E_\infty}{E_1} = 10^{-3}$.



Voigt element



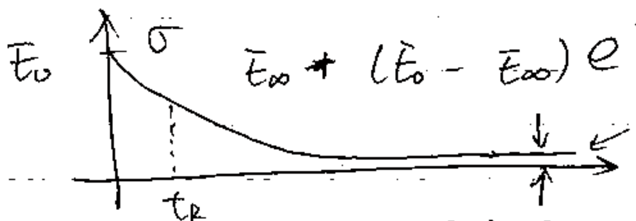
$\varepsilon(t) = H(t)$.



→ fluid.

$Y(t) = E e^{-\frac{E}{\eta}t} H(t)$.

Für standard solid:



$E_\infty + (E_0 - E_\infty) e^{-\frac{t}{t_R}}$

$t_R = \cancel{\frac{E_1 + E_2}{\eta}}$, $E_0 = E_1$

$\hookrightarrow \frac{\eta}{E_1 + E_2}$ (typo)

Boltzmann superposition principle

① Assume system is linear.

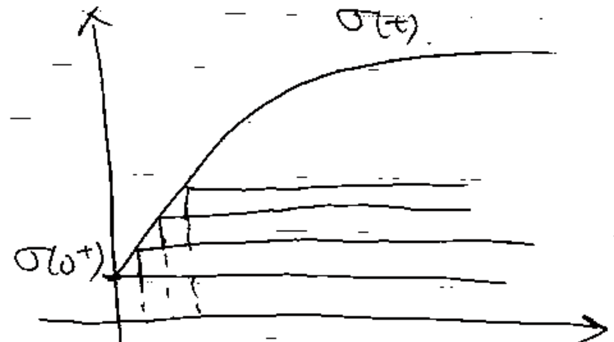
② Assume Causuality.

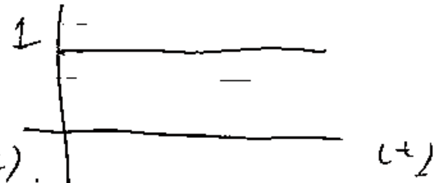
③ Non-Aging.

Creep.

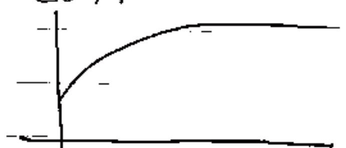
Given $C(t)$

Response of system due to a unit stress step function.





$\sigma(t) = \sigma(0^+) H(t) + \sigma'(\Delta t) H(t-\Delta\tau) + \sigma'(\Delta\tau)\Delta\tau H(t+\Delta\tau)$.



$\varepsilon(t) = \sigma(0^+) C(t) + \sigma'(0)\Delta\tau\, C(t-\Delta\tau) + \sigma'(\Delta\tau)\Delta\tau\, C(t-2\Delta\tau) H(t-2\Delta\tau) + \ldots$

$= \sigma(0^+) C(t) + \int_{0^+}^{t} \sigma'(\tau) C(t-\tau)\, d\tau$.

Nov. 10, Wed, Week 12.

Review: Boltzmann superposition –

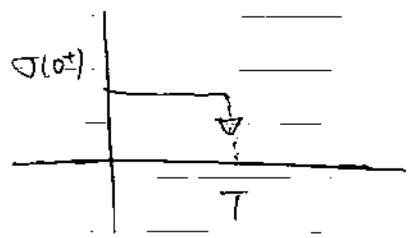
Any stress history can be broken down into sum of step functions.

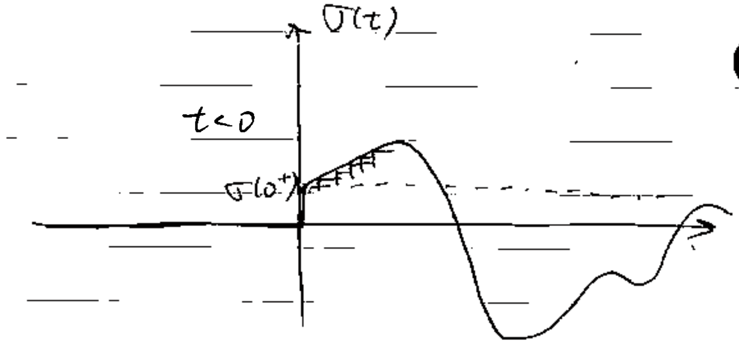
$\downarrow$ Each step fct. $H(t-\tau)$

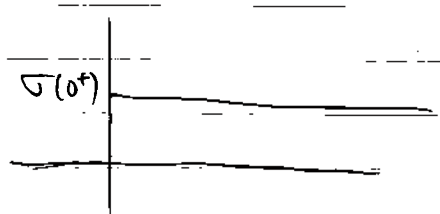
$C(t-\tau)$.

Real linear viscoelastic model for uniaxial tension $\Rightarrow$ $\varepsilon(t) = \sigma(0^+)C(t) + \int_{0^+}^{t} C(t-\tau)\frac{d\sigma}{d\tau}\,d\tau$

Do a simple example

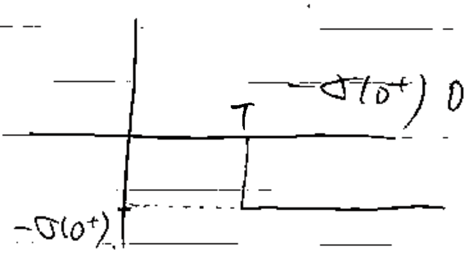




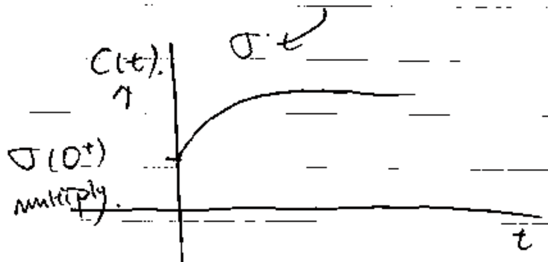


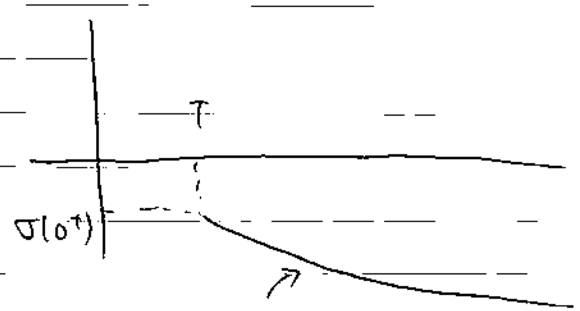
$\sigma(0^+)H(t)$

σ

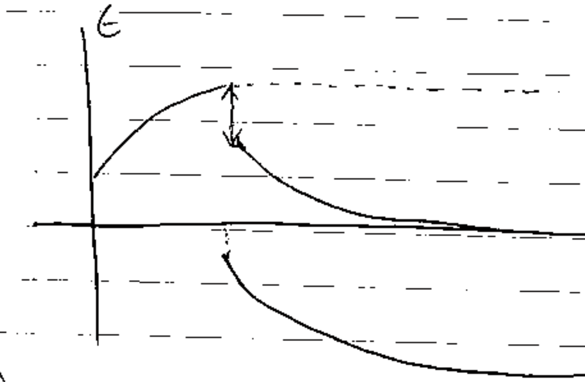


$-\sigma(0^+)\,0\,(t-T)H(t-T)$.





$-\sigma(0^+)\,C(t-\tau)H(t-\tau)$.



[Convolution product $f(t)$, $g(t)$ define for $t \in (0, \infty)$.

$$f * g = \int_0^t f(t-\tau)\,g(\tau)\,d\tau = \int_0^t g(t-\tau)\,f(\tau)\,d\tau.$$

(1) $\boxed{\varepsilon = \sigma(0^+)C(t) + C * \sigma'}$ $\quad \sigma' \equiv \frac{d\sigma}{dt}$

The Laplace transform of a function f defined in zero to infinity

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt.$$

$\Downarrow$

$\tilde{f}(s)$ $\Rightarrow$ is a function of s.

$\hookrightarrow$ s in the transform variable

$\hookrightarrow$ s is in general a complex variable

Properties.

$\mathcal{L}[\dot{f}(t)] = s\tilde{f}(s) - f(0^+)$

$\mathcal{L}[f * g] = \tilde{f}(s)\,\tilde{g}(s)$ $\leftarrow$ convolution Theorem

Laplace transform ① :

$$\tilde{\varepsilon}(s) = \sigma(0^+)\tilde{Y}(s) + \underbrace{\mathcal{L}[Y * \sigma']}_{\tilde{C}(s)\,\underbrace{\mathcal{L}[\sigma'(t)]}_{[s\tilde{\sigma}(s) - \sigma(0^+)]}}$$

clear & linear → $\boxed{\tilde{\varepsilon}(s) = s\tilde{C}(s)\tilde{\sigma}(s)}$ ②

transform domain

Messy => $\varepsilon(t) = \sigma(0^+)C(t) + \int_{0^+}^{t} C(t-\tau)\frac{d\sigma}{d\tau}\,d\tau$.

Do the same thing with Relaxation function $Y(t)$.

Apply a strain history $\varepsilon(t)$ ← given

$$\sigma(t) = \varepsilon(0^+)Y(t) + \int_{0^+}^{t} Y(t-\tau)\frac{d\varepsilon}{d\tau}\,d\tau.$$

$\boxed{\tilde{\sigma}(s) = s\tilde{Y}(s)\tilde{\varepsilon}(s)}$ ③ on the transform plane, — it's trivial.

Combine ② & ③

$$\hat{\sigma}(s) = s\hat{Y}(s).\, s\tilde{C}(s)\,\hat{\sigma}(s).$$

=> $s\tilde{Y}(s)\cdot\tilde{C}(s) = 1$ ← related

$$\tilde{Y}(s) = \frac{1}{s^2\tilde{C}(s)}.$$

$$Y(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2\hat{C}(s)}\right]$$

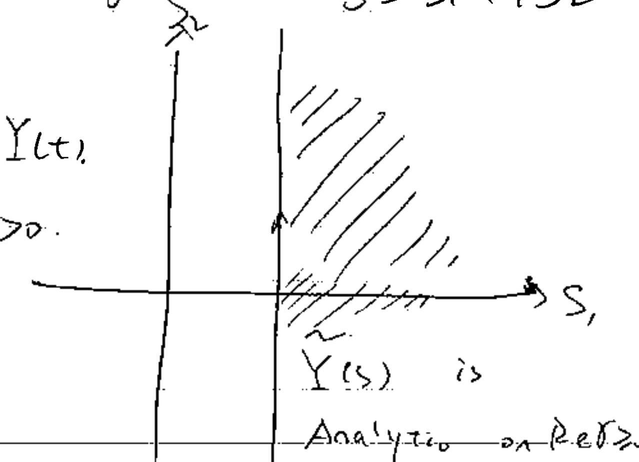
↳ $\tilde{Y}(s).\hat{C}(s) = \frac{1}{s^2}$

$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$.

$$\int_0^t Y(t-\tau)C(\tau)\,d\tau = t \qquad t \geq 0.$$

(Bromwich Integral) $\qquad S = S_1 + iS_2$

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \tilde{Y}(s)\,e^{st}\,ds = Y(t), \quad t>0.$$



Isotropic Linear viscoelastic solid.

$$\sigma_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}.$$

MEMORIZE THIS !!!

OR

$$\varepsilon_{ij} = \frac{\sigma_{ij}}{2G} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \qquad E = \frac{G}{2(1+\nu)}$$

deviatoric stress tensor

$$S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}$$

strain deviator tensor

in literature, $\frac{1}{3}\sigma_{kk} = \sigma$ (volumetric part) ; $\varepsilon_{kk} = e$ (change of volume)

e_{ij}

shear Modulus (relaxation modulus)

$$\boxed{\begin{cases} S_{ij} = 2G\, e_{ij} \\ \sigma_{kk} = 3K\, \varepsilon_{kk} \end{cases}}$$

(relaxation) Bulk Modulus.

exactly the SAME

$$\begin{cases} \sigma_{ij} = 2G\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\delta_{ij} \\ \varepsilon_{ij} = \dfrac{\sigma_{ij}}{2G} - \dfrac{\nu}{E}\sigma_{kk}\delta_{ij} \end{cases}$$

$$S_{ij} = e_{ij}(0^+)\,Y_1(t) + \int_{0^+}^{t} Y_1(t-\tau)\frac{\partial e_{ij}}{\partial \tau}\,d\tau$$

$Y_1(t) \longleftrightarrow 2G$

$Y_2 \longleftrightarrow 3K$

$$\sigma_{kk} = \varepsilon_{kk}(0^+)\,Y_2(t) + \int_{0^+}^{t} Y_2(t-\tau)\frac{\partial \varepsilon_{kk}}{\partial \tau}\,d\tau$$

Consider $\mathcal{L}$

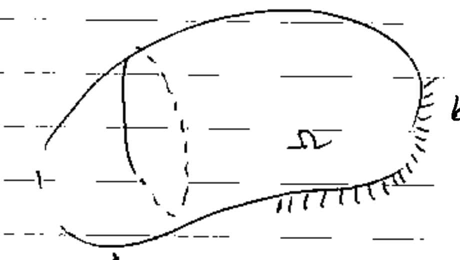
$$\boxed{\begin{aligned} \tilde{S}_{ij} &= s\,\hat{Y}_1(s)\,\tilde{e}_{ij} \\ \tilde{\sigma}_{kk} &= s\,\hat{Y}_2(s)\,\tilde{\varepsilon}_{kk} \end{aligned}}$$

Linear Elasticity: $\sigma_{ij,j} = 0$ — No body Force

$\varepsilon_{ij} = \dfrac{u_{i,j} + u_{j,i}}{2}$, kinematics

Constitutive model.
$$\begin{cases} S_{ij} = e_{ij}(0^+)\,Y_1(t) + Y_1 * \acute{e}_{ij} \\ \sigma_{kk} = \varepsilon_{kk}(0^+)\,Y_2(t) + Y_2 * \acute{\varepsilon}_{kk} \end{cases}$$

General Problem



given

$u(x,t) = f(x,t)$

$\sigma_{ij}n_j = T_i(x,t)$ — given — involve time

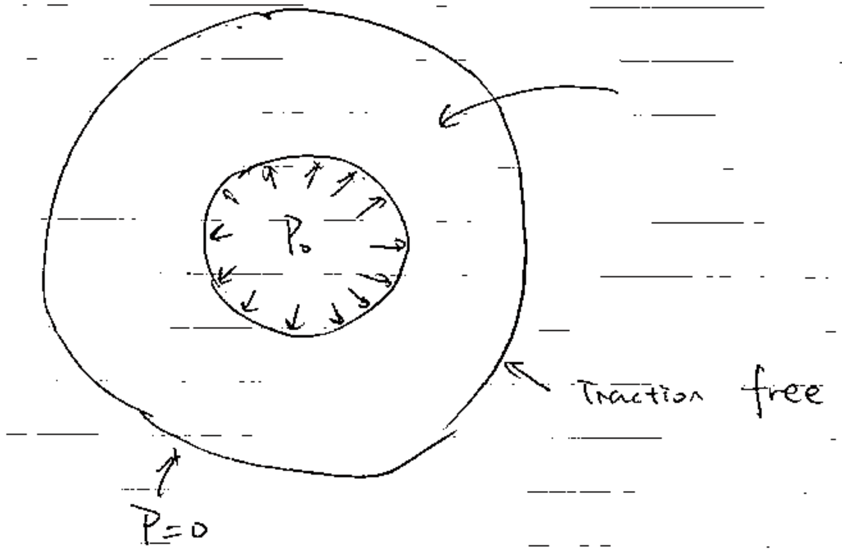
Linear Elasticity
$$\begin{cases} \tilde{\sigma}_{ij,j} = 0 \\ \tilde{\varepsilon}_{ij} = \dfrac{\tilde{u}_{i,j} + \tilde{u}_{j,i}}{2} \\ \tilde{S}_{ij} = \underbrace{s\,\tilde{Y}_1(s)}_{2G}\,\tilde{e}_{ij}(s) \end{cases}$$

