COURSE NOTES

FOUNDATIONS OF SOLID MECHANICS

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2021

Week 1: Mon. 8/29/2021

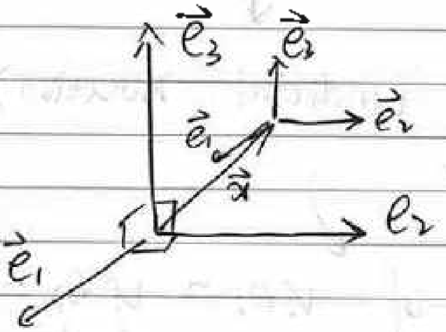
Vectors & tensors. → Cartesian.

in Physics, we are familiar with

$$\begin{cases} \nabla \times \vec{E} = 0 \\ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \end{cases} \Rightarrow \text{Electro-statics.}$$

↳ which is, independent of coordinate system

in a RH coordinate,



$\|\vec{e}_i\| = 1.$

$$\vec{V} = \sum_{i=1}^{3} V_i \vec{e}_i$$

index subscript ← $\vec{e}_i \cdot \vec{e}_j \equiv \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$

⇓

called Kronecker delta.

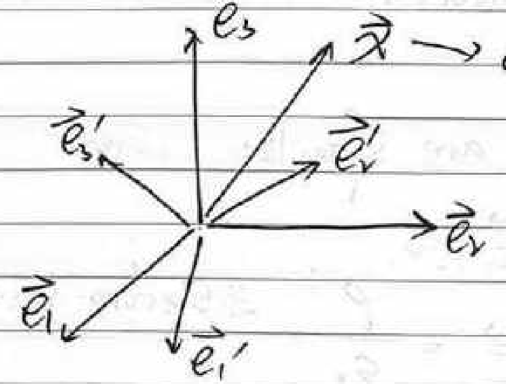
(in orthonormal basis).

Here, v_i is component of $\vec{v}$

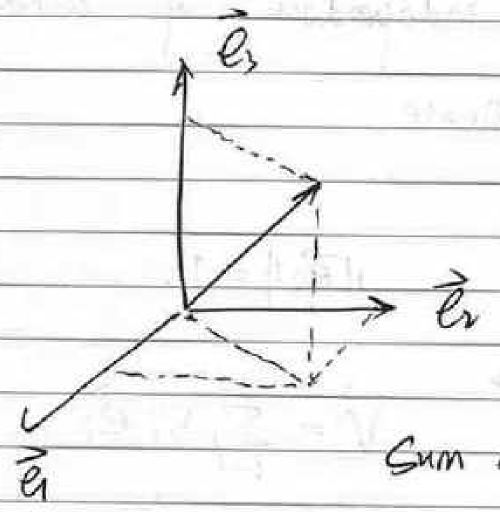
with a basis $\{\vec{e}_i\}$.

$$= \sum_{j=1}^{3} V_j' \vec{e}_j'$$

transformation of basis.



→ cooridinate dependent.



Summation convection

↓

(indicial notation).

↓

Sum of $v_i e_i = \underbrace{v_i e_i}_{\text{repeated indices.}} = v_k e_k$

term: subscripts. $\underbrace{a_{ij} b_k c_{jm}}_{\text{summing over } j} \rightarrow$ free indices.

contraction, if $i=m$.

$a_{ip} b_k c_{pi} \rightarrow$ double summation.

*** A dummy index cannot repeat more than 2 !!!

$\delta_{ij}\delta_{jk} = d_{ik} = \delta_{ik}$.

e.g. $\delta_{ij}\delta_{jk} = \delta_{11}\delta_{1k} + \underset{0}{\underset{\shortparallel}{\delta_{12}\delta_{2k}}} + \underset{0}{\underset{\shortparallel}{\delta_{13}\delta_{3k}}}$.

$= \delta_{1k}$.

$\delta_{2k}\ (i=2),\ \delta_{3k}\ (i=3)$.

$\vec{v} \cdot \vec{w} = v_i \vec{e}_i \cdot w_j \vec{e}_j = v_i w_j \underbrace{(\vec{e}_i \cdot \vec{e}_j)}_{\delta_{ij}}$

$= v_i w_i = v_j w_j$.

usual dot product.

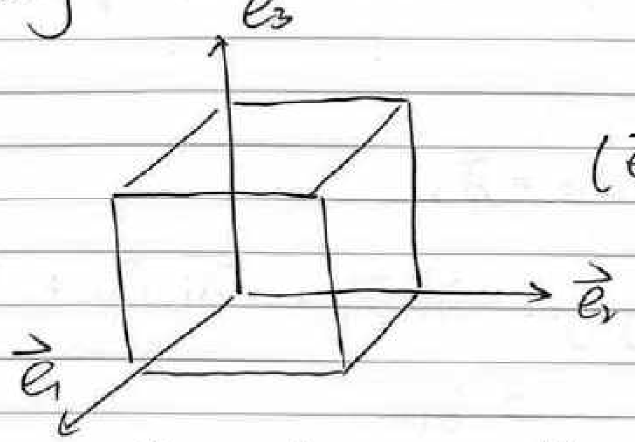
Cross product.

$\vec{v} \times \vec{w} = v_i \vec{e}_i \times w_i \vec{e}_j = v_i w_j\ \vec{e}_i \times \vec{e}_j$ (a).

$\vec{e}_i \times \vec{e}_j = [\underbrace{(\vec{e}_i \times \vec{e}_j) \cdot \vec{e}_k}_{= \varepsilon_{ijk}}] \vec{e}_k$

free index on two sides of Eqn. must be equal !!!

$= \varepsilon_{ijk}$ (Permutation symbol) e.g.



$(\vec{e}_1 \times \vec{e}_2) \cdot \vec{e}_3 = 1.$

$\epsilon_{ijk} = 1, \quad (1,2,3), (3,1,2), (2,3,1).$

$= -1, \quad (2,1,3), (1,3,2), (3,2,1).$

$= 0, \quad$ otherwise .