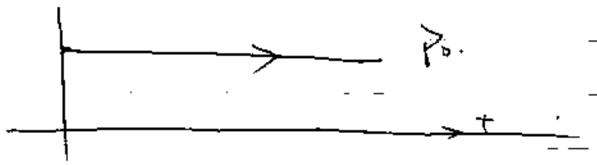
$\Downarrow$ correspondence principle

$$\tilde{\sigma}_{kk} = \overbrace{s\,\tilde{Y}_2(s)}^{3K}\,\tilde{\varepsilon}_{kk}(s)$$

BC
$$\begin{cases} \hat{u}(x,s) = \tilde{f}(x,s) \\ \tilde{\sigma}_{ij}n_j = \tilde{T}_i(x,s) \end{cases}$$

SIMPLE Example



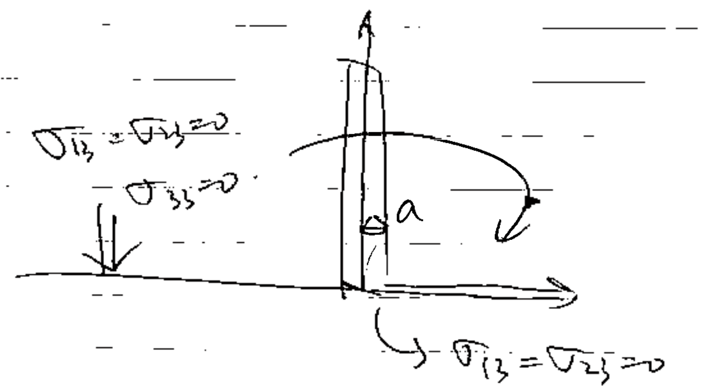


$$\text{BCs}\begin{cases}\sigma_{rr}(r=a,\ t>0)=P_0\\ \sigma_{rr}(r=b,\ t\geq 0)=0\end{cases}$$

$$sY(s)\,C_1(s)\,s=1$$

$$\tilde{e}_{ij}(s)=sC_1(s)\tilde{S}_{ij}$$

$$e_{ij}=C_1(t)\tilde{S}_{ij}(0^+)+\int_{0^+}^{t}C_1(t-\tau)\underbrace{\frac{\partial S_{ij}}{\partial\tau}}_{=0}d\tau$$



$$u_r=\Delta g\left(\frac{r}{a},0\right)$$

$$u_3(x_1,x_2,x_3=0)=-\Delta$$

$$|x_1^2+x_2^2|<1$$

$$\sigma_{ij}=\frac{G\Delta}{a}f\left(\frac{r}{a},0\right)$$

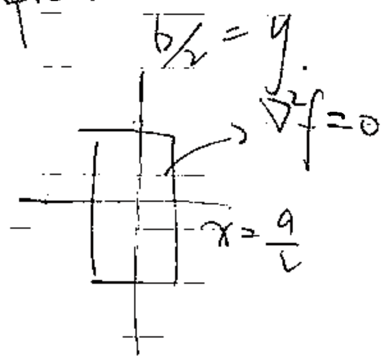
Office Hour.

$$\psi\to\phi$$

$$\nabla^2\phi=0 \qquad \phi=\frac{x^2+y^2}{2}$$

formulate

$f \to$ BCs to simple.



$$f\big|_{BCs}=0$$

$f\to$ harmonic

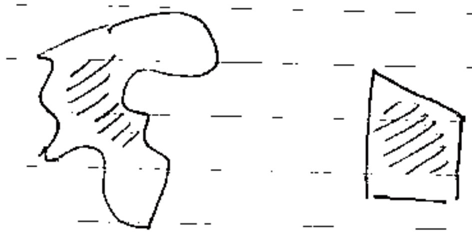
$\hookrightarrow$ separation of variables

Differential EQs.

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0\Rightarrow\frac{\partial^2\phi}{\partial x^2}=-\frac{\partial^2\phi}{\partial y^2}$$

$$f=\frac{\partial^2\phi}{\partial x^2}+1$$

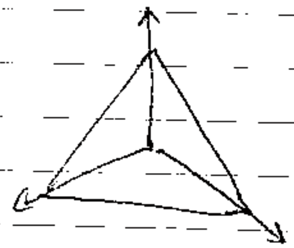
Example:



PRANDTL: Stress function

Three common formulations for solving torsion: w, ϕ, Φ_p.

Solution for Laplacian → Harmonic Analysis.



November 15, 2021. Week 13. Mon.

REVIEW: Correspondence principle.

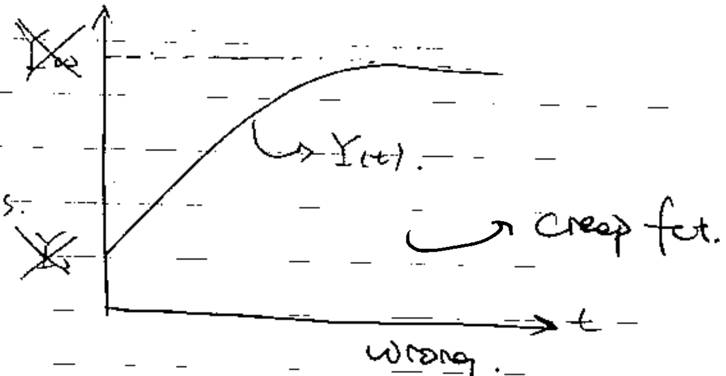
$$Y(t) = Y_\infty + (Y_0 - Y_\infty)\sum_{j=1}^{n} a_j e^{-t/t_j}$$

$Y_\infty = Y(t=\infty)$ long time Modulus.

$Y_0 = Y(t=0)$. Instantaneous Modulus.

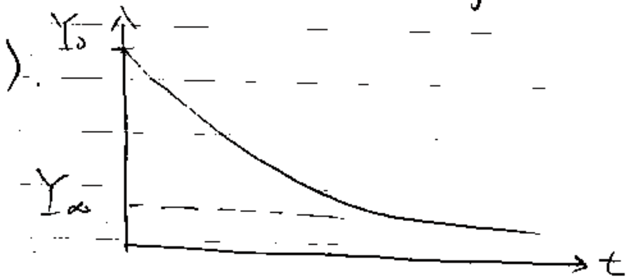
$$\sum_{j=1}^{n} a_j = 1$$

t_j = Relaxation times.

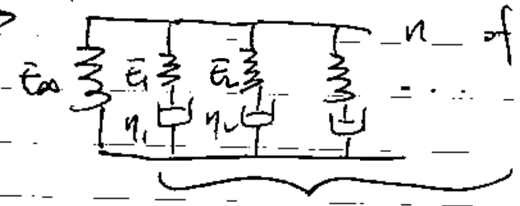


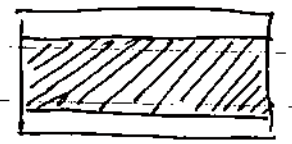
Power law model.

$$Y(t) = Y_\infty + (Y_0 - Y_\infty)\frac{1}{1+(t/t_0)^\alpha}$$



Corresponds to



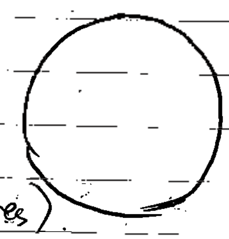


Rheology.

Shear strain: $\varepsilon = \varepsilon_0 e^{i\omega t}$.

* What is the response.

(long time / steady state res)



Shear Stress $\sigma(t) = \varepsilon(0^+)\, Y_1(t) + \int_{0^+}^{t} Y_1(t-\tau)\frac{d\varepsilon}{d\tau}\, d\tau$

ε_0

$\varepsilon_0 i\omega e^{i\omega\tau}$

$= \varepsilon_0 Y_1(t) + i\omega\varepsilon_0 \int_0^t Y_1(t-\tau)\, e^{i\omega\tau}\, d\tau$

$\varepsilon_0 Y_1(t) + i\omega\varepsilon_0 \int_{0^+}^{t} \left[Y_1(t-\tau) - Y_1(t) \right] e^{i\omega\tau}\, d\tau$

$$\underbrace{+\, i\omega\varepsilon_0 Y_1(t) \int_{0^+}^{t} e^{i\omega\tau}\, d\tau}_{i\omega\varepsilon_0 Y_1(t)\cdot \frac{e^{i\omega\tau}}{i\omega}\Big|_{0^+}^{t}}$$

$= \varepsilon_0 Y_1(t) e^{i\omega t} - \varepsilon_0 Y_1(t)$

$= \varepsilon_0 Y_1(t)\, e^{i\omega t} + i\omega\varepsilon_0 \int_0^t \left[Y_1(t-\tau) - Y_1(t) \right] e^{i\omega\tau}\, d\tau$

$\int_0^t Y_1(t-\tau)\, e^{i\omega\tau}\, d\tau$. $\quad \eta = t-\tau$.

$\tau = t-\eta$.

$= \int_t^0 Y_1(\eta)\, e^{i\omega(t-\eta)}\, (-d\eta)$

$= \int_0^t Y_1(\eta)\, e^{i\omega(t-\eta)}\, d\eta = e^{i\omega t}\int_0^t Y_1(\eta)\, e^{-i\omega\eta}\, d\eta$

$\int_0^t Y_1(t)\cdot e^{i\omega\tau}\, d\tau$.

$\to Y_1(t)\cdot e^{i\omega t} \int_0^t e^{i\omega\eta}\, d\eta$.

$\sigma(t) = \varepsilon_0 Y_1(t)\cdot e^{i\omega t} + i\omega e^{i\omega t}\int_0^t \left[Y_1(\eta) - Y_1(t) \right] e^{i\omega\eta}\, d\eta$

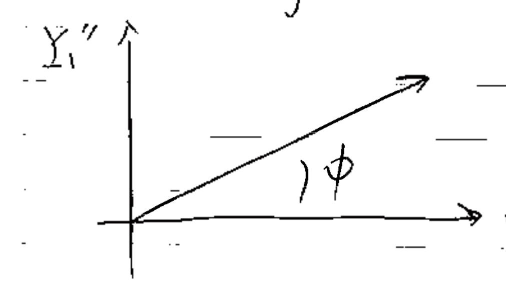
$= e^{i\omega t}\left[\varepsilon_0 Y_1(t) + i\omega \int_0^t \left[Y_1(\eta) - Y_1(t) \right] e^{-i\omega\eta}\, d\eta \right]$

$\xrightarrow{t\to\infty}$ Converge to $\underbrace{\varepsilon_0 Y_1(\infty) + i\omega \int_0^\infty \left[Y_1(\eta) - Y_1(\infty) \right] e^{-i\omega\eta}\, d\eta}_{\hat{Y}_1(\omega)}$

Complex Modulus

$\sigma(t\to\infty) = \varepsilon_0\, e^{i\omega t}\, \hat{Y}_1(\omega)$ $\to \cos\omega t + i\sin\omega t$

$\hat{Y}_1 = \underbrace{\hat{Y}_1'(\omega)}_{\text{Storage modulus}} + i\, \underbrace{\hat{Y}_1''(\omega)}_{\text{lost modulus}}$



$\tan\phi = \frac{\hat{Y}_1'(\omega)}{\hat{Y}_1''(\omega)}$ $\to$ loss tangent

$$\hat{Y}_1'(\omega) = Y_1(\infty) + \omega \int_0^{\infty} [Y_1(\eta) - Y_1(\infty)] \sin(\omega\eta)\, d\eta.$$

$$\hat{Y}_1''(\omega) = \omega \int_0^{\infty} [Y_1(\eta) - Y_1(\infty)] \cos(\omega\eta)\, d\eta.$$

$\hat{Y}_1'(\omega) = \hat{Y}_1'(-\omega) \rightarrow$ even fct. of ω

$\hat{Y}_1''(\omega) =$ odd fct. of ω

Change of energy in a cycle.

$$W = \int_{cycle} \sigma\, d\varepsilon$$

We apply $\varepsilon = \varepsilon_0 \left[\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right]$

$= \varepsilon_0 \cos(\omega t).$

$\varepsilon_0 \frac{e^{i\omega t}}{2} \rightarrow \varepsilon_0 \hat{Y}_1(\omega) e^{i\omega t} = \sigma(t)$

$\varepsilon_0 \frac{e^{-i\omega t}}{2} \rightarrow \frac{\varepsilon_0 \hat{Y}_1(-\omega) e^{-i\omega t}}{2} = \sigma(t).$

$\hat{Y}_1(-\omega) = \overline{\hat{Y}_1(\omega)}$

$$\sigma = \frac{\varepsilon_0}{2}\left[\hat{Y}_1(\omega) e^{i\omega t} + \overline{\hat{Y}_1(\omega) e^{i\omega t}}\right]$$

$$= \varepsilon_0 \left[\hat{Y}_1'(\omega) \cos\omega t - \hat{Y}_1''(\omega) \sin\omega t\right].$$

$\because e^{i\omega t}\hat{Y}_1(\omega) = [\hat{Y}_1'(\omega) + i\hat{Y}_1''(\omega)][\cos\omega t + i\sin\omega t].$

$Re(\quad) = \hat{Y}_1'(\omega) \cos\omega t - \hat{Y}_1''(\omega) \cdot \sin\omega t$

$d\varepsilon = \omega \varepsilon_0 \sin\omega t \cdot dt$

$$-\omega\varepsilon_0^2 \int_{cycle} [\hat{Y}_1'(\omega)\cos\omega t - \hat{Y}_1''(\omega)\sin\omega t] \sin\omega t\, dt$$

work = W in a cycle.

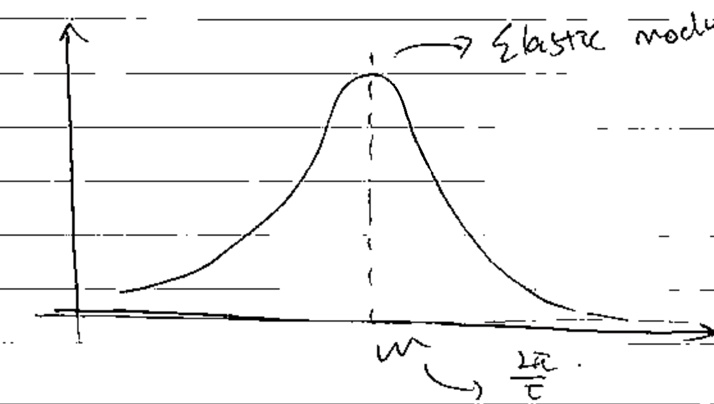
we already show: $\varepsilon_0 e^{i\omega t} \rightarrow \varepsilon_0 \hat{Y}_1(\omega) e^{i\omega t}$.

1st integral

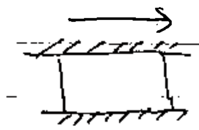
$$\omega\varepsilon_0^2 \hat{Y}_1''(\omega) \underbrace{\int_{cycle} \sin^2(\omega t)\, d\, t\omega}_{\text{Some number}}$$

$$\hat{Y}_1'(\omega) \underbrace{\int_{cycle} \cos\omega t\, d\sin\omega t}_{\frac{d\cos^2\omega t}{2\omega} = 0}$$

if you plot the Lost modulus @ this freq.



Nov. 17, 2021. Wed., Week 13.



$\varepsilon_0 e^{i\omega t}$.

Long time stress

$$\sigma = \varepsilon_0 \underbrace{\hat{Y}_1(\omega)}_{\text{complex modulus}} e^{i\omega t}$$

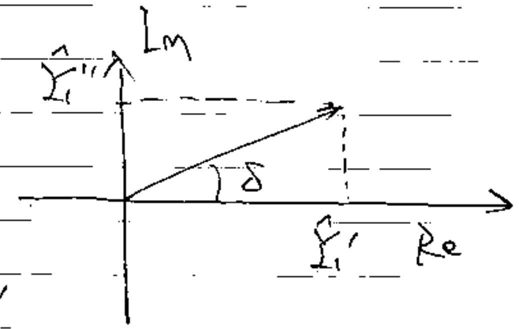
$$\hat{Y}_1(\omega) = \hat{Y}_1'(\omega) + i\,\hat{Y}_1''(\omega)$$

$i = \sqrt{-1}$

$\hat{Y}_1''$: loss modulus → odd function.

$\hat{Y}_1'$: Storage modulus → even function.

Energy loss per cycle $= \pi \varepsilon_0^2 \left| \hat{Y}_1''(\omega) \right|$



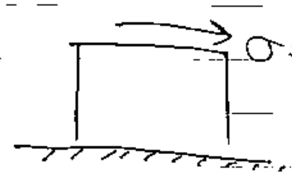
loss tangent

$$\hat{Y}_1 = |\hat{Y}_1| e^{i\delta}$$

$$\underbrace{\tan\delta}_{\text{loss tangent.}} = \frac{\hat{Y}_1''}{\hat{Y}_1'}$$

$$\underbrace{\hat{Y}_1(\omega)}_{\text{complex modulus.}} = \underbrace{Y_1(\infty)}_{\text{relaxation function in time domain}} + i\omega \int_0^\infty \left[Y_1(\eta) - Y_1(\infty) \right] e^{-i\omega\eta}\, d\eta$$

creep modulus.

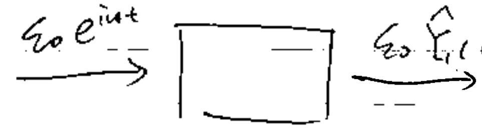


$\sigma = \sigma_0 e^{i\omega t}$.

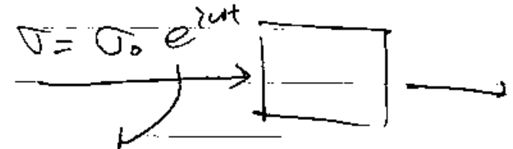
long time shear strain

$$\varepsilon = \sigma_0 \hat{C}_1(\omega) e^{i\omega t}.$$

$$\hat{C}_1(\omega) = C_1(\infty) + \int_0^\infty \left[C_1(\eta) - C_1(\infty) \right] e^{-i\eta\omega}\, d\eta$$



$\varepsilon_0 e^{i\omega t} \longrightarrow \varepsilon_0 \hat{Y}_1(\omega) e^{i\omega t} = \sigma$.



$\sigma = \sigma_0 e^{i\omega t} \longrightarrow \sigma_0 \hat{C}_1(\omega) e^{i\omega t}$.

$\sigma_0 = \varepsilon_0 \hat{Y}_1(\omega)$

$\hookrightarrow \varepsilon_0 \underbrace{\hat{Y}_1(\omega) \hat{C}_1(\omega)}_{1} e^{i\omega t}$

$$\hat{Y}_1(\omega) = \frac{1}{\hat{C}_1(\omega)}.$$

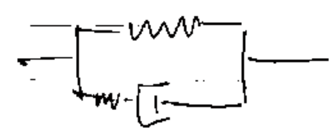
$$\int_0^t C_1(t-\tau) Y_1(\tau)\, d\tau = t.$$

$$s^2\, \tilde{C}_1 \tilde{Y}_1 = 1$$

↳ Laplace transform

$$\int_0^\infty e^{-st} C_1(t)\, dt = \tilde{C}_1(s)$$

Standard model

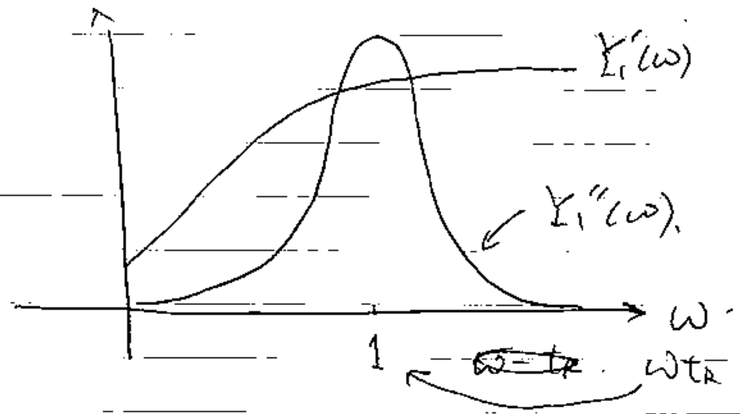


$$Y_1(t) = Y_\infty + (Y_0 - Y_\infty) e^{-t/t_R}$$

$Y_\infty \equiv Y_1(t \to \infty)$ → long time shear modulus

$Y_0 = Y_1(t \to 0)$ → short time

$$\begin{cases} \hat{Y}_1(\omega) = Y_\infty + \dfrac{i\omega(Y_0 - Y_\infty)}{t_R^{-2} + \omega^2} \\ \hat{Y}_1'(\omega) = Y_\infty + \dfrac{\omega^2 t_R^2 (Y_0 - Y_\infty)}{1 + \omega^2 t_R^2} \\ \hat{Y}_1''(\omega) = \dfrac{\omega t_R (Y_0 - Y_\infty)}{1 + \omega^2 t_R^2} \end{cases}$$



$[\omega] \sim \frac{1}{[t]}$

▷ Time Temp. Superposition

ω : 10^{-2} Radium/s + o 10^2 rad/s

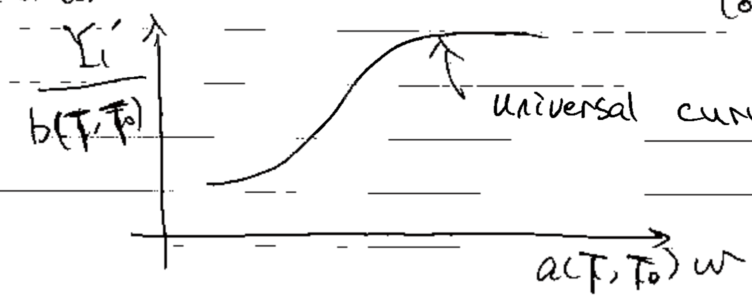
$$\begin{cases} Y_1'(\omega, T) = b(T, T_0)\,\phi_1(a(T,T_0)\,\omega) \\ Y_2'(\omega, T) = \underbrace{b(T, T_0)}\,\phi_2\,(\underbrace{a}(T, T_0)\,\omega) \end{cases}$$

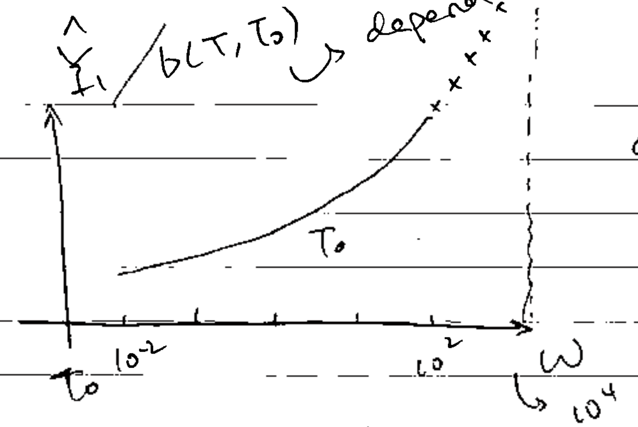
// Ref. Temp.

Shift Factors.

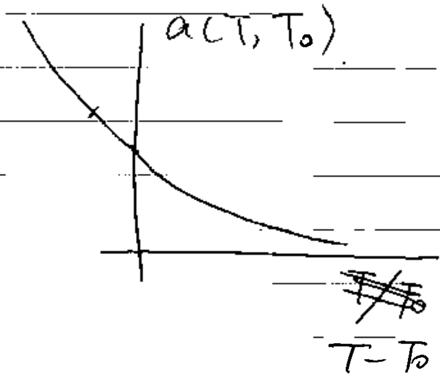
$$\frac{Y_1'}{b(t, t_0)} = \phi_1(a(t, t_0)\,\omega)$$

$\log \frac{\hat{Y}_1}{b(T, T_0)} = \log \hat{Y} - \log b(T - T_0)$





$$a(T, T_0) = \begin{cases} > 1, & T < T_0 \\ = 1, & T = T_0 \\ < 1, & T > T_0 \end{cases}$$



$\log[a(T, T_0)\,\omega]$

$= \log a(T, T_0) + \log(\omega)$.

$$\hat{Y}_1(\omega) = Y(\infty) + i\omega \int_0^\infty [Y_1(\eta) - Y_1(\omega)]\,e^{-i\omega\eta}\,d\eta$$

complex modulus

time domain (relaxation fct.)

Gwein

WLF - shift factor.

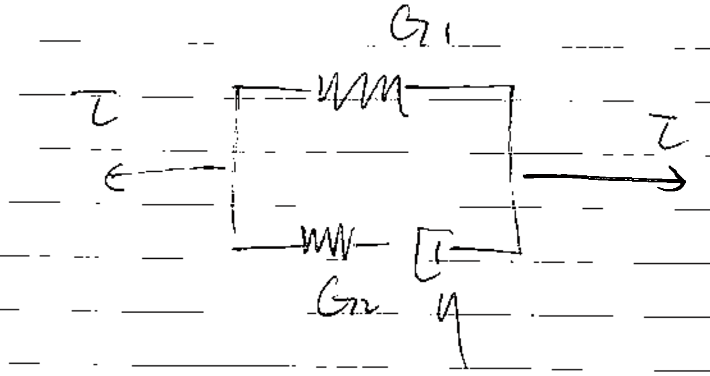
$$a(T, T_0) = \cancel{C_1(T)T_0} \quad \frac{-C_1(T_0)(T - T_0)}{C_2(T_0) + (T - T_0)}$$

Normally, $T_0 \rightarrow$ glass transition temp. of polymer

Y_1' & Y_2' are not independent to each other

if u know one, you know the other.

$$\frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{\hat{Y}_1'(r) - Y_1(\infty)}{r\omega}$$



$\longleftarrow \gamma \Longrightarrow$

↳ shear strain.
Engineering

usually. $\gamma = 2\varepsilon$.

$2G\varepsilon = \tau$.

$\tau = G\gamma$.

γ,

$e_{ij} = sY_1^*$

$S_{ij} = sY_1^* e_{ij}$

$\sigma_{kk} = sY_2^* \varepsilon_{kk}$

$E = 2G(1+\nu)$

Linear Elasticity.

G, ν.

Linear Viscoelasticity.

G, ν, E, K.

bulk (relaxation).

OH: superposition preciple

want to stress in tension test.

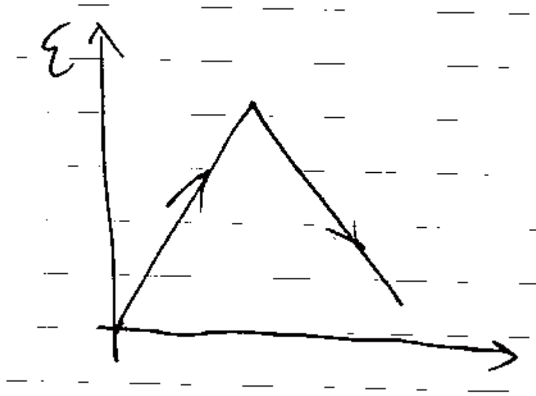
$$\sigma(t) = \varepsilon(0^+)\cdot Y(t) + \int_{0^+}^{t} \underbrace{Y(t-\tau)}_{\to 2a} \frac{d\varepsilon}{d\tau}\, d\tau$$

$$= \varepsilon(0^+)\, Y(t) + Y * \frac{d\varepsilon}{d\tau}$$

strain history

⇓

stress history

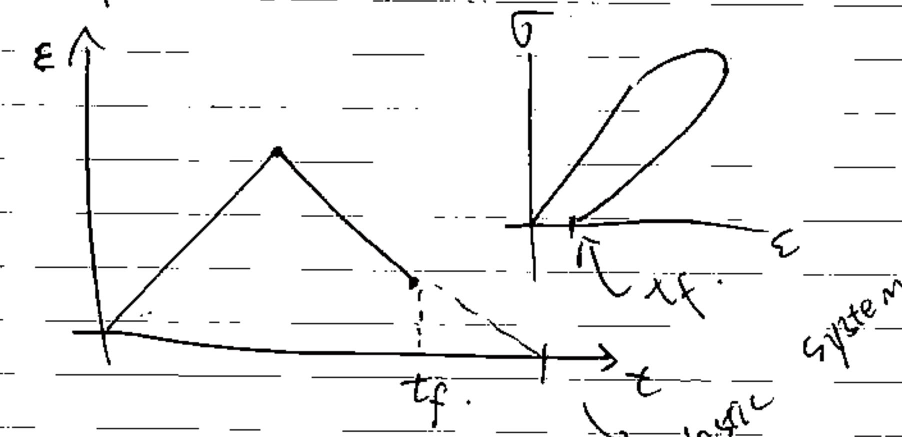


asked to evaluate the stress history.

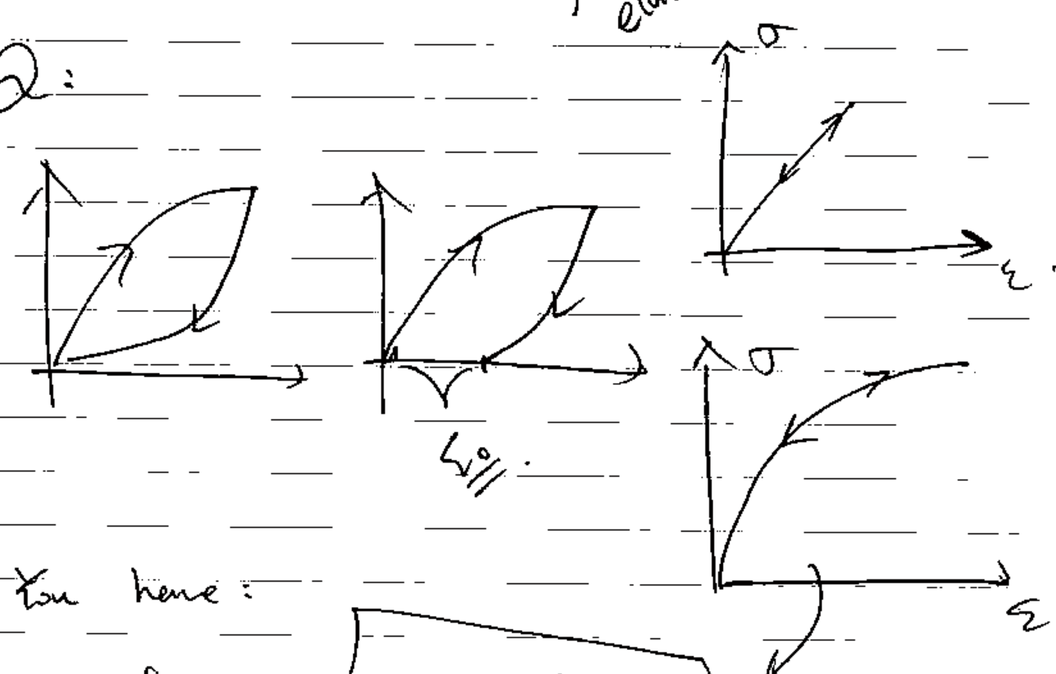
load fast → strain rate → high →

End: should not have stess in Spring 1 & Spring 2.

After 2b. you should be able to see.



Q:

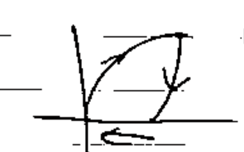


You have:

creep function $\boxed{C(t). \quad Y(t).}$

find out equation of how

⇓.



Only one disp. field (r)

↓

Strain ⟼ 3 ε_{rr} & ε

u_r

$u_r \longrightarrow \varepsilon_{rr}, \varepsilon_{\theta\theta} = \varepsilon_{\phi\phi}$

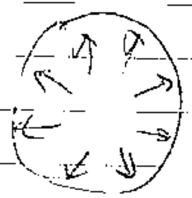
$\Downarrow$ consti. model

$\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{\phi\phi}, \phi,$

Equilibrium Eq.

Incompressible.

$\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\phi\phi} = 0$



$\Downarrow$

pressure = const.

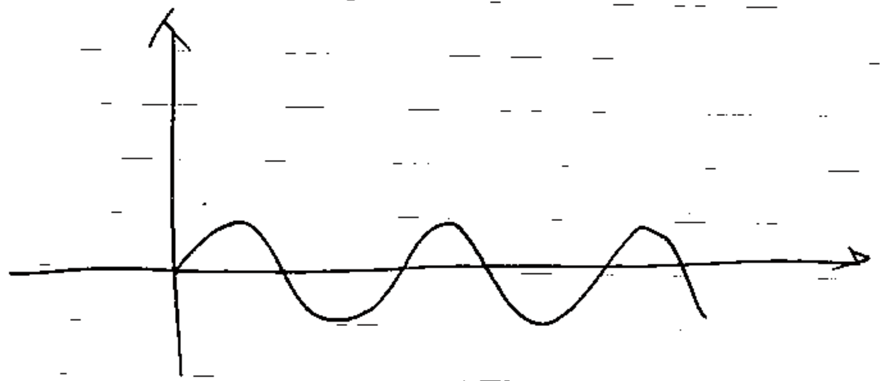
$\rightarrow u \xrightarrow{\infty} 0.$

$r = r_0 \quad , \sigma_{rr} = -P.$

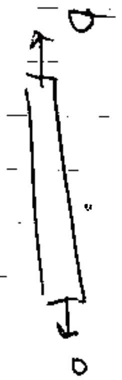
integrate incompressibility.