Review ~~ong~~ undergrad linear algebra

det(~).

eq. (a) writes. $v_i w_j \epsilon_{ijk} \vec{e}_k$

$\Rightarrow \epsilon_{ijk} v_i w_j \vec{e}_k$

$= \epsilon_{kij} v_i w_j \vec{e}_k$

Q. $\vec{a} \times (\vec{b} \times \vec{c}) = ?$

$= a_k \vec{e}_k \times (b_i c_j \vec{e}_i \times \vec{e}_j)$

$= a_k \vec{e}_k \times (b_i c_j \epsilon_{ijm} \vec{e}_m).$

$= a_k b_i c_j (\delta_{iw} \delta_{jk} - \delta_{ik} \delta_{jw}) \vec{e}_w$

$= (a_i \vec{e}_i) \times [\epsilon_{pjk} b_j c_k \vec{e}_p].$

$= (\delta_{sj} \delta_{ik} - \delta_{sk} \delta_{ij}) a_i b_j c_k \vec{e}_s$

$= (b_s a_k c_k - b_i a_i c_s) \vec{e}_s$

$= [b_s (\vec{a} \cdot \vec{c}) - (\vec{a} \cdot \vec{b}) c_s] . \vec{e}_s$

$= (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot b) . \vec{c}$

- How vectors transform? (on basis) .

$\vec{v} = v_i \vec{e}_i = v'_j \vec{e}'_j$

$v'_j = (\vec{v} \cdot \vec{e}'_j) = (v_i \vec{e}_i \cdot \vec{e}'_j).$

$= v_i (\vec{e}_i \cdot \vec{e}'_j).$

$P_{ji} = \vec{e}'_j \cdot \vec{e}_i .$

↙

projection of one basis on another basis.

$\vec{v}'_j = P_{ji} \cdot \cancel{v_j} \vec{v}_i$

$\vec{v}' = \begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$\vec{P} = [P_{ji}]$

$\vec{v}' = \vec{P} \vec{v}$

$\vec{v} = \vec{P}^{-1} \vec{v}'$

☆☆☆ $\hookrightarrow \vec{P}^{-1} = \vec{P}^{t}$

off-course supplementary :

Week 1:, Wed. 9/1/2021.

Second order Tensor $\underline{\underline{A}}$.

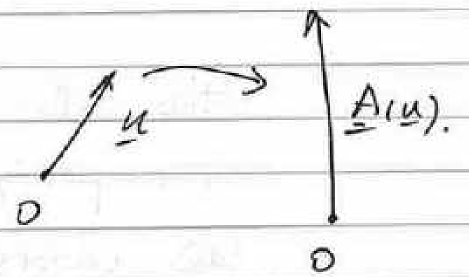
A 2nd order tensor is a linear transformation from $\mathbb{E}^3$ to $\mathbb{E}^3$

LT = a special kind of mapping $\mathbb{E}^3 \to \mathbb{E}^3$

properties:

$\underline{\underline{A}}(\underline{u})$ to some vector

$\underline{\underline{A}} : (\underline{u}) \longrightarrow \underline{\underline{A}}(\underline{u})$.



$\to \underline{\underline{A}}(\underset{\nwarrow \text{ real No.}}{a}\underline{u}) = a\,\underline{\underline{A}}(\underline{u})$.

$\to \underline{\underline{A}}(\underline{u}+\underline{w}) = \underline{\underline{A}}(\underline{u}) + \underline{\underline{A}}(\underline{w})$.

$\Updownarrow$

$\underline{\underline{A}}(a\underline{u}+b\underline{w}) = a\underline{\underline{A}}(\underline{u}) + b\,\underline{\underline{A}}(\underline{w})$.

$\forall\, \underline{u}, \underline{w}$ w/ $\mathbb{E}^3$ and a, b.

$\underline{\underline{A}}(\underline{0}) = \underline{0}$.

Example rigid body rotation about a fixed point.

{Defination gradient tensor.
Stress tensor

$\underline{x} = x_i \vec{e}_i$

$\underline{\underline{A}}(\underline{x}) = \underline{\underline{A}}(x_j \vec{e}_j)$.

$= \cancel{A_j}\ x_j\ \underline{\underline{A}}(\underline{e}_j)$

this tells us that a linear transformation is completely determined by its action on the basis vectors.

$\underline{\underline{A}}(\underline{e}_j) = a_{ij}\,\underline{e}_i$

$a_{ij}\, e_i\, (e_j \cdot \underline{x})$

$\underline{\underline{A}}(x_j e_j) = \cancel{a_{ij}}\ x_j\, \underline{\underline{A}}\cdot(e_j) = a_{ij} x_j\, \underline{e}_i$

$\underline{\underline{A}}(\underline{e}_1)\cdot \underline{e}_1 = a_{11}$

$\underline{\underline{A}}(\underline{e}_1)\cdot \underline{e}_2 = a_{21}$

$\underline{\underline{A}}(\underline{e}_1)\cdot \underline{e}_3 = a_{31}$

$$A = \begin{bmatrix} \cancel{A}\, a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\underline{\underline{A}}(\underline{x}) = a_{ij}\, \underline{e}_i (\underline{e}_j \cdot \underline{x})$

$= \underbrace{a_{ij}\, \underline{e}_i\, \underline{e}_j}_{\underline{\underline{A}}} \cdot \underline{x}$

Define $\underline{a}\underline{b}$ as the linear transformation.

$(\underline{a}\underline{b})(\underline{x}) = \underline{a}(\underline{b}\cdot\underline{x})$. ??

↓

check: this is a LT.

$= \underline{\underline{A}}\cdot\underline{x}$ (we can skip the dot). ??

$\underline{a}\underline{b}$ → dyad

*** Any linear transformation can be written as sum of dyad

$\underline{a}\underline{b} \Leftrightarrow \underline{a} \otimes \underline{b} \rightarrow$ linear transformation.

A simple representation: $\underline{\underline{A}} = a_{ij}\, e_i\, \underline{e}_j$

$\underline{a}\underline{b} \neq \underline{b}\underline{a}$.

$\underline{e}_i \to \underline{e}'_k$.

$$\underline{\underline{A}} = a_{ij}\, \underline{e}_i \underline{e}_j = a'_{rs}\, \underline{e}'_r \underline{e}'_s$$

$$= a_{ij} \underbrace{(\underline{e}_i \cdot \underline{e}'_r)\, \underline{e}'_r}_{\underline{e}_i} \underbrace{(\underline{e}_j \cdot \underline{e}'_s)\, \underline{e}'_s}_{\underline{e}_j}$$

$$= a_{ij} (\underbrace{\underline{e}_i \cdot \underline{e}'_r}_{P_{ri}}) (\underbrace{\underline{e}_j \cdot \underline{e}'_s}_{P_{sj}})$$

$$= a'_{rs}\, \underline{e}'_r \underline{e}'_s$$

$a'_{rs} = a_{ij} P_{rs} P_{ij}$.

$[PA]_{rj} [P^T]_{js}$.