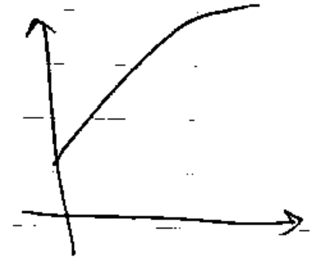
$\sigma_{ij} = 2G\varepsilon_{ij} + P\delta_{ij}$ hydrostatic. $\downarrow$ incompressibility

$\varepsilon_{kk} = 0$



$\sigma_{rr} = P_0 \sin \omega \theta.$

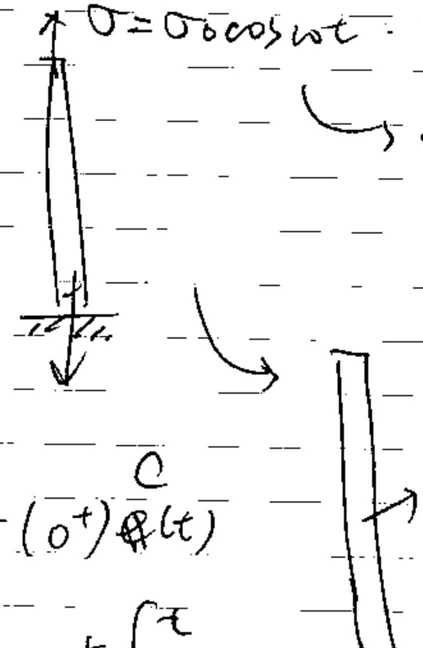




$\sigma = \frac{(\cos\omega t)e^t}{(1+t^3)}$

only difference: $\downarrow$ strain.

$\sigma = \sigma_0 \cos \omega t$



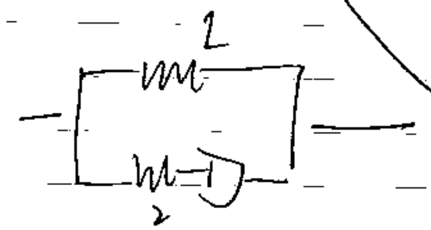
$\hookrightarrow \varepsilon = \frac{\sigma_0 \cos \omega t}{E}$

$\Downarrow$

elastic.

$\varepsilon = \sigma(0^+) C(t) + \int_0^t C(t-\tau)\frac{\partial \sigma}{\partial \tau} d\tau$

$\varepsilon = \sigma_1 / E_1$



$\sigma = \sigma_1 + \sigma_2$

$\sigma_2 = \sigma - \sigma_1$

$\dot{\varepsilon}$

$$\dot{\varepsilon} = \frac{\dot{\sigma}_2}{E_2} + \frac{\sigma_2}{\eta}$$

$$\int_0^\infty \frac{e^{-i\omega\eta}}{1+\eta/t_R}\, d\eta \quad G_0 - G_\infty$$

$$\frac{\eta}{t_R} = u$$

Exponential Integral

$$= \underbrace{\int_0^\infty \frac{e^{-i\hat{\omega}\eta}\, du}{1+u}}$$

$d\eta = u\, t_R$

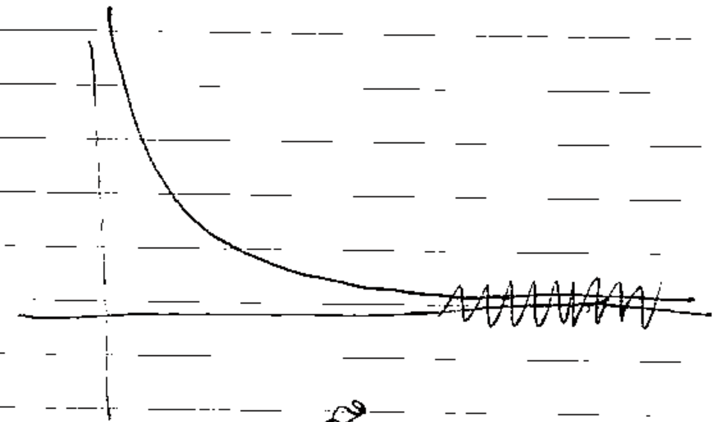
$\hat{\omega} = \omega t_R$

basic idea $\mapsto$ integrate this term.

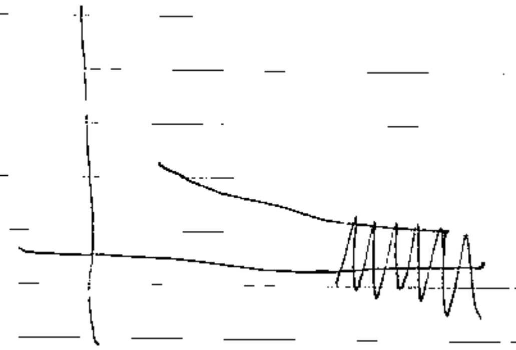
Compute integral by picking ~~integral~~ infinite value

$$\int_0^\infty \frac{\cos \hat{\omega} u\, du}{(1+u)} - i \int_0^\infty \frac{\sin \omega u\, du}{(1+u)}$$

Riemann - Lamé theory



$$\int_0^\infty \frac{\cos \omega x\, dx}{1+x}$$



Nov, 22, Mon, 2021. Wk 14.

Linear visco - isotropic material.

$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij} = \frac{1}{2G} S_{ij}$ Elasticity.

$\varepsilon_{kk} = \frac{1}{K}\cdot\frac{\sigma_{kk}}{3}$ $\qquad \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$

$\Downarrow$ bulk deformation $\qquad \Downarrow$ shear deformation

Visco. $\quad e_{ij} = C_1(t)\, S_{ij}(0^+) + \underbrace{\int_{0^+}^{t} C(t-\tau)\frac{\partial S_{ij}}{\partial \tau}d\tau}_{C_1^* S_{ij}'}$

$\varepsilon_{kk} = C_2(t)\,\sigma_{kk}(t=0^+) + \underbrace{\int_{0^+}^{t} C_2(t-\tau)\frac{\partial \sigma_{kk}}{\partial \tau}d\tau}_{C_2^* \sigma_{kk}'}$

$\downarrow$ transform variable.

$\tilde{e}_{ij} = s\,\tilde{C}_1(s)\,\tilde{S}_{ij}$

$\tilde{\varepsilon}_{kk} = s\,\tilde{C}_2(s)\,\tilde{\sigma}_{kk}$

$\frac{1}{2G} \longleftrightarrow s\,\tilde{C}_1(s)$

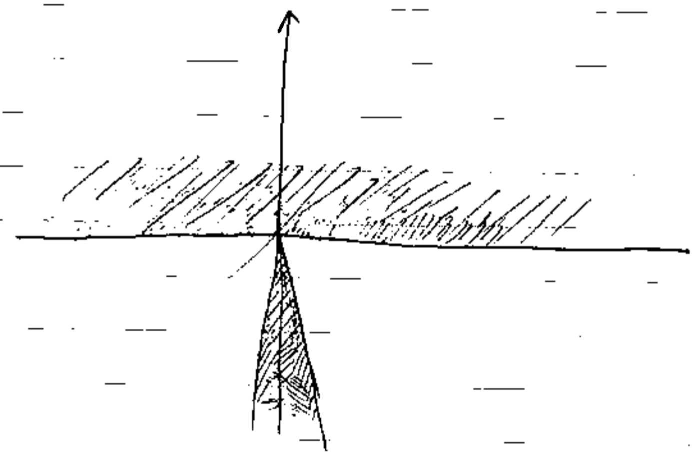
$\frac{1}{3k} \longleftrightarrow s\,\tilde{C}_2(s)$.

$$\frac{s3\tilde{C}_2(\tilde{C}_1/2)}{\tilde{C}_2 + (\tilde{C}_1/2)}$$

$\uparrow$

$$E = \frac{9kG}{3k+G}$$

$$= \frac{s3\tilde{C}_1\tilde{C}_2}{2\tilde{C}_2 + \tilde{C}_1} \equiv s\,\tilde{E}(s)$$



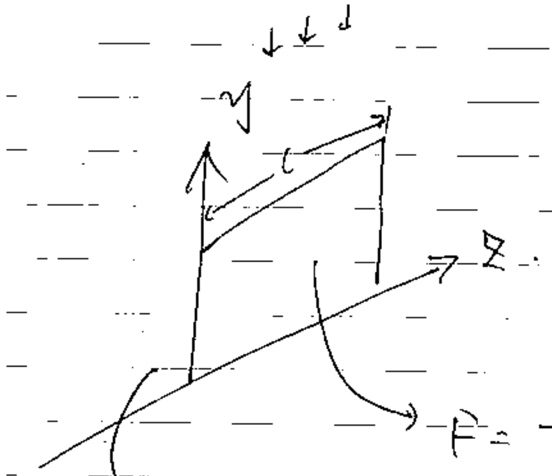
plane strain problem.

$t_{22} = \frac{F}{A} \to 0$. > fundamental sol. for Elas.

line force F.

(force per unit length).

in direction of blade.



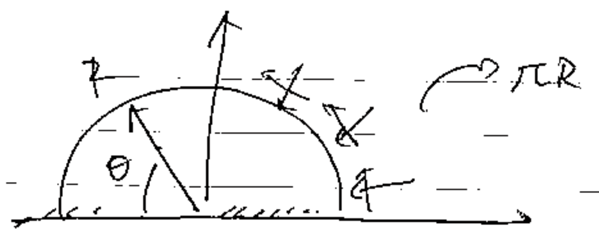
$F = \frac{P}{L}$.

assume infinitely long

F, G, r, θ (go away)

$\sigma = \frac{F}{r} f(\cancel{\frac{F}{Gr}}, \theta)$

force per unit length.



plane strain

solution has to be,

$\underline{\sigma} = \frac{F}{r} \underline{f}(\theta) = 0$

$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \frac{1}{2E}(\sigma_{22} + \sigma_{33})$ $\underline{\varepsilon}$ $\underbrace{\varepsilon_{12},\ \varepsilon_{22},\ \varepsilon_{12}}$

$= \frac{\sigma_{11}}{E} - \frac{1}{2E}(\sigma_{22} + \frac{1}{2}\sigma_{11}$

~~$= \frac{\sigma_{22}}{E} - \frac{1}{2E}(\sigma_{11} + \frac{1}{2}\sigma_{22})$~~

$\mathrm{tr}(\underline{\varepsilon}) = \varepsilon_{11} + \varepsilon_{22} = 0$

$\Leftarrow$ incompressibility

$\Downarrow$

$= \sigma_{11} \left[1 - \frac{1}{4}\right] - \frac{1}{2E}\left[\frac{3\sigma_{22}}{2}\right]$

$\frac{3}{4E} = \frac{1}{4G} = \frac{3\sigma_{11}}{4E} - \frac{3\sigma_{22}}{4E}$

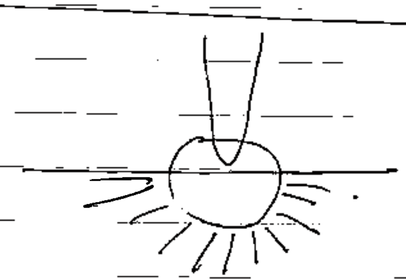
$= \frac{1}{4G}\left[\sigma_{11} - \sigma_{22}\right]$

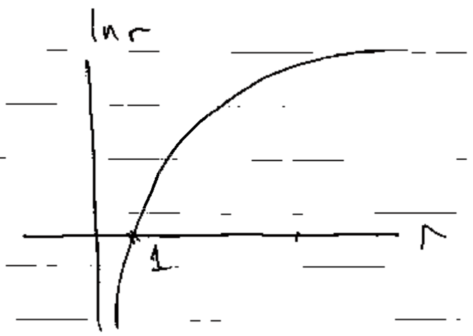
$\underline{\sigma} = \frac{F}{r} \underline{f}(\theta)$

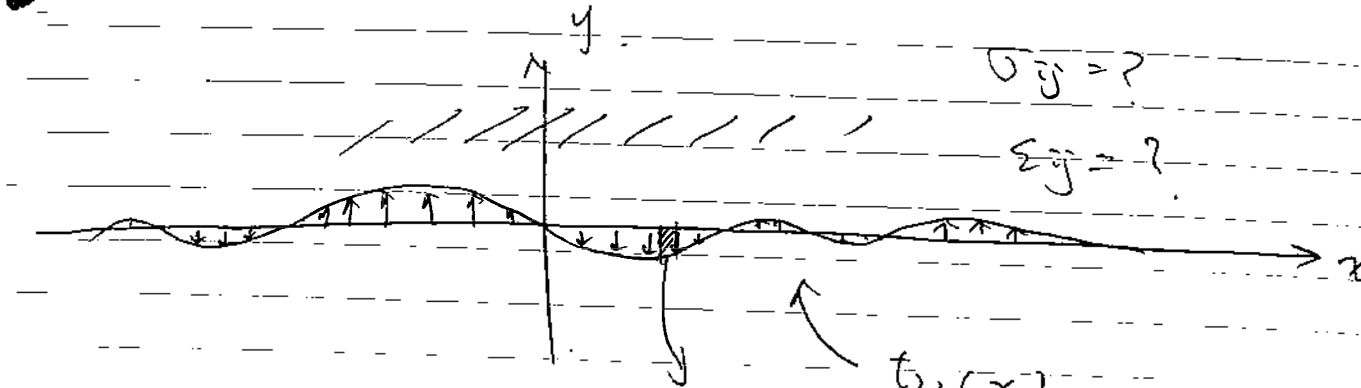
$\underline{\varepsilon} = \frac{F}{Gr} \underline{g}(\theta)$