$A' = PAP^T \qquad P^T A' P = A$.

$\Downarrow$

$[a'_{rs}]$.

$\underline{\underline{A}}(\underline{x}) = \underline{y}$.

$\det \underline{\underline{A}} \equiv \det A$.

$$\det A = \det(P^T A' P) = \det P^T \det A' \det P$$

$$= \det(\underbrace{P^T P}_{I}) \det A'$$

$$= \det A'$$

*** det is invariant

$$(a\underline{\underline{A}} + b\underline{\underline{B}})(\underline{x}) \equiv a\underline{\underline{A}}(\underline{x}) + b\underline{\underline{B}}(\underline{x})$$

composition $\boxed{\underline{\underline{A}}^{-1}}$ of mapping.

$$(\underline{\underline{A}} \circ \underline{\underline{B}})(\underline{x}) \equiv \underline{\underline{A}}(\underline{\underline{B}}(\underline{x}))$$

$$= \underline{\underline{A}} \cdot (\underline{\underline{B}} \cdot \underline{x}).$$

$\underline{\underline{B}} \cdot \underline{x} = b_{ij}\, \underline{e}_i \underline{e}_j \cdot \underline{x}$

$$\underline{\underline{A}} \cdot (\underline{\underline{B}} \cdot \underline{x}) = (a_{rs}\, \underline{e}_r \underline{e}_s) \cdot b_{ij}\, \underline{e}_i \underline{e}_j \cdot \underline{x}.$$

$$= a_{rs}\, \underline{e}_r\, b_{ij} \underbrace{(\underline{e}_s \cdot e_i)}_{\delta_{si}} \underline{e}_j \cdot \underline{x}.$$

$$= a_{rs} b_{sj}\, \underline{e}_r \underline{e}_j \cdot \underline{x}.$$

$[\underline{\underline{C}}] = C \qquad C = AB$.

$\underline{\underline{A}}^{-1} \circ \underline{\underline{A}} = \underline{\underline{I}}.$

$\underline{\underline{I}} = \delta_{ij}\, \underline{e}_i \underline{e}_j = \underline{e}_i \underline{e}_i$

↓

identity tensor.

$\underline{\underline{A}}^T$: transpose of $\underline{\underline{A}}$.

$\underline{v} \cdot \underline{\underline{A}}^T \cdot \underline{u} = \underline{u} \cdot \underline{\underline{A}} \cdot \underline{v}$

for all $\underline{u}$ & $\underline{v}$ in $\mathbb{E}^3$

$\underline{u} = \underline{e}_j$

$\underline{v} = \underline{e}_i$

$$\underline{e}_j \cdot \underline{\underline{A}} \cdot \underline{e}_i$$

$$= \underline{e}_j \cdot a_{rs}\, \underline{e}_r \underbrace{\underline{e}_s \cdot \underline{e}_i}_{\delta_{si}}$$

$$= a_{rs}\, \delta_{jr} \delta_{si} = a_{ji}.$$

$$\underline{e}_i \cdot \underline{\underline{A}}^T \cdot \underline{e}_j = \underbrace{a^T_{ij} = a_{ji}}.$$

⇓

true for 1 coor,

true for all ~

$(\underline{\underline{A}} + \underline{\underline{B}})^T = \underline{\underline{A}}^T + \underline{\underline{B}}^T$

$(\underline{\underline{A}}^T)^T = \underline{\underline{A}}$ $\quad (\underline{\underline{A}}^{-1})^T = (\underline{\underline{A}}^T)^{-1}$

$(\underline{\underline{A}} \circ \underline{\underline{B}})^T = \underline{\underline{B}}^T \circ \underline{\underline{A}}^T$

$\underline{\underline{A}}^T = \underline{\underline{A}}$ symmetric

$\underline{\underline{A}}^T = -\underline{\underline{A}}$ asymmetric → mechanics of solids.

*** Eigenvalue of asymmetric tensor

$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}})$ → independent of basis

$$= \det(A' - \lambda I)$$

$$= \det[\underbrace{PAP^T}_{A'} - \lambda \underline{\underline{I}}].$$

$PP^T = I$

$$= \det[P(A - \lambda I)P^T]$$

$$= \underbrace{\det[PP^T]}_{1} \det(A - \lambda I]$$

$$\equiv \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}).$$

$\det(A - \lambda I)$

$= P_3(\lambda) = (-\lambda)^3 + I_1 \lambda^2 - I_2 \lambda + \det A.$

$I_1 = \frac{1}{2}[(tr(A))^2 - tr(A^2)]$

$I_2 = A_{ii} = tr(A).$

↳ $a_{11} + a_{22} + a_{33}.$

I_1 , I_2 , det A

are scalar movement of the Tensor $\underline{\underline{A}}$

$$\underline{\underline{A}} = \lambda_1 \underline{E}_1 \underline{E}_1 + \lambda_2 \underline{E}_2 \underline{E}_2 + \lambda_3 \underline{E}_3 \underline{E}_3$$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix} \qquad \{E_i\}$$

↑ eigenvectors of $\underline{\underline{A}}$

$$\det A = \lambda_1 \lambda_2 \lambda_3$$

$$I_2 = \lambda_1 + \lambda_2 + \lambda_3 .$$

$$I_1 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3 .$$

☆ how to diagonize 3x3 matrix.

(Labor day - Monday).

Week 2: Wed.

(Review)

$$\underline{\underline{A}}(\underline{v}) - \underline{\underline{A}} \cdot \underline{v}$$

$$\underline{\underline{A}} = a_{ij} \underline{e}_i \underline{e}_j$$

$$\underline{\underline{A}}(\underline{v}) = a_{ij} v_j \underline{e}_i$$

$$= a_{ij} \underline{e}_i \underbrace{\underline{e}_j \cdot \underline{v}}_{v_j}$$

$$(\underline{\underline{A}} \circ \underline{\underline{B}})(\underline{v}) = \underline{\underline{A}}(\underline{\underline{B}}(\underline{v}))$$

$$= (a_{ij} \underline{e}_i \underline{e}_j)(b_{kl} V_l \underline{e}_k).$$

$$= a_{ij} \underline{e}_i b_{kl} V_l \delta_{jk}$$

$$= a_{ij} b_{jl} V_l \underline{e}_i$$

in other words,

$$\underline{\underline{A}} \circ \underline{\underline{B}} = a_{ij} b_{jl} \underline{e}_i \underline{e}_l$$

$$\wedge$$

$$= \underline{\underline{A}} \otimes \underline{\underline{B}} =$$

$$\underline{\underline{A}} = a_{ij} \underline{e}_i \underline{e}_j \ , \quad \underline{\underline{B}} = b_{kl} \underline{e}_k \underline{e}_l$$

$$\underline{\underline{A}} \cdot \underline{\underline{B}} = a_{ij} \underline{e}_i \underline{e}_j \cdot b_{kl} \underline{e}_k \underline{e}_l$$

$$\underline{\underline{A}}\underline{\underline{B}} = \underline{\underline{A}} \circ \underline{\underline{B}}$$

$\underline{\underline{AB}}$

$$\underline{a}\,\underline{b} : \underline{c}\,\underline{d} \equiv (\underline{a}\cdot\underline{c})(\underline{b}\cdot\underline{d})$$

$$\underline{a}\,\underline{b} \cdot\cdot\, \underline{c}\,\underline{d} \equiv (\underline{a}\cdot\underline{d})(\underline{c}\cdot\underline{b})$$

$$\underline{\underline{A}} = a_{ij}\,\underline{e}_i\,\underline{e}_j$$

$$\underline{\underline{B}} = b_{kl}\,\underline{e}_k\,\underline{e}_l$$

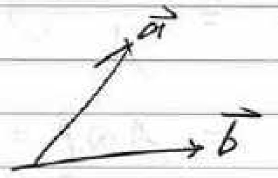
$$\underline{\underline{A}} : \underline{\underline{B}} = (a_{ij}\,\underline{e}_i\,\underline{e}_j) : (b_{kl}\,\underline{e}_k\,\underline{e}_l)$$

$$\underline{\underline{A}} : \underline{\underline{B}} = a_{ij} b_{kl}\,\underbrace{(\underline{e}_i\cdot\underline{e}_k)}_{\delta_{ik}}\underbrace{(\underline{e}_j\cdot\underline{e}_l)}_{\delta_{jl}}$$

$$= a_{kj}\,b_{kj} \rightarrow \text{scalar}$$

↳ Can be extended to 2 angular relations of linear transformations

e.g. For vectors



$\underline{a}\cdot\underline{b}$

$$tr(\underline{\underline{A}})$$

$$= a_{ij}\underbrace{(\underline{e}_i\cdot\underline{e}_j)}_{\delta_{ij}} = a_{ii}$$

$$tr(\underline{\underline{A}}+\underline{\underline{B}}) = tr\,\underline{\underline{A}} + tr\,\underline{\underline{B}}.$$

$$tr(a\underline{\underline{A}}) = a\,tr(\underline{\underline{A}}).$$

$$tr(\underline{\underline{A}}^T) = tr(\underline{\underline{A}}).$$

$$tr(\underline{\underline{A}}\circ\underline{\underline{B}}) = tr(\underline{\underline{B}}\circ\underline{\underline{A}})$$

Tensor Field

Scalar field $\quad f(\underline{x})$

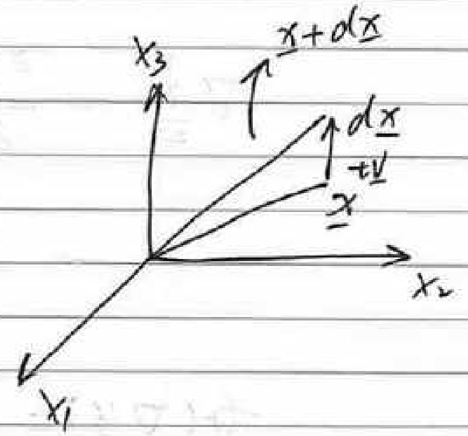
$$dx_i = d\underline{x}\cdot\underline{e}_i$$

$$df = \frac{\partial f}{\partial x_i}\,dx_i.$$

$$= \frac{\partial f}{\partial x_i}(\underline{e}_i\cdot d\underline{x}).$$

$$\nabla f \equiv \frac{\partial f}{\partial x_i}\,\underline{e}_i$$

$$= \nabla f\cdot d\underline{x}$$



$$\nabla\underset{\sim}{u} = ?$$

$$\lim\left(\frac{\underline{u}(\underline{x}+t\underline{v}) - \underline{u}(\underline{x})}{t}\right) \Longleftarrow (\nabla\underline{u})\cdot\underline{v} \equiv \lim\frac{\underline{u}(\underline{x}+t\underline{v}) - \underline{u}(\underline{x})}{t}$$

$$= \frac{\left[\underline{u}(\underline{x}) + \frac{\partial u}{\partial x_i} + v_i\right] - u_i(\underline{x})}{t}$$

$$= \frac{\partial \underline{u}}{\partial x_k} V_k$$

$$= \frac{\partial (u_i \underline{e}_i)}{\partial x_k} V_k = \frac{\partial u_i}{\partial x_k} \underline{e}_i (\underline{e}_k \cdot \underline{V}).$$

$$= \underbrace{\left(\frac{\partial u_i}{\partial x_k} \underline{e}_i \underline{e}_k \right)}_{\nabla \underline{u}} \cdot \underline{V}$$

$\nabla \underline{u} = \frac{\partial u_i}{\partial x_k} \underline{e}_i \underline{e}_k$ → bump up by 1 order (w/ gradients).

$$\nabla \underline{\underline{P}} = \frac{\partial \underline{\underline{P}}}{\partial x_k} \underline{e}_k$$

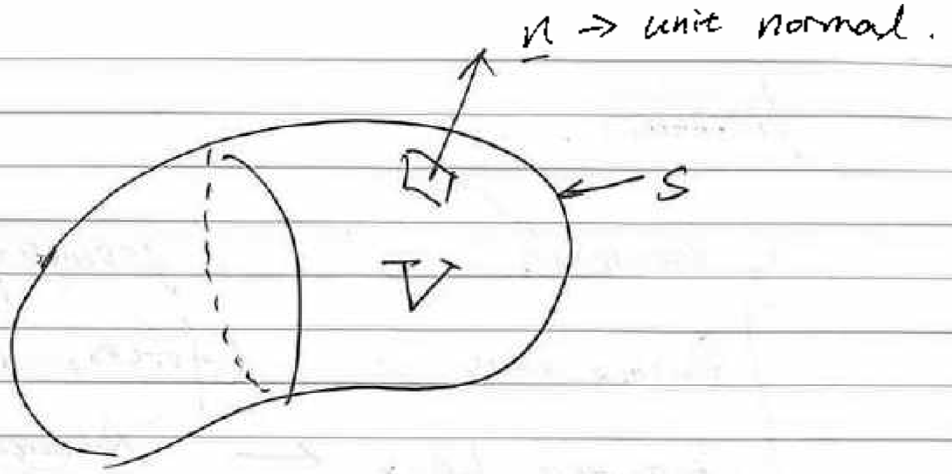
$$= \frac{\partial P_{ij}}{\partial x_k} \underline{e}_i \underline{e}_j \underline{e}_k$$