$\Downarrow$

displacement $\underline{u}$





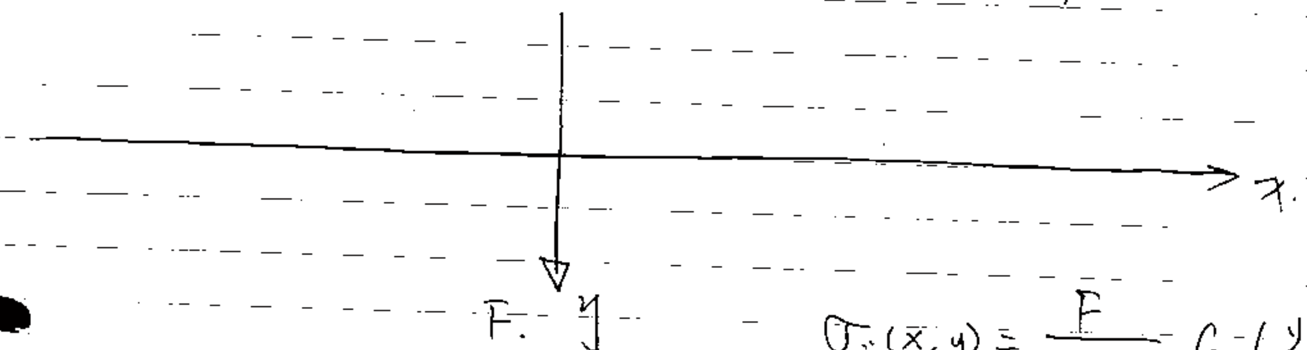


$\sigma_{ij} = ?$

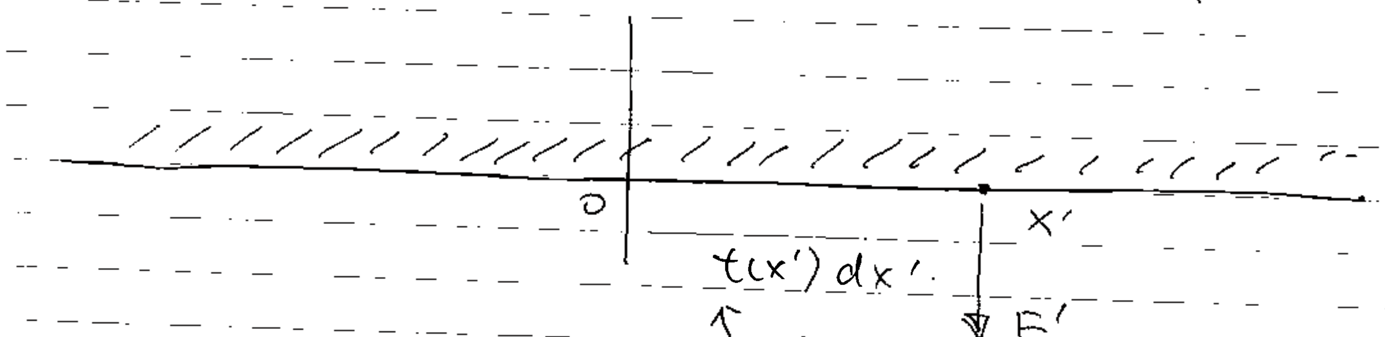
$\varepsilon_{ij} = ?$

$t_{22}(x)$

$dF(x') = t_{22}(x')\,dx'$



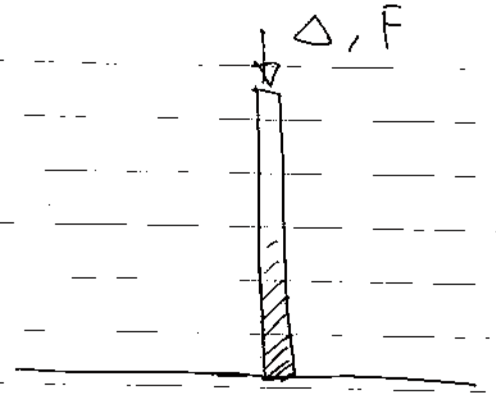
$\sigma_{ij}(x, y) = \frac{F}{\sqrt{x^2+y^2}} f_{ij}\left(\frac{y}{x}\right)$

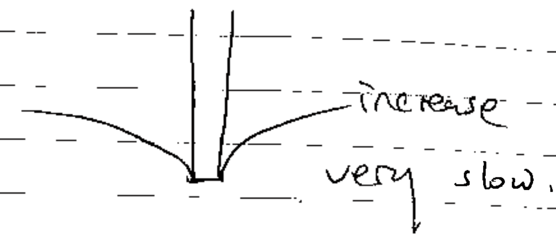


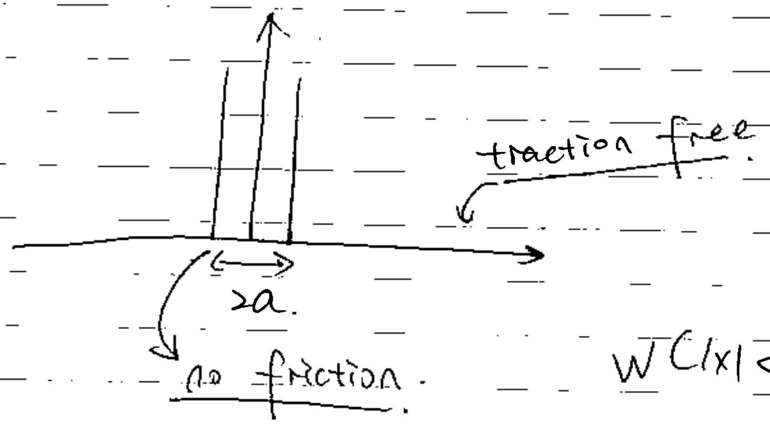
$d\sigma_{ij}(x, y, x') = \frac{F'}{\sqrt{(x-x')^2 + y^2}} f_{ij}\left(\frac{y}{x-x'}\right)$

$\sigma_{ij}(x, y) = \int_{-\infty}^{\infty} \frac{t(x') f_{ij}\left(\frac{y}{x-x'}\right) dx'}{\sqrt{(x-x')^2 + y^2}}$

$\rightarrow$ general solution $\rightarrow$ you need to find

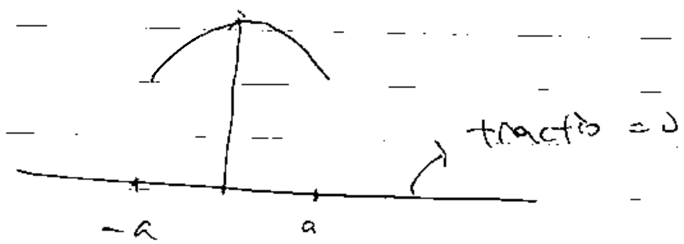


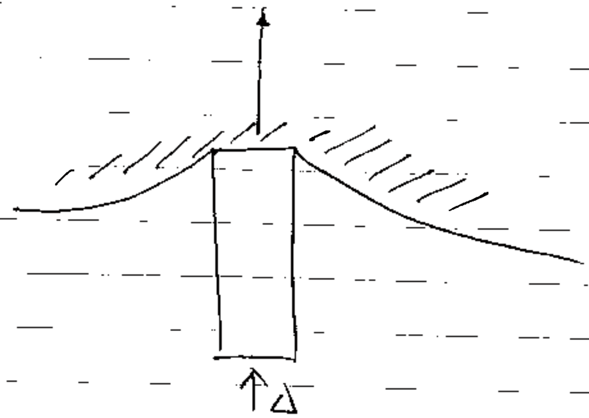




$\sigma_{12} = \sigma_{22} (|x| > a, y=0) = 0$

$W(|x| < a, y=0) = 0$





$$\frac{1}{2\pi G} \int_{-a}^{a} \frac{t(x')\,dx'}{x - x'} = 0$$

principal value integrate.

$$V_{,x} = -\frac{1}{2\pi G} \int_{-a}^{a} \frac{t(x')\,dx'}{x - x'} = 0.$$

gradient of disp.

SOLUTION: $t(x) = \dfrac{A}{\sqrt{a^2 - x^2}} = \dfrac{F}{\pi\sqrt{a^2 - x^2}}$

HW 11. Q3.

First, ~~substitute~~ given the BCs:

$\sigma_{rr}(r=b) = -p$

Compatibility Eq.

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \quad \cdots (1)$$

Constitutive Eq.

$$\sigma_{ij} = 2G\varepsilon_{ij} + P\delta_{ij} \quad \cdots (2)$$

Equilibrium Eq.

$$\sigma_{ij,j} = 0 \quad \cdots (3)$$

Substitute the BCs into Eqs. (1), (2), (3), we can hence compute the stress (radial).

▷ Taking BG's advice, we take ~ from Eq. (2). $\varepsilon_{ij} = (\sigma_{ij} - P\delta_{ij})/2G$.

From compatibility Eq. (1):

$\sigma_{rr,\theta\theta} - P\delta_{rr,\theta\theta} + \sigma_{\theta\theta,rr} - P\delta_{\theta\theta,rr} = \sigma_{r\theta,r\theta} - P\delta_{r\theta,r\theta}$

▷ Equilibrium:

$$\sigma_{r\theta,\theta} + \sigma_{\theta\theta,r} + \sigma_{r\theta,r} + \sigma_{\theta\theta,\theta} = 0 \quad \dots (4)$$

▷ Modified compatibility

$$\sigma_{rr,\theta\theta} + \sigma_{\theta\theta,rr} - \sigma_{r\theta,r\theta} = P \quad \dots (5)$$

↳ ~~Now, Substitute BCs~~ ▷ Nah

→ Equilibrium: $\int(\sigma_{\theta r} + \sigma_{rr})\,d\theta - \int(\sigma_{\theta\theta} + \sigma_{r\theta})\,dr$

Laplace trans.

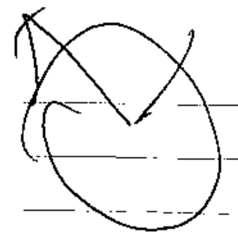
$$\begin{cases} \hat{\sigma}_{ij,i} = 0 \\ \tilde{\varepsilon}_{ij} = (\tilde{u}_{i,j} + \tilde{u}_{j,i})/2 \\ \tilde{\varepsilon}_{kk} = s\hat{C}_2\tilde{\sigma}_{kk}, \quad \tilde{e}_{ij} = s\hat{C}_1\tilde{S}_{ij} \end{cases}$$

$$\tilde{\sigma}_{rr}(r=b) = -\frac{P}{s}.$$

$$\frac{d\sigma_{rr}}{dr} + \frac{2}{r}(\sigma_{rr} - \sigma) = 0.$$

Same in Laplace space:

$$\frac{d\tilde{\sigma}_{rr}}{dr} + \frac{2}{r}(\tilde{\sigma}_{rr} - \tilde{\sigma}) = 0.$$



$$\frac{\partial^2\varepsilon_{xx}}{\partial y^2} + \frac{\partial^2\varepsilon_{yy}}{\partial x^2} = \frac{\partial^2\varepsilon_{xy}}{\partial x\partial y}$$

$$\sigma_{ij} = 2G\varepsilon_{ij} + P\delta_{ij}$$

$$\frac{\partial^2\varepsilon_{ii}}{\partial x_j^2} + \frac{\partial^2\varepsilon_{jj}}{\partial x_i^2} = 2\frac{\partial^2\varepsilon_{ij}}{\partial x_i\partial x_j}$$

$$\begin{cases} \sigma_{ii} = 2G\varepsilon_{ii} + P\delta \\ \sigma_{jj} = 2G\varepsilon_{jj} + P\delta \\ \sigma_{ij} = 2G\varepsilon_{ij} + P\delta \end{cases}$$

$$\begin{cases} \varepsilon_{ii} = \dfrac{\sigma_{ii} - P\delta_{ii}}{2G} \\ \varepsilon_{jj} = \dfrac{\sigma_{jj} - P\delta_{jj}}{2G} \\ \varepsilon_{ij} = \dfrac{\sigma_{ij} - P\delta_{ij}}{2G} \end{cases}$$

spherical: $\frac{\partial^2\tilde{\sigma}_{rr}}{\partial\tilde{\theta}^2} + \frac{\partial\tilde{\sigma}_{\theta\theta}}{\partial \tilde{r}^2} = \frac{\partial\tilde{\sigma}_{r\theta}}{\partial\tilde{r}\partial\theta}$

$$\frac{\partial^2\sigma_{ii}}{\partial x_j^2} + \frac{\partial^2\sigma_{jj}}{\partial x_i^2} = \frac{\partial^2\sigma_{ij}}{\partial x_i\partial x_j}.$$

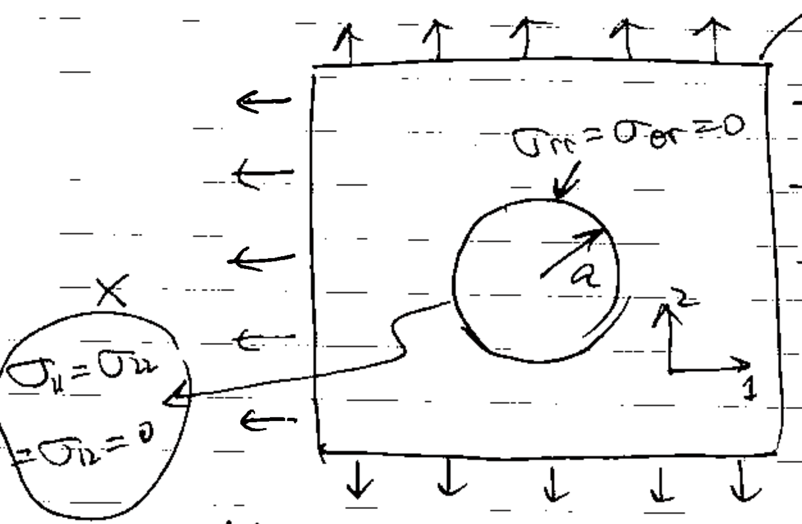
in Laplace domain: $\frac{\partial^2\tilde{\sigma}_{ii}}{\partial\tilde{x}_j^2} + \frac{\partial^2\tilde{\sigma}_{jj}}{\partial\tilde{x}_i^2} = \frac{\partial^2\tilde{\sigma}_{ij}}{\partial\tilde{x}_i\partial\tilde{x}_j}$

Nov. 29, Mon. Wk 15

☆ Exam: Dec. 11. 9am - 9pm.

HW Review

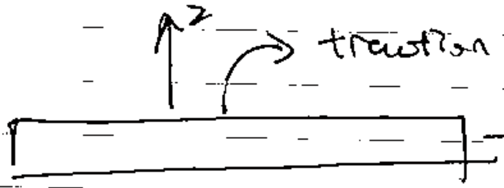
σ_{22}^{∞}



→ no shear $\sigma_{12} = 0$, @ $|y| \to \infty$

$\sigma_{21} = 0$ @ $|x| \to \infty$

☆☆☆ Setup the traction free BCs



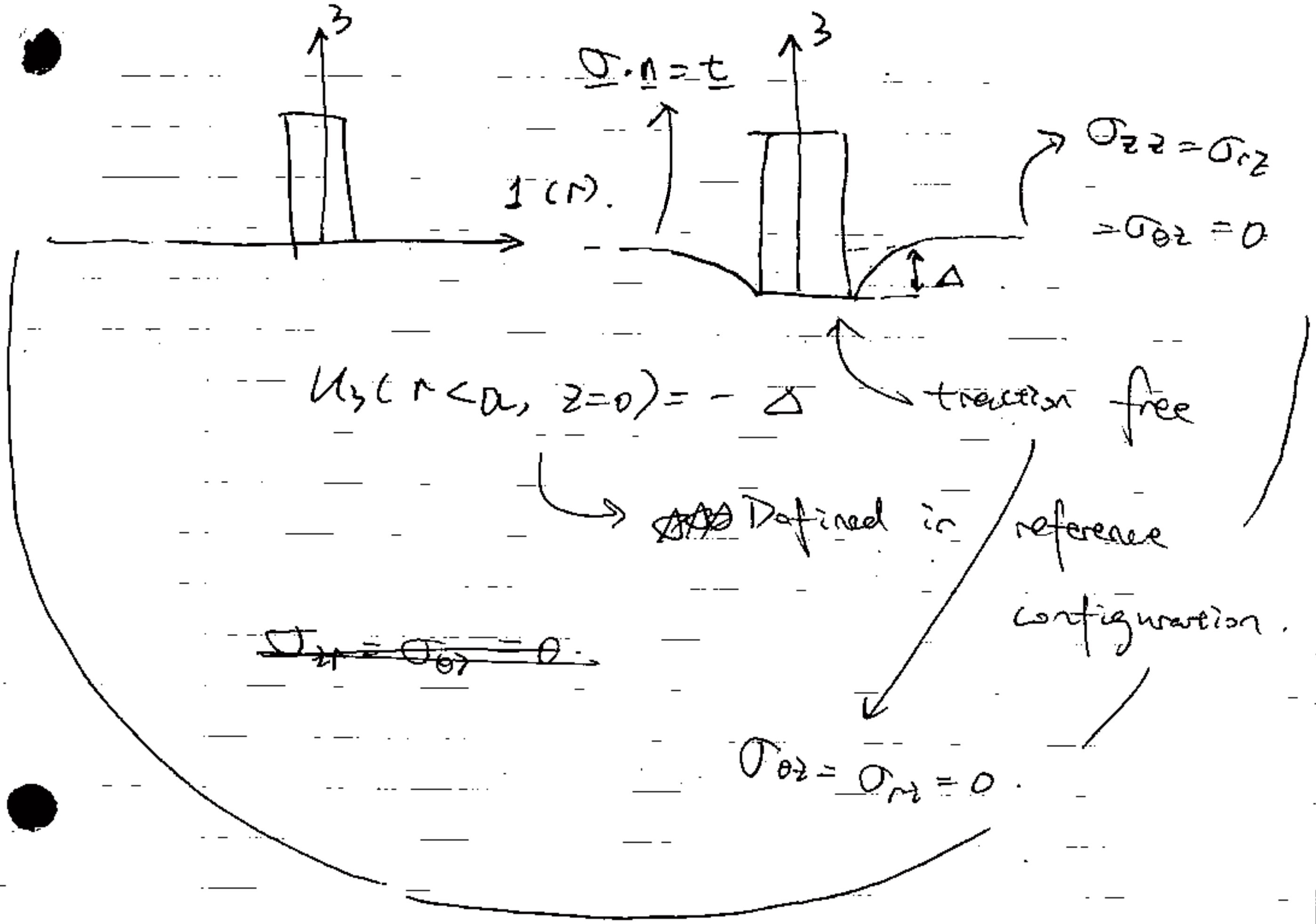
traction free : $\underline{\sigma} \cdot \underline{n} = 0$

$\sigma_{12} = \sigma_{22} = 0$

Δ remember how the setup the traction free BCs

$\underline{n} = \cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_2$

☆☆☆ $\underline{n}$ should always normal to the surface BECAUSE it identifies the surface.



$\underline{\sigma} \cdot \underline{n} = \underline{t}$

$\sigma_{zz} = \sigma_{rz} = \sigma_{\theta z} = 0$

$u_3 (r < a, z = 0) = -\Delta$ → ☆☆☆ Defined in reference configuration

traction free

~~$\sigma_{zr} = \sigma_{\theta z} = 0$.~~

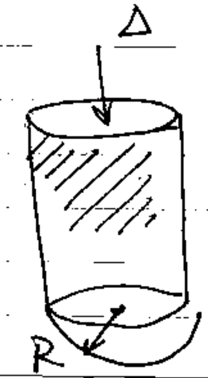
$\sigma_{\theta z} = \sigma_{rz} = 0$.

$\rho = \sqrt{r^2 + z^2}$

$\underline{\sigma} (\rho \to \infty) \to 0$.

frictionless BCs : Cannot take any load.

☆☆☆ there are no shear



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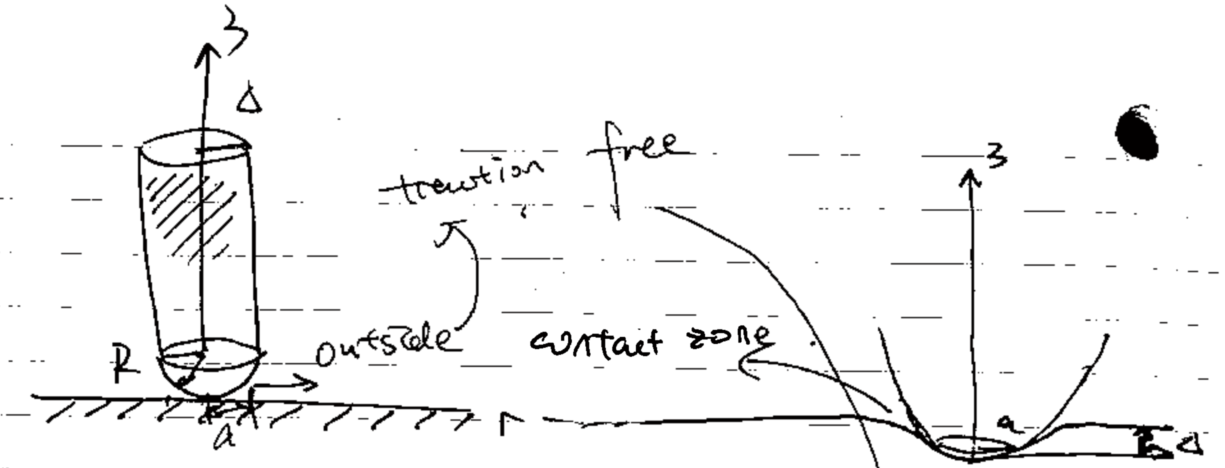


What is the displacement

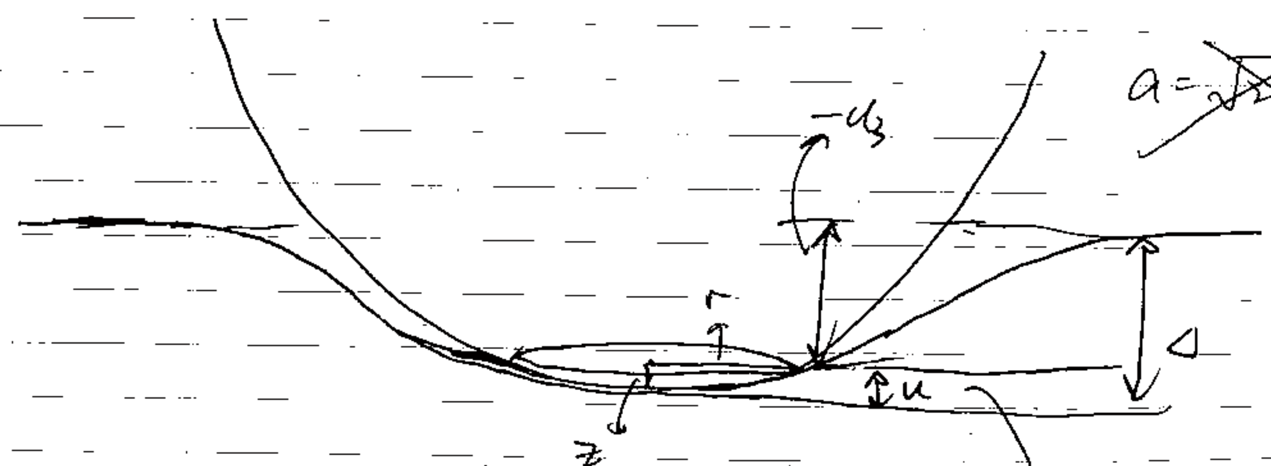
What is the contact condition

BC $\sigma_{rz} = \sigma_{\theta z} = \sigma_{zz} = 0, \quad r > a, \ z = 0$

traction free

inside the contact region: $\sigma_{rz} = \sigma_{\theta z} = 0, \quad r < a, \ z = 0$

$a = \cancel{\sqrt{2R\Delta}}$



$r^2 + (z-R)^2 = R^2$

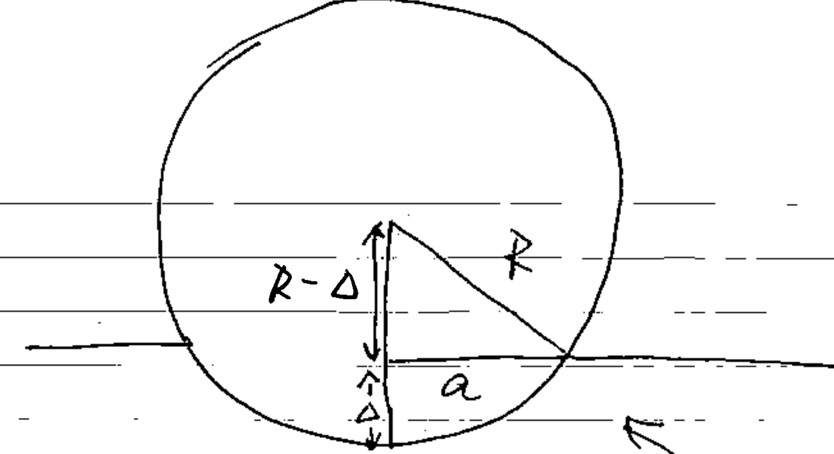
z

$2Rz \cong r^2$

$z \cong \frac{r^2}{2R}$

$z - \frac{r^2}{2R} = 0$

$-u_3 \ (r<a, z=0)$



$(R-\Delta)^2 + a^2 = R^2$

wrong

$R^2 - 2R\Delta + \Delta^2 + a^2 = 0.$

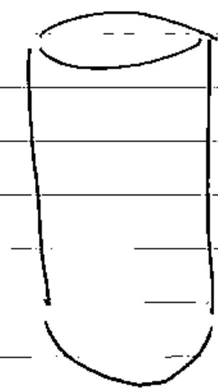
⇓

$a^2 \approx 2R\Delta$

like a fluid.

$\Delta = \frac{a^2}{2R}.$

Actual Hertz soln. $\Delta = \frac{a^2}{R}.$



$\Delta = \frac{P}{G \pi a} \cdot \alpha$ numerical const.

← measure the load F

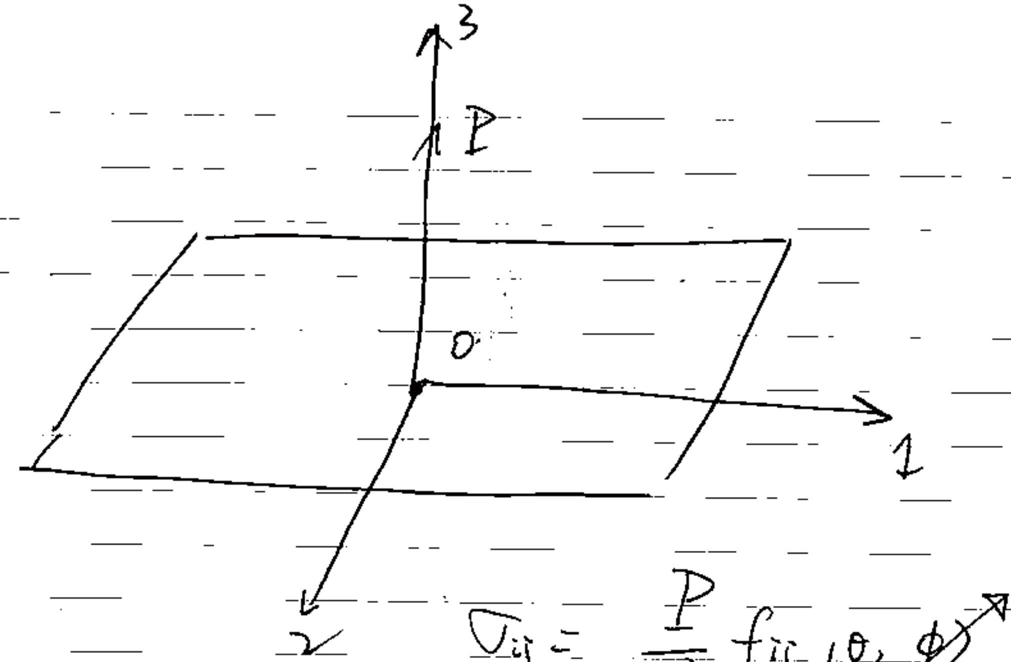
$$\boxed{\Delta = \left[\frac{9}{16R\,(4G)^2} \right] F^{2/3}}$$

⇓

incompressible solid.

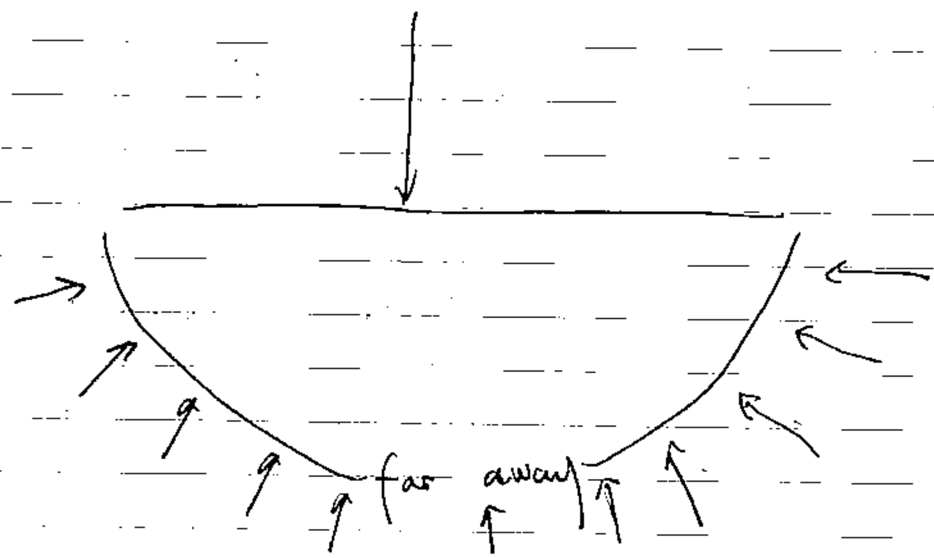
$4G \longrightarrow E^* = \frac{E}{1-\nu^2}$

shear modulus



$\sigma_{ij} = \frac{P}{r^2} f_{ij}(\theta, \cancel{\phi})$

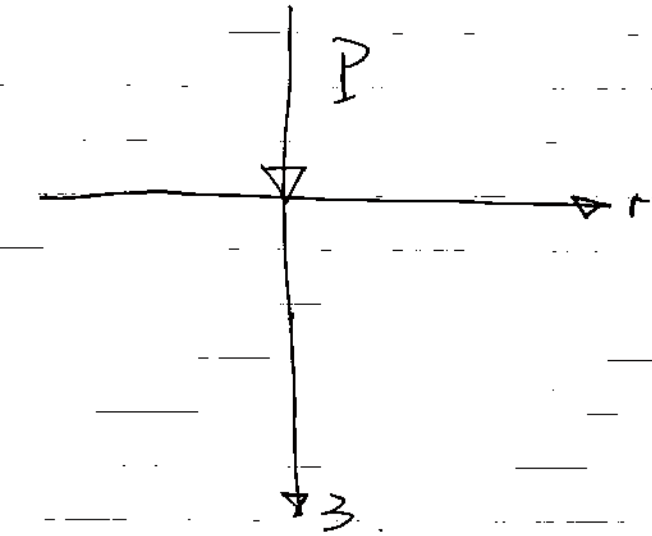
independent of ϕ.

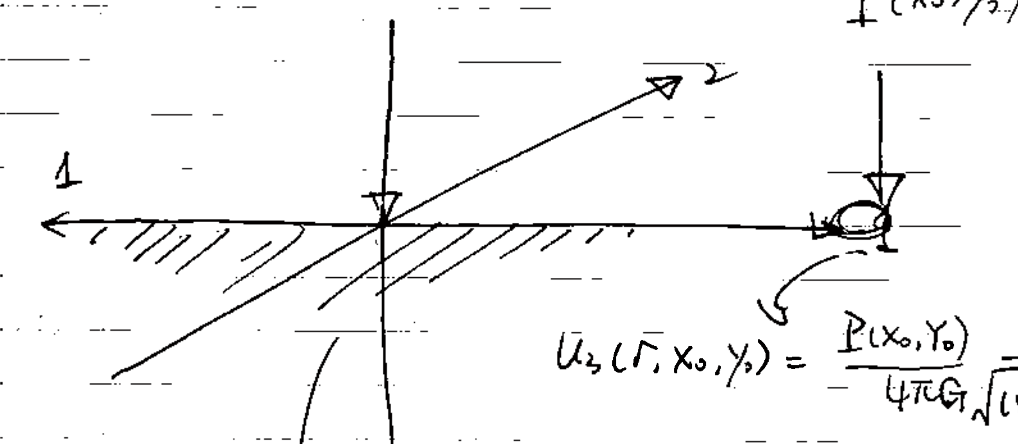


$4\pi r^2 \sigma_{rr} \approx P$

$r >> 1$ $\quad u_i \approx \frac{P}{Gr} \hat{u}_i(\theta)$

Standard Bushiness sln. $\underset{(r,\, z=0)}{u_3} = \frac{P}{4\pi G} \cdot \frac{2(1-\nu)}{r}$, $r \to \sqrt{x^2+y^2}$





$u_3(r, x_0, y_0) = \frac{P(x_0, y_0)}{4\pi G} \frac{2(1-\nu)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$

line load

$u_3(r, z=0) = \frac{P}{4\pi G} \frac{2(1-\nu)}{\sqrt{x^2+y^2}}$

distributed load

Superposition: $P(x_0, y_0) \Rightarrow p(x_0, y_0)\, dx_0\, dy_0$. very small area

$u_3 = \frac{1-\nu}{2\pi G} \iint_A \frac{p(x_0, y_0)\, dx_0\, dy_0}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}$

wdd.

Nov. 24, Wed., 2021. Wk 13.

Introduction to anistropic Elasticity

Isotropic Elasticity: 2 materials consts

anisotropic:

▷ worst: 21 materials consts.

linear elasticity: $\boxed{\underline{\sigma} = \underline{\underline{K}}\, \underline{\varepsilon}}$ (1)

fourth order tensor

(stiffness tensor).

$\underline{K} = K_{ijkl}\, \underline{e}_i\, \underline{e}_j\, \underline{e}_k\, \underline{e}_l$

$K_{ijkl} = K_{jikl} = K_{jilk}$

(due to the symmetry of stress & strain tensors).