$$\mathrm{tr}(\nabla \underline{u}) = \frac{\partial u_i}{\partial x_k} (\underline{e}_i \cdot \underline{e}_k)$$

$$= \frac{\partial u_i}{\partial x_i}$$

$$= \nabla \cdot \underline{u}.$$

$$\nabla \cdot \underline{\underline{P}} = \frac{\partial P_{ij}}{\partial x_k} \underline{e}_i \underbrace{(\underline{e}_j \cdot \underline{e}_k)}_{\delta_{jk}} = \frac{\partial P_{ij}}{\partial x_j} \underline{e}_i$$



$$\iiint_V f_{,i} \, dV = \iint_S f \, n_i \, dS.$$

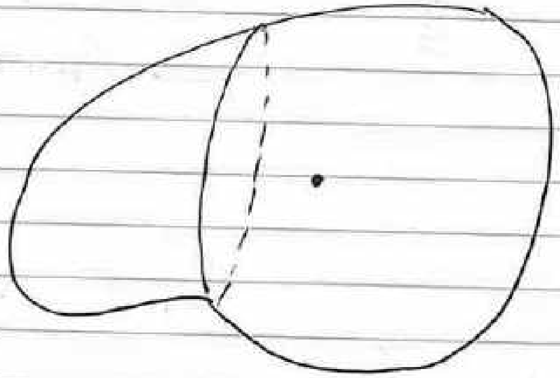
→ Green's theorem.

$$f_{,i} = \frac{\partial f}{\partial x_i}$$

$$\left. \begin{aligned} \int_V u_{j,i} \, dV &= \int_S u_j n_i \, dS \\ \int_V T_{kl,i} \, dV &= \int_S T_{kl} n_i \, dS. \end{aligned} \right\}.$$

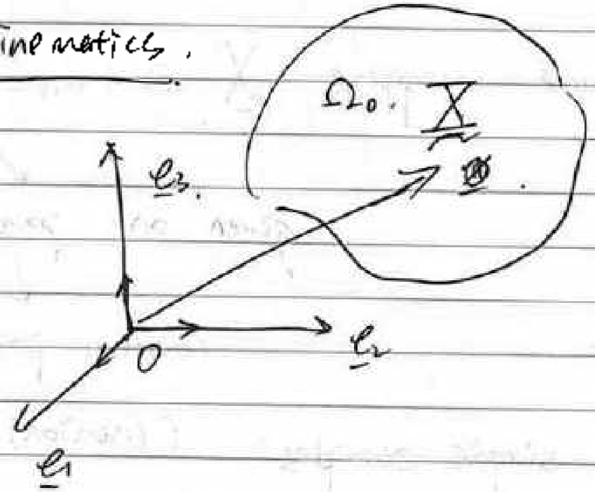
Mechanics.

- Kinematics. ✓ (geometry)
- Balance laws ✓ (forces, moments, ...)
- Contitutive laws ← "Research". connect & .



Week 3: Mon.

Kinematics.

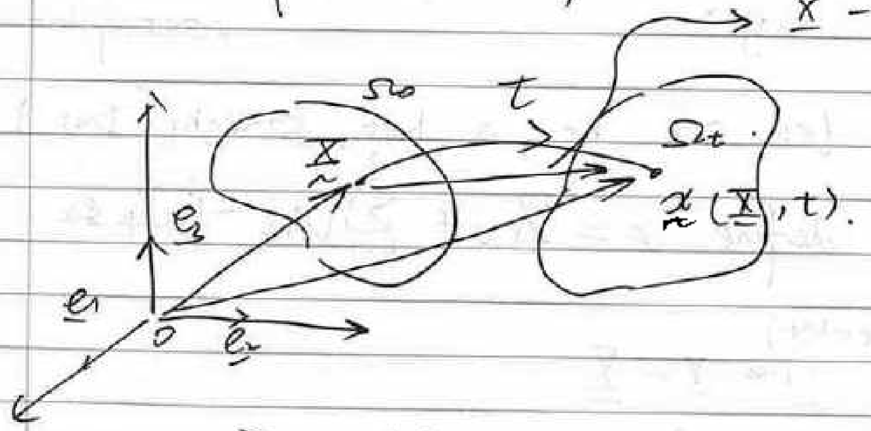


Material point is labeled by its coordinate $\underline{X}_i$ or $\underset{\sim}{X}$.

(Ref. config.).

Ω_0 is the configuration of the Body at $t=0$.

Normally we choose Ω_0 to be the undeformed State of the body.



$\underline{x} - \underline{X} = \underline{u}(\underline{X}, t)$.

$t>0$. Body deforms at occupies Ω_t.

$\underset{\sim}{X} \rightarrow \underset{\sim}{x}$

mapping.

function "kai".

this is a mapping: $\underset{\sim}{x} = \chi(\underset{\sim}{X}, t)$.

$= \underline{x}(\underline{X}, t)$.

$\underline{u} = \underline{x} - \underline{X}$ → displacement vector.

Always assume mapping $\underline{\chi}$ is one-one points

↓

given one point

↓

another point associated w/ it.

*** Some simple examples: (motion).

$\underline{x} = \underline{X} + \underline{c}(t)$. Rigid body translation.

X interesting cuz no defamation.

***. e.g. 2.

rectangular crosssection.

let Ω_0 be a bar, (straight bar)

we define $\underline{x} = \underline{X} + \sum_{k=1}^{3} (\lambda_k - 1) X_k \underline{e}_k$.

(remember)

$\underline{u} = \underline{x} - \underline{X}$

$\underline{u} = \sum_{k=1}^{3} (\lambda_k - 1) X_k \underline{e}_k$

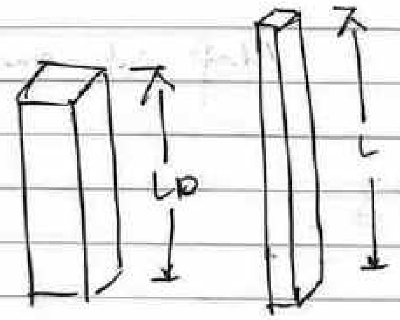
↓

real positive numbers.