36 characteristics.

Existence of strain energy density W.

$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}$ $\underline{K}$ has 21 independent consts.

$\sigma_{ij} \longrightarrow \underline{\sigma} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix}$

$\underline{\sigma} = \sigma_{ij}\, \underline{e}_i\, \underline{e}_j.$

9.

$\varepsilon_{ij} \longrightarrow \underline{\varepsilon} = \begin{pmatrix} \varepsilon_{11} = \varepsilon_1 \\ \varepsilon_{22} = \varepsilon_2 \\ \varepsilon_{33} = \varepsilon_3 \\ \varepsilon_{23} = \varepsilon_4 \\ \varepsilon_{13} = \varepsilon_5 \\ \varepsilon_{12} = \varepsilon_6 \end{pmatrix}$

Eq. (1) $\Leftrightarrow$ $\underline{\sigma} = \underline{\underline{K}}\, \underline{\varepsilon}$

$$K = \begin{pmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ & & K_{33} & K_{34} & K_{35} & K_{36} \\ & & & K_{44} & & \\ & & & & K_{55} & \\ & & & & & K_{66} \end{pmatrix}$$

$$\sigma_{11} = K_{1111}\varepsilon_{11} + K_{1112}\varepsilon_{12} + K_{1113}\varepsilon_{13} + K_{1121}\varepsilon_{21} + K_{1122}\varepsilon_{22} + K_{1123}\varepsilon_{23} + K_{1131}\varepsilon_{31} + K_{1132}\varepsilon_{32} + K_{1133}\varepsilon_{33}.$$

$$\sigma_1 = K_{11}\varepsilon_1 + K_{16}\varepsilon_6 + K_{15}\varepsilon_5 + K_{14}\varepsilon_4 + K_{12}\varepsilon_2 + K_{13}\varepsilon_3 \implies \sigma = K\varepsilon.$$

Plane of symmetry: (material)

anisotropic

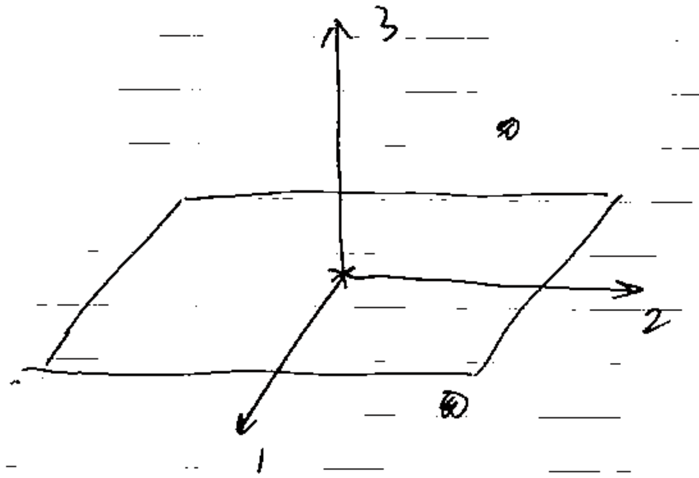
2 bases: $\{e_1, e_2, e_3\} = B_1$

$\{e_1', e_2', e_3'\} = B_2$.

$e_1' \rightarrow e_1$

$e_2' = e_2$

$e_3' = -e_3$



(Reflection)

$\rightarrow K_{ijkl} \mapsto K'_{rstu}$

$\underline{\underline{K}}$ is the same for both basis. (B_0, B_0')

transform into B' basis

$K'_{ijkl} = K_{ijkl}$.

$$K'_{rstu} = K_{ijkl}\underbrace{(e_i \cdot e_r')}_{P_{ri}}\underbrace{(e_j \cdot e_s')}_{P_{sj}}\underbrace{(e_k \cdot e_t')}_{P_{tk}}\underbrace{(e_l \cdot e_u')}_{P_{lu}}.$$

general transformation formula

P matrix is simple

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

$$K'_{1111} = K_{1111} = K_{ijkl}\underbrace{P_{1i}}_{\delta_{1i}}\underbrace{P_{1j}}_{\delta_{1j}}\underbrace{P_{1k}}_{\delta_{1k}}\underbrace{P_{1l}}_{\delta_{1l}}$$

(Hypothesis)

$$= K_{1111}$$

$$K'_{2222} = K_{2222} = K_{ijkl}P_{2i}P_{2j}P_{2k}P_{2l} = K_{2222}.$$

$$K'_{1122} = K_{ijkl}\underbrace{P_{1i}}_{\delta_{1i}}\underbrace{P_{1j}}_{\delta_{1j}}\underbrace{P_{2k}}_{\delta_{2k}}\underbrace{P_{2l}}_{\delta_{2l}}. = K_{1122}.$$

in the same way you can show for all.

$K'_{1123} = K_{ijkl} P_{1i} P_{1j} P_{2k} P_{3l} = -K_{1123}$

~~$\delta_{1i}\ \delta_{1j}$~~

$\delta_{1i}\ \delta_{1j}\ \delta_{2k}\ (-\delta_{3l})$

we know in prior: $K'_{1123} = K_{1123}$

Hence: $K_{1123} = -K_{1123}$.

$\therefore K_{1123} = 0$.

$K_{14} = 0$

$K_{24} = K_{25} = K_{34} = K_{35} = K_{46} = K_{56} = 0$

reduce the num. of consts. to 8.

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} & 0 & 0 & K_{16} \\ & K_{22} & K_{23} & 0 & 0 & K_{26} \\ & & K_{33} & 0 & 0 & K_{36} \\ & & & K_{44} & K_{45} & \cancel{K_{46}}\ 0 \\ & & & & K_{55} & \cancel{K_{56}}\ 0 \\ & & & & & K_{66} \end{bmatrix}$$

13 Material constants.

I already have material plane of symmetry

(3)

$\sigma_1 = K_{11}\varepsilon_1 + K_{12}\varepsilon_2 + K_{13}\varepsilon_3 + K_{16}\varepsilon_6.$

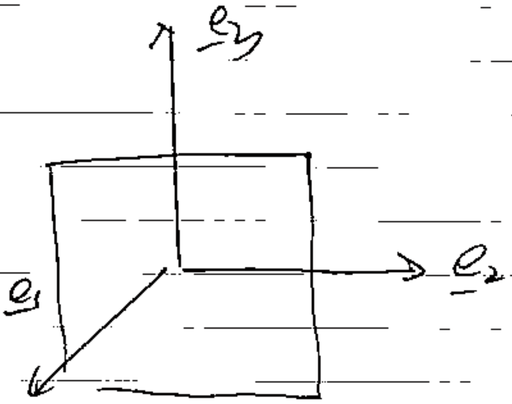
$\sigma_2 = K_{12}\varepsilon_1 + K_{22}\varepsilon_2 + K_{23}\varepsilon_3 + K_{26}\varepsilon_6$

$\sigma_3 = K_{13}\varepsilon_3 + K_{23}\varepsilon_2 + K_{33}\varepsilon_3 + K_{36}\varepsilon_6.$

$\sigma_4 = K_{44}\varepsilon_4 + K_{45}\varepsilon_5.$

$\sigma_5 = K_{54}\varepsilon_4 + K_{55}\varepsilon_5.$

$\sigma_6 = K_{61}\varepsilon_1 + K_{62}\varepsilon_2 + K_{63}\varepsilon_3 + K_{66}\varepsilon_6.$



define a new basis:

$e_1' = -e_1$

$e_2' = e_2.$

$e_3' = e_3$

put an additional plane of symmetry.

$\sigma_{ij} \to \sigma'_{ij}$

$\varepsilon_{ij} \to \varepsilon'_{ij}$

$$[\sigma'_{ij}] = \begin{bmatrix} \sigma_{11} & -\sigma_{12} & -\sigma_{13} \\ -\sigma_{12} & \sigma_{22} & \sigma_{23} \\ -\sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

$\sigma'_4 = \sigma_4 = K_{44}\varepsilon'_{23} + K_{45}\varepsilon'_{13}$

$\sigma'_4 = K_{44}\varepsilon_{23} - K_{45}\varepsilon_{13}$

$K_{45} = 0 \quad K_{36} = 0$

$K_{16} = 0 \quad K_{26} = 0$

$$
K = \begin{pmatrix} K_{11} & K_{12} & K_{13} & 0 & 0 & 0 \\ & K_{22} & K_{23} & 0 & 0 & 0 \\ & & K_{33} & 0 & 0 & 0 \\ & & & K_{44} & 0 & 0 \\ & & & & K_{55} & 0 \\ & & & & & K_{66} \end{pmatrix}
$$

HW 11

In torsion rheology test, circular cylinder

R, h $\quad \gamma(t) = \gamma_0 e^{i\omega t}$.

Use cylinder coordinate,

only stress exist: $\sigma_{\theta z}$.

Initial condition: $\varepsilon_{ij} = \sigma_{ij} = 0$, $\quad t = 0$

boundary condition:

$$
\begin{cases} u_i(r, \theta, z=0, t>0) = 0, \\ u_r(r, \theta, z=h, t>0) = 0, \; u_z(r, \theta, z=h, t>0) = 0, \\ u_\theta(r \le R, \theta, z=h, t>0) = rh\gamma = rh\gamma_0 e^{i\omega t} \\ \sigma_{rr}(r=R, \theta, 0<z<h, t>0) \ldots \\ = \sigma_{zr}(r=R, \theta, 0<z<h, t>0) = 0. \end{cases}
$$

(traction free on side walls)

governing eqs. for torsion:

$u_r = u_z = 0$, $\quad u_\theta = \gamma r z$.

the only non-vanishing strain: $\underline{\varepsilon_{z\theta} = \frac{r\gamma}{2}}$

In cylindrical coor., all equilibrium satisfied!

* constitutive model: here linear viscoelasticity comes in

$$
\sigma_{z\theta}(r, t) = 2G(t)\varepsilon_{z\theta}(r, t=0^+) + 2\int_{0^+}^{t} G(t-\tau)\frac{\partial \varepsilon_{z\theta}(r, \tau)}{\partial \tau} d\tau
$$

$$
\rightarrow \sigma_{z\theta}(r, t) = G(t)\, r\gamma(t=0^+) + r\int_{0^+}^{t} G(t-\tau)\frac{d\gamma(\tau)}{d\tau} d\tau
$$

$$= G(t)\,r\,\gamma_0 + r\int_{0^+}^{t} G(t-\tau)\frac{d\gamma_0 e^{i\omega\tau}}{d\tau}\,d\tau.$$

$$= \left[G(t) + i\omega\int_{0^+}^{t} G(t-\tau)\,e^{i\omega\tau}d\tau\right] r \equiv \varphi(\omega,t)\gamma_0 r$$

the torque $M(t)$:

$$M(t) = 2\pi\int_0^R \sigma_{\theta 3}\, r^2 dr = \pi\gamma_0\,\varphi(\omega,t)\int_0^R r^3 dr$$

$$= \frac{\pi\varphi(\omega,t)R^4\gamma_0}{2}$$

1b. $\varphi(\omega,t) = G(t) + i\omega\underbrace{\int_{0^+}^{t} G(t-\tau)\,e^{i\omega\tau}d\tau}_{\text{as } t\to\infty}.$

integral term:

$$\int_{0^+}^{t} G(t-\tau)\,e^{i\omega\tau}d\tau = e^{i\omega t}\int_0^t G(t-\tau)\,e^{-i\omega(t-\tau)}d\tau$$

$$= e^{i\omega t}\int_0^t G(\eta)\,e^{-i\omega\eta}d\eta = e^{i\omega t}\left[\int_0^t [G(\eta)-G_\infty]\right.$$

$$\left. e^{-i\omega\eta}d\eta + \int_0^t G_\infty e^{-i\omega\eta}d\eta\right]$$

$$= e^{i\omega t}\left[\int_0^t [G(\eta)-G_\infty]e^{-i\omega\eta}d\eta\right.$$

$$\left. + \frac{G_\infty}{-i\omega}e^{-i\omega\eta}\Big|_0^t\right]$$

$$= e^{i\omega t}\int_0^t [G(\eta)-G_\infty]e^{-i\omega\eta}d\eta + \frac{G_\infty}{i\omega}(e^{i\omega t}-1)$$

then we have φ:

$$\varphi(\omega,t) = G(t) + i\omega\left[e^{i\omega t}\int_0^t [G(\eta)-G_\infty]e^{-i\omega\eta}d\eta\right.$$

$$\left. + \frac{G_\infty}{i\omega}(e^{i\omega t}-1)\right]$$

$$= (G(t)-G_\infty) + G_\infty e^{i\omega t} + i\omega e^{i\omega t}\int_0^t [G(\eta)-G_\infty]e^{-i\omega\eta}d\eta$$

We already know:

$$\int_0^t [G(\eta)-G_\infty]\,e^{-i\omega\eta}d\eta$$

$$= \int_0^\infty [G(\eta)-G_\infty]\,e^{-i\omega\eta}d\eta - \int_t^\infty [G(\eta)-G_\infty]e^{-i\omega\eta}d\eta$$

$$\varphi(\omega,t) = \left[(G(t)-G_\infty) - i\omega e^{i\omega t}\int_t^\infty [G(\eta)-G_\infty]\right.$$

$$\left. e^{-i\omega\eta}d\eta\right] + \left\{G_\infty + i\omega\int_0^\infty [G(\eta)-G_\infty]e^{-i\omega\eta}d\eta\right\}e^{i\omega t}$$

Since $G(\eta\to\infty) - G_\infty = 0$.

$$\varphi(\omega, t\to\infty) = \left\{G_\infty + i\omega\int_0^\infty [G(\eta)-G_\infty]e^{-i\omega\eta}d\eta\right\}e^{i\omega t}$$

$$= \mu(\omega)\,e^{i\omega t}.$$

$$M_{ss}(\omega) = \frac{\pi R^4\gamma_0}{2}\mu(\omega)\,e^{i\omega t}.$$

Assuming $G(t) = G_\infty + \dfrac{G_0 - G_\infty}{(1+\frac{t}{t_R})^n}$, find storage & loss modulus.

- Storage modulus:

$$\mu'(\omega) = Re[\mu(\omega)] = Re\left[G_\infty + i\omega\int_0^\infty \frac{G_0 - G_\infty}{(1+\eta/t_R)^n} e^{-i\omega\eta}\, d\eta\right]$$

$$= G_\infty + (G_0 - G_\infty)\,\omega\, Re\left[i\int_0^\infty (1+\eta/t_R)^{-n} e^{-i\omega\eta}\, d\eta\right]$$

- loss modulus:

$$\mu''(\omega) = Im[\mu(\omega)]$$

$$= (G_0 - G_\infty)\,\omega\, Im\left[i\int_0^\infty (1+\eta/t_R)^{-n} e^{-i\omega\eta}\, d\eta\right]$$

to evaluate the integrals, let $\eta/t_R = p$, so that $\int_0^\infty (1+\eta/t_R)^{-n} e^{-i\omega\eta}\, d\eta$

$$= t_R\int_0^\infty (1+p)^{-n} e^{-i\Omega p}\, dp$$

For our case, $n=1$.

$$\int_0^\infty (1+p)^{-1} e^{-i\Omega p}\, dp = \int_0^\infty (1+p)^{-1}\cos(\Omega p)\, dp - i\int_0^\infty (1+p)^{-1}\sin(\Omega p)\, dp$$

$$= \int_0^\infty (\Omega+p)^{-1}\cos q\, dq - i\int_0^\infty (\Omega+q)^{-1}\sin q\, dp$$

$$= \{-Ci(\Omega)\cos\Omega - si(\Omega)\sin\Omega\} - i\{Ci(\Omega)\sin\Omega - si(\Omega)\cos\Omega\}.$$

Ci & Si: sine and cosine integrals.

normalize the storage & loss modulus:

$$\bar{\mu}'(\omega) = 1 + \left(\frac{G_0}{G_\infty} - 1\right)\Omega\, Re\left[i\int_0^\infty (1+p)^{-n} e^{-i\Omega p}\, dp\right]$$

$$= 1 + \left(\frac{G_0}{G_\infty} - 1\right)\Omega\{Ci(\Omega)\sin\Omega - si(\Omega)\cos\Omega\}.$$

$$\bar{\mu}''(\omega) = \left(\frac{G_0}{G_\infty} - 1\right)\Omega\, Im\left[i\int_0^\infty (1+p)^{-n} e^{-i\Omega p}\, dp\right]$$

$$= -\left(\frac{G_0}{G_\infty} - 1\right)\Omega\{Ci(\Omega)\cos\Omega + Si(\Omega)\sin\Omega\}.$$

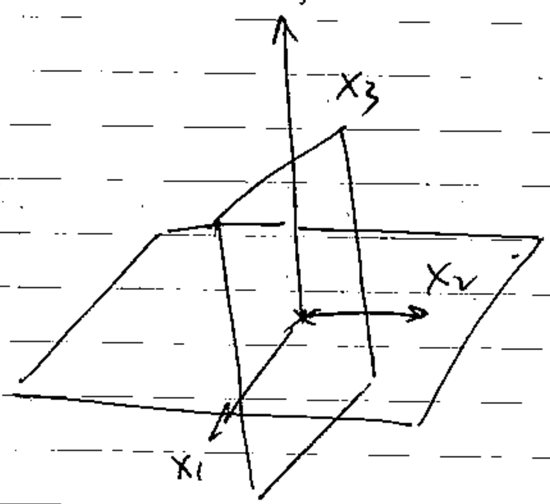
$$\therefore \tan\delta = \frac{-\left(\frac{G_0}{G_\infty} - 1\right)\Omega\{Ci(\Omega)\cos\Omega + Si(\Omega)\sin\Omega\}}{1 + \left(\frac{G_0}{G_\infty} - 1\right)\Omega\{Ci(\Omega)\sin\Omega - Si(\Omega)\cos\Omega\}}$$

Dec. 6., Mon. 2021 Wk 16.

orthotropic material. 9 constants.