$\lambda_k = 1$: no displacements → body remain initial state.

$\lambda_k \neq 1$: Strech & compress in $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ directions

↳ Stretch ratios

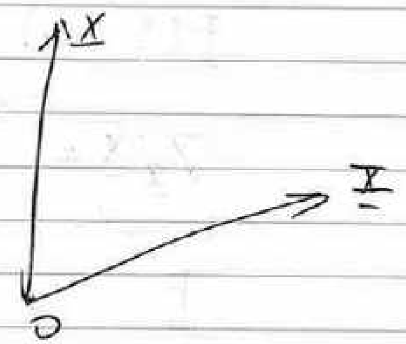


$\lambda_k = \frac{L}{L_0}$.

☆ you can always impose a displacement field on a body.

Rigid body rotation.

$\underline{x} = \underline{R}(\underline{X})$.

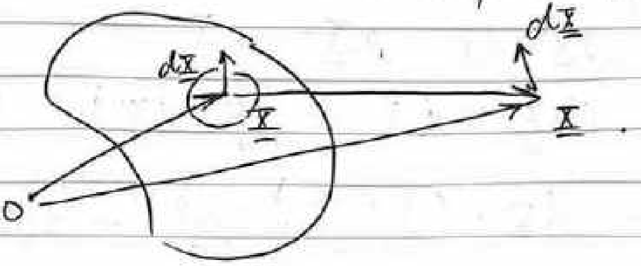


$\underline{e}_k \longrightarrow \underbrace{\underline{R}(\underline{e}_k)}_{\underline{n}_k}$

$\boxed{\underline{R} = \underline{n}_k \underline{e}_k.}$ ⟷ rotation.

linear trans. → completely det. by action on its basis

does not increase any defination



$$d\underline{x} = \frac{\partial \underline{x}}{\partial \underline{X}_j} d\underline{X}_j$$ (def of grad.).

$$= \underbrace{\nabla_{\underline{X}} \underline{x}}_{\underline{\underline{F}}} \cdot d\underline{X}.$$

$\underline{\underline{F}} = \nabla_{\underline{X}}$ → deformation gradient tensor.

↳ contains all the information on local deformation.

$\underline{\underline{F}}(\underline{X}, t)$

$$\underbrace{\nabla_{\underline{X}} \underline{x}}_{\underline{\underline{F}}} = \frac{\partial (x_i \underline{e}_i)}{\partial \underline{X}_j} \underline{e}_j$$

$$= \frac{\partial x_i}{\partial \underline{X}_j} \underline{e}_i \underline{e}_j$$

$$[\underline{\underline{F}}] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \vdots & \vdots & \vdots \end{bmatrix}.$$

↓

respect to the basis $\underline{e}_i$, $\underline{e}_j$

$$d\underline{x} = \underline{\underline{F}} \cdot d\underline{X}.$$

<u>change of length</u> (fiber)

$$d\underline{x} \cdot d\underline{x} - d\underline{X} \cdot d\underline{X}$$

$$= (\underline{\underline{F}} \cdot d\underline{X}) \cdot (\underline{\underline{F}} \cdot d\underline{X}) - d\underline{X} \cdot d\underline{X}.$$

$$= d\underline{X} \cdot (\underbrace{\underline{\underline{F}}^T \cdot \underline{\underline{F}}}_{\underline{\underline{C}}}) \cdot d\underline{X} - d\underline{X} \cdot d\underline{X}$$

$\underline{\underline{C}}$ is the Cauchy - Green Tensor

$$= d\underline{X} \cdot (\underbrace{\underline{\underline{C}} - \underline{\underline{I}}}_{\underline{\underline{E}}}) \cdot d\underline{X}.$$

$\underline{\underline{E}}$ ↓

$$\underline{\underline{I}} \cdot d\underline{X} = d\underline{X}.$$

$$\frac{\|d\underline{x}\|^2}{\|d\underline{X}\|^2} = \left(\frac{\|d\underline{x}\|}{\|d\underline{X}\|}\right)^2$$ Lagrangian Strain Tensor

$$\frac{d\underline{x} \cdot d\underline{x}}{\|d\underline{X}\|^2} - 1$$

$$= \frac{d\underline{X}}{\|d\underline{X}\|} \cdot \underline{\underline{C}} \cdot \frac{d\underline{X}}{\|d\underline{X}\|} - 1$$

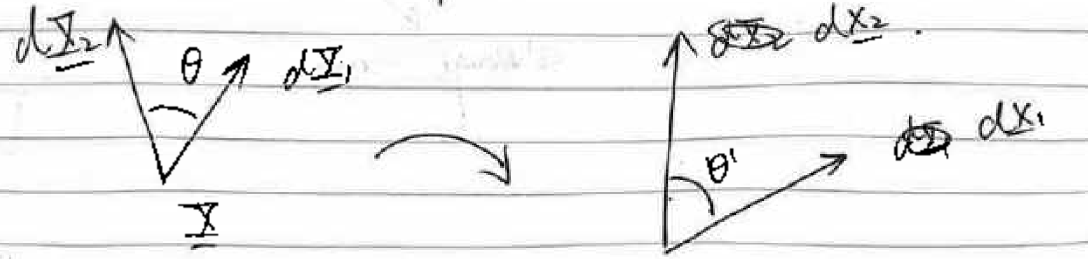
$$= \underline{N} \cdot \underline{\underline{C}} \cdot \underline{N} - 1$$ stretch ratio

$\underline{N}$ ↓ unit vector

$\underline{X}$, t.

Solid: $\underline{X}$ → reference configuration.

Fluid: $\underline{x}$ ↳ spatial → current coordinates.



don't use summation with λ !!!

Ex 1: $\underline{u} = \sum_{k=1}^{3} (\lambda_k - 1) \cancel{\theta} X_k \underline{e}_k$ (3k).

$$[\underline{\underline{F}}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad \underline{F} = \lambda \underline{e}_1\underline{e}_1 + \lambda_2 \underline{e}_2\underline{e}_2 + \lambda_3 \underline{e}_3\underline{e}_3$$

$$\underline{\underline{F}} = \lambda_1 \underline{e}_1\underline{e}_1 + \lambda_2 \underline{e}_2\underline{e}_2 + \lambda_3 \underline{e}_3\underline{e}_3.$$

$\lambda_1 = \lambda_2 = \lambda_3 > 1.$

uniform expansion:

$$[\underline{\underline{F}}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 \cdot & \lambda_3 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = \lambda_3 < 1$

Uniform compression

$\det \underline{\underline{F}} = J$

invariant.

$= \lambda_1 \lambda_2 \lambda_3.$

Rubber is almost incompressible, so $J \simeq 1$.

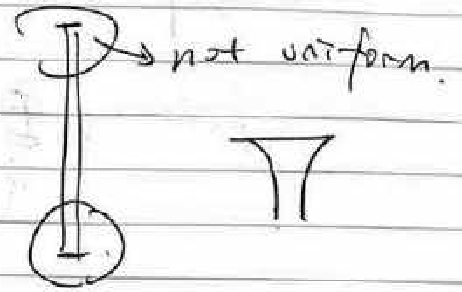
In general, $\det \underline{\underline{F}} = \frac{dV}{dV_0}$ the deform of the volume over the reference (original) volume.

always true.

$= \frac{V}{V_0}$ ← new

reference volume

in a tension bar:



$\underline{\underline{E}} = E_{ij} \underline{e}_i \underline{e}_j$

$$E_{ij} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)}_{\varepsilon_{ij}} + \frac{1}{2}\left[\frac{\partial u_k}{\partial X_i} \cdot \frac{\partial u_k}{\partial X_j}\right].$$

small strain tensor

(1% ~ 2%).

effect of large defor. quadratic. term.

→ (10^{-2}% ~ $4 \cdot 10^{-2}$%)

$10^{-2} \cdot 10^{-2} = 10^{-4}.$

Week 3.

Sep. 15th (Wed.)

Review. $\underline{\underline{F}}$ Deformation Gradient Tensor

$\uparrow$ completely characterize the local deformation at a point $\underline{X}$.

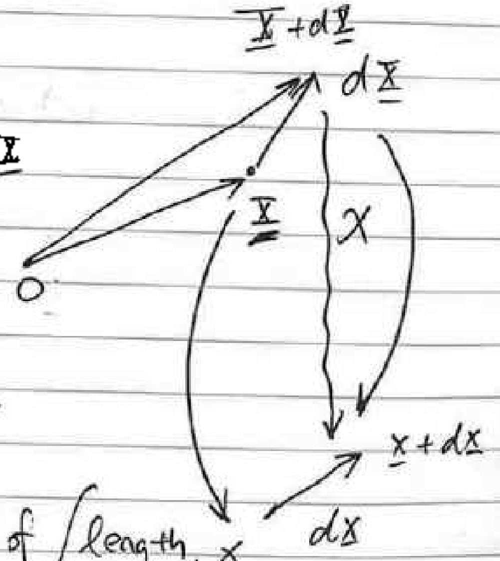
$\underline{\underline{F}}(\underline{X}, t)$, $\underline{X}$, t, independent variables.

$\hookrightarrow$ Material description.

$\underline{\underline{C}} = \boxed{\underline{\underline{F}}^T \underline{\underline{F}}}\ .\ \underline{\underline{F}}^T \underline{\underline{F}}$ Right Caughty - Green Tensor

$\underline{\underline{E}} = \underline{\underline{F}}^T \underline{\underline{F}} - \underline{\underline{I}}$.

$d\underline{x} \cdot d\underline{x} = d\underline{X} \cdot \underline{\underline{C}} \cdot d\underline{X}$



$\underline{N}$ is a unit vector.

$$\frac{\|d\underline{x}\|^2}{\|d\underline{X}\|^2} = \overbrace{\frac{d\underline{X}}{\|d\underline{X}\|}}^{\underline{n}} \cdot \underline{\underline{C}} \cdot \overbrace{\frac{d\underline{X}}{\|d\underline{X}\|}}^{\hat{}}.$$

$$\frac{\|d\underline{x}\|}{\|d\underline{X}\|} = \lambda.$$ → Ratio of length $\underline{x}$ ↪ of material line element.

$$\text{Stretch Ratio} = \frac{\text{length of material line element in the current configuration.}}{\text{length of mat. line ele. in the Ref. configuration.}}$$

$$\lambda_n^2 = \underline{N} \cdot \underline{\underline{C}} \cdot \underline{N}.$$

$\underline{N} \cdot \underline{\underline{E}} \cdot \underline{N} = \lambda_n^2 - 1$. ← measure the deformation.

↓

Lagrangian Strain tensor.

$$E_{ij} = \underbrace{\frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right]}_{\varepsilon_{ij}.} + \frac{1}{2}\left[\frac{\partial u_k}{\partial X_i} \cdot \frac{\partial u_k}{\partial X_j}\right].$$

Remember:

$\underline{x} = \underline{X} + \underline{u}(\underline{X}, t)$.

Both $\underline{\underline{C}}$ & $\underline{\underline{E}}$ are symmetric Tensors.

$\underline{\underline{C}}^T = \underline{\underline{C}}$. Recall $\underline{\underline{F}}$ is revertible.

$\underline{\underline{F}}^T \underline{\underline{F}} = \underline{\underline{C}}$.

↓ revertible ↗ $\underline{\underline{C}}$ is positive definite.

$d\underline{X} \cdot \underline{\underline{C}} \cdot d\underline{X} \geq 0$. only when $d\underline{X} = 0$.

identical (exactly).

$\|\underline{\underline{F}} \cdot d\underline{X}\|^2 > 0$. $d\underline{X} \neq 0$.

Real.

$\underline{\underline{C}}$ symmetric implies that $\underline{\underline{C}}$ has eigen values $\lambda_1^2, \lambda_2^2, \lambda_3^2$.

$\lambda_1^2 > \lambda_2^2 > \lambda_3^2 \leftarrow$

$\underline{\underline{C}}$ positive definite implies $\lambda_i^2 > 0.$

$i=1,2,3.$

$\underline{\underline{C}}$ Canbe diaganiolized

that is, $\underline{\underline{C}}$ can be written as

$$\underline{\underline{C}} = \lambda_1^2 \underline{n}_1\underline{n}_1 + \lambda_2^2 \underline{n}_2\underline{n}_2 + \lambda_3^2 \underline{n}_3\underline{n}_3$$

$\underline{n}$ are orthornormal eigen vectors of $\underline{\underline{C}}$

that is $\underline{n}_i \cdot \underline{n}_j = \delta_{ij}$

THIS $\lambda_1, \lambda_3, \lambda_2$ are called principal stretches.