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} & 0 & 0 & 0 \\ & K_{22} & K_{23} & & & \\ & & K_{33} & & & \\ & & & K_{44} & & \\ & & & & K_{55} & \\ & & & & & K_{66} \end{bmatrix}$$

Transversely Isotropic

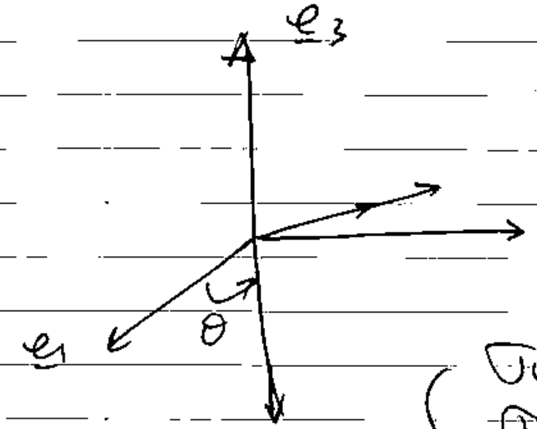


e_3 is an axis of symmetry,

every plane containing this axis is a plane of reflection symmetry

invariant to Rotation above this axis.

$$[R] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{cases} e_1' = \cos\theta\, e_1 + \sin\theta\, e_2 \\ e_2' = -\sin\theta\, e_1 + \cos\theta\, e_2 \\ e_3' = e_3 \end{cases}$$



$$[P] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

standard isotropic material

$$\begin{cases} \sigma_1 = K_{11}\varepsilon_1 + K_{12}\varepsilon_2 + K_{13}\varepsilon_3 \\ \sigma_2 = K_{21}\varepsilon_1 + K_{22}\varepsilon_2 + K_{23}\varepsilon_3 \\ \sigma_3 = K_{31}\varepsilon_1 + K_{32}\varepsilon_2 + K_{33}\varepsilon_3 \\ \sigma_4 = K_{44}\varepsilon_4 \\ \sigma_5 = K_{55}\varepsilon_5 \\ \sigma_6 = K_{66}\varepsilon_6 \end{cases}$$

$$\underline{\sigma}' = \underline{P}\,\underline{\sigma}\,\underline{P}^T$$

$$\underline{\varepsilon}' = \underline{P}\,\underline{\varepsilon}\,\underline{P}^T$$

$$\begin{cases} \sigma_1' = \sigma_2 \quad \sigma_2' = \sigma_1 \quad \sigma_3' = \sigma_3 \\ \sigma_4' = -\sigma_5 \quad \sigma_5' = \sigma_4 \\ \sigma_6' = -\sigma_6 \end{cases}$$

~~Same~~ same thing for strain.

$$\sigma_1' = K_{11}\varepsilon_1' + K_{12}\varepsilon_2' + K_{13}\varepsilon_3'$$

$$\to \sigma_2 = K_{11}\varepsilon_2 + K_{12}\varepsilon_1 + K_{13}\varepsilon_3$$

$$\sigma_2' = K_{21}\varepsilon_1' + K_{22}\varepsilon_2' + K_{23}\varepsilon_3$$

$$\sigma_2' = K_{21}\varepsilon_1' + K_{22}\varepsilon_2' + K_{23}\varepsilon_3'$$

$$\sigma_1 = K_{21}\varepsilon_2 + K_{22}\varepsilon_1 + K_{23}\varepsilon_3$$

$\sigma_1 = k_{12}\varepsilon_1 + k_{22}\varepsilon_2 + k_{23}\varepsilon_3$

$k_{12}\varepsilon_2 + k_{22}\varepsilon_1 = k_{11}\varepsilon_1 + k_{12}\varepsilon_2$

$\cancel{k_{11}\varepsilon_2 = k_{22}\varepsilon_2}$

$k_{11}\varepsilon_2 + k_{13}\varepsilon_3 = k_{22}\varepsilon_2 + k_{23}\varepsilon_3$

$(k_{11} - k_{22})\varepsilon_2 + (k_{13} - k_{23})\varepsilon_3 = 0$

$k_{11} = k_{22}$
$k_{13} = k_{23}$

$\sigma_4 = k_{44}\varepsilon_4$

$\sigma_4' = k_{44}\varepsilon_4'$
$\sigma_5' = k_{55}\varepsilon_5'$

$\sigma_4' = -\sigma_5 = -k_{44}\varepsilon_5'$

$\vee$

$\sigma_5 = k_{44}\varepsilon_5'$

$k_{44} = k_{55}$

$k_{44} = k_{55} \Rightarrow \left.\begin{aligned} k_{11} &= k_{22} \\ k_{13} &= k_{23} \end{aligned}\right\}$

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & 0 & 0 & 0 \\ & k_{11} & k_{13} & & & \\ & & k_{23} & & & \\ & & & k_{44} & & \\ & & & & k_{44} & \\ & & & & & k_{66} \end{bmatrix}$$

$\theta = 45° \rightarrow \pi/4$

$$[R] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\sigma_{12}', \ \varepsilon_{12}'$

calculation. this has to be true.

$$\underbrace{k_{66} = \tfrac{1}{2}(k_{11} - k_{12})}$$

$\frac{1}{2}[k_{11} - k_{12}]$

$k^{-1} \cdot \sigma = \varepsilon$

S matrix compliance index

Poisson's ratio for anisotropic elastic material. Can leave no bounds. Tricky TCT.

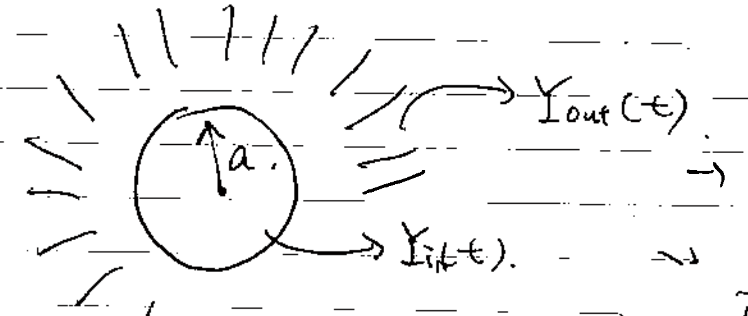
Linear viscoelasticity

— correspondence principle

↳ stress dependent of modulus

$$e_{ij} = S_{ij}(0^+)\, C_1(t) + \int_{0^+}^{t} C_1(t-\tau)\, \frac{\partial S_{ij}}{\partial \tau}\, d\tau$$

$$\varepsilon_{kk} = \sigma_{kk}(0^+)\, C_2(t) + \int_{0^+}^{t} C_2(t-\tau)\, \frac{\partial \sigma_{kk}}{\partial \tau}\, d\tau$$



$P_\infty(t)$.

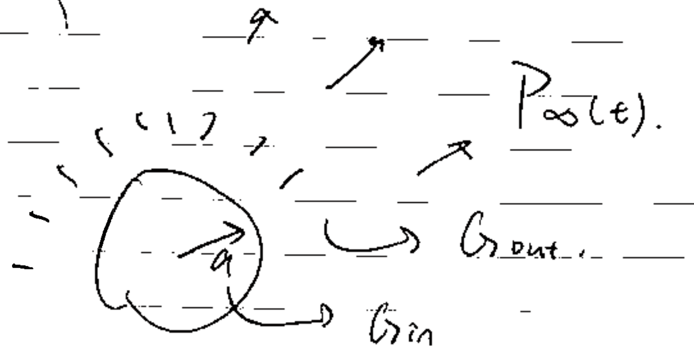
creep modulus.

↳ remote hydrostatic tension.

$P_\infty(t<0) = 0$

Elastic problem:

Plane stress



$$\boxed{\begin{cases} \sigma_{rr} = \dfrac{A}{r^2} + P_\infty \\ \sigma_{\theta\theta} = -\dfrac{A}{r^2} + P_\infty \\ \sigma_{r\theta} = 0 \end{cases} \qquad r > A}$$

$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{in}$.

$\Downarrow$ $\qquad r < A$.

Continuity of traction.

$\dfrac{A}{r^2} + P_\infty = \sigma_{in}$. $\qquad A, P_\infty \to$ unknown.

$3G_{out}\, \varepsilon_{\theta\theta} = \sigma_{\theta\theta} - \frac{1}{2}\sigma_{rr}$. $\qquad r > a$.

$$\underset{(r>a)}{\varepsilon_{\theta\theta}} = \frac{1}{3G_{out}}\left[\frac{-3A}{2r^2} + \frac{P_\infty}{2}\right].$$

$\Downarrow$

$\varepsilon_{\theta\theta} = \dfrac{u}{r}$

$r < a$, $\varepsilon_{in} = \varepsilon_{\theta\theta} = \varepsilon_{rr} = \dfrac{u}{r}$.

$u = \varepsilon_{in}\, r$

$3G_{in}\varepsilon_{in} = \sigma_{\theta\theta} - \frac{1}{2}\sigma_{rr}$

$= \sigma_{in}/2$.

$$\therefore \varepsilon_{in} = \frac{\sigma_{in}}{6 G_{in}} = \frac{u}{r}.$$

continuity of Hooke's strain.

$$\frac{r}{2G_{out}}\left[\frac{-3A}{2a^2} + \frac{P_\infty}{2}\right] = \frac{\sigma_{in}}{6 G_{in}}$$

$$\boxed{\frac{A}{a^2} = \frac{(\rho-1)P_\infty}{(1+3\rho)}, \quad \rho = \frac{G_{in}}{G_{out}} \qquad \sigma_{in} = \frac{4\rho}{1+3\rho} P_\infty}$$

$$\frac{G_{in}}{G_\infty} \to \infty \quad \Rightarrow \quad \frac{A}{a^2} = \frac{1}{3} P_\infty$$

$$\sigma_{\theta\theta}(r = a+) = -\frac{2}{3} P_\infty.$$

$$G_{in} = \frac{1}{2} s \tilde{Y}_{in}(s). \quad ☆☆☆$$

$$G_{out} = \frac{1}{2} s \tilde{Y}_{out}(s)$$

$$\tilde{\sigma}_{in} = \frac{4 \frac{\tilde{Y}_{in}(s)}{\tilde{Y}_{out}(s)}}{1 + 3 \frac{\tilde{Y}_{in}(s)}{\tilde{Y}_{out}(s)}} \tilde{P}_\infty(s)$$

$$Y_{in}(t) = Y_{\infty in} + (Y_{0 in} - Y_{\infty in}) e^{-t/t_{in}}$$

$$\mathcal{L}(Y_{in}(t)) = \tilde{Y}_i(s) = \int_0^\infty e^{-st} Y_i(t)\,dt.$$

$$\tilde{Y}_{in}(s) = \frac{Y_{in\infty}}{s} + \frac{(Y_{in\,0} - Y_{in\infty})}{s + t/t_{in}}$$

$$\tilde{Y}_{out}(s) = \frac{Y_{out\,\infty}}{s} + \frac{Y_{out\,0} - Y_{out\,\infty}}{s + t/t_{out}}$$

$$\sigma_{in}(t) = \frac{1}{2\pi i}\int_{\delta-\infty}^{\delta+\infty} e^{st}\,\tilde{\sigma}_\infty(t)\,ds.$$

★ Solve ODE with MATLAB.

HW 10. Review:

a. problem formulation.

$$\nabla^2 \phi = 0, \quad \text{in } |x| < \frac{a}{2} \ \& \ |y| < \frac{b}{2}$$

BCs:

$$\phi(x = \pm a/2,\ |y| < \tfrac{b}{2}) = \tfrac{1}{2}\left(\tfrac{a^2}{4} + y^2\right)$$

$$\phi(|x| < \tfrac{a}{2},\ y = \pm \tfrac{b}{2}) = \tfrac{1}{2}\left(\tfrac{b^2}{4} + x^2\right)$$

b. $f = \nabla_{xx}\phi + 1$.

on ~~the~~ boundary $y = \pm \frac{b}{2}$,

$$\Rightarrow f(|x| < a/2,\ y = \pm \tfrac{b}{2}) = 2$$

on the boundary $x = \pm \frac{a}{2}$.

we know $\partial_{xx}\phi = -\partial_{yy}\phi$.

$$\Rightarrow f(x = \pm \tfrac{a}{2},\ |y| < \tfrac{b}{2}) = 0$$

c. find f:

$$f(x,y) = X(x)\, Y(y).$$

Substitute into $\nabla^2 f = 0$:

$$\ddot{X}(x) Y(y) + X(x)\ddot{Y}(y) = 0.$$

$$\Rightarrow \frac{\ddot{X}(x)}{X(x)} = -\frac{\ddot{Y}(y)}{Y(y)} = C = -k^2.$$

we look for solution: satisfy $x = \pm \frac{a}{2}$.

$$\begin{cases} \ddot{X}(x) + k^2 X(x) = 0 \\ \ddot{Y}(y) - k^2 Y(y) = 0 \end{cases}$$

$$\begin{cases} X(x) = B\sin kx + A\cos kx \xrightarrow{x = \pm\frac{a}{2},\ X = 0} \\ Y(y) = C\cosh k_n y + D\sinh k_n y. \end{cases}$$

$$f(x,y) = \sum_i a_n \cos(k_n x)\cosh(k_n y).$$

BCs: $f(x < \frac{a}{2},\ y = \pm \frac{b}{2}) = 2.$

$$\sum_i a_n \cos(k_n x)\cosh(k_n b/2) = 2.$$

method of Fourier series:

$$a_n = \frac{2}{a\cosh(k_n b/2)} \int_{-a/2}^{a/2} 2\cos(k_n x)\,dx$$

$$= \frac{8(-1)^n}{\pi(2n+1)\cosh(k_n b/2)}.$$

Thus:

$$f(x,y) = \frac{8}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cdot\frac{\cosh(k_n y)}{\cosh(k_n b/2)}\cos(k_n x),$$

$$k_n = \frac{2n+1}{a}\pi$$

d: find max stress:

Shear stresses:

$$\begin{cases} \sigma_{13} = G\gamma(-y + \phi_{,2}) \\ \sigma_{23} = G\gamma(x - \phi_{,1}) \\ \frac{\sigma_{13}}{G\gamma} + y = \phi_{,2} \\ -\frac{\sigma_{23}}{G\gamma} + x = \phi_{,1} \end{cases}$$

on the boundary $x = \pm\frac{a}{2}$, $\phi_{,2}\Big|_{x=\pm\frac{a}{2}} = y$.

$\sigma_{13} = 0$ on $x = \pm\frac{a}{2}$.

$\sigma_{23} = 0$ on $y = \pm\frac{b}{2}$.

$\phi_{,1}\Big|_{y=\pm\frac{b}{2}} = x.$