~~Ni~~ 's are the principal direction.
$\underline{n}_i$

Polar Decomposition Theorem

$$\boxed{\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}} = \underline{\underline{V}}\,\underline{\underline{R}}.$$

$\underline{\underline{R}}$ is a rigid body rotation tensor, $\underline{\underline{R}}^T = \underline{\underline{R}}^{-1}$

$$\underline{\underline{R}}^T\underline{\underline{R}} = \underline{\underline{R}}\,\underline{\underline{R}}^T = \underline{\underline{I}}.$$

$\underline{\underline{U}}$ is symmetric, positive definite.

and $\underline{\underline{U}}^2 = \underline{\underline{C}}$ & $\underline{\underline{U}} = \sqrt{\underline{\underline{C}}}$.

$$\underline{\underline{U}} = \lambda_1 \underline{N}_1\underline{N}_1 + \lambda_2 \underline{N}_2\underline{N}_2 + \lambda_3 \underline{N}_3\underline{N}_3$$

check $\underline{\underline{U}} \cdot \underline{\underline{U}} = \underline{\underline{U}}^2 = \underline{\underline{C}}.$

$\underline{\underline{F}}$ can be decompose into two simple tensor,

where first tensor, $\underline{\underline{U}} \rightarrow$ stretch tensor.

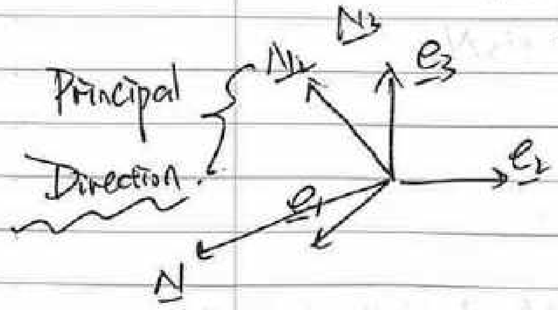
$\downarrow$

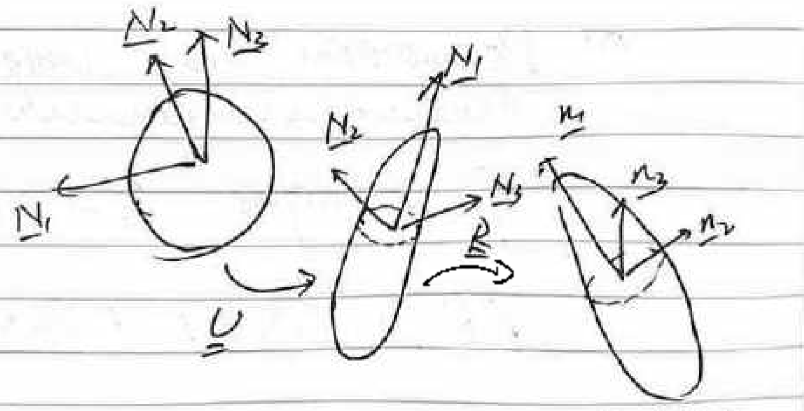
it stretch the material point in principal directions

the it rotate with $\underline{\underline{R}}$.

$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} \cdot \underline{X}$. → this a local theorem

$\curvearrowleft$ $\underline{\underline{U}}$ comes first. and $\underline{\underline{R}}$ comes second.





$\underline{n}_i = \underline{\underline{R}}(\underline{N}_i)$

☆ Only need to prove

$$\underline{\underline{R}} \equiv \underline{\underline{F}}\,\underline{\underline{U}}^{-1} \text{ is a rotation.}$$

$$\underline{\underline{R}}^T\underline{\underline{R}} = \underline{\underline{I}}.$$

$$\underline{\underline{R}}^T\underline{\underline{R}} = (\underline{\underline{F}}\,\underline{\underline{U}}^{-1})^T(\underline{\underline{F}}\,\underline{\underline{U}}^{-1}). \qquad \underline{\underline{C}} = \underline{\underline{U}}^2$$

$$= (\underline{\underline{U}}^{-T}\underline{\underline{F}}^T)(\underline{\underline{F}}\,\underline{\underline{U}}^{-1}) = \underline{\underline{U}}^{-1}\underbrace{\underline{\underline{F}}^T\underline{\underline{F}}}_{}\underline{\underline{U}}^{-1}$$

symmetric $\downarrow$ $\underline{\underline{U}}^{-1}$

$$= \overbrace{\underline{\underline{U}}^{-1}\underline{\underline{U}}}^{\underline{\underline{I}}}\;\overbrace{\underline{\underline{U}}\,\underline{\underline{U}}^{-1}}^{\underline{\underline{I}}}$$

$$= \underline{\underline{I}}$$

Therefore we prove: $\underline{\underline{R}}\,\underline{\underline{R}}^T = \underline{\underline{R}}^T\underline{\underline{R}} = \underline{\underline{I}}$.

$$\underline{n}_i = \underline{\underline{R}}(\underline{N}_i).$$

$$\underline{\underline{R}} = \underline{n}_i\underline{N}_i = \underline{n}_1\underline{N}_1 + \underline{n}_2\underline{N}_2 + \underline{n}_3\underline{N}_3.$$

*** Decomposition is unique.

if we define: $\underline{\underline{V}} = \lambda_1\underline{n}_1\underline{n}_1 + \lambda_2\underline{n}_2\underline{n}_2 + \lambda_3\underline{n}_3\underline{n}_3.$

$$\underline{\underline{V}}\,\underline{\underline{R}} = (\lambda_1\underline{n}_1\underline{n}_1 + \lambda_2\underline{n}_2\underline{n}_2 + \lambda_3\underline{n}_3\underline{n}_3)\cdot(\underline{n}_1\underline{N}_1 + \underline{n}_2\underline{N}_2 + \underline{n}_3\underline{N}_3).$$

$$= \lambda_1\cdot\underline{n}_1\underline{N}_1 + \lambda_2\underline{n}_2\underline{N}_2 + \lambda_3\underline{n}_3\underline{N}_3$$

$\Downarrow$ then we can check:

$$\underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}.$$

[IN CASE] in mechanics theorem paper,

Someone writes:

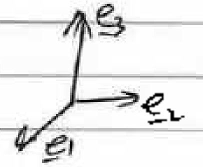
$\underline{\underline{F}} =$ ~~F_{iA}~~ e_i

$F_{iA}\,\underline{e}_i\,\underline{E}_A$ → two point tensor

$$\underline{e}_i\cdot\underline{e}_j = \delta_{ij}$$

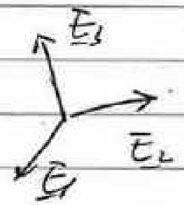
$$\underline{E}_A\cdot\underline{E}_B = \delta_{AB}$$

in the ref. config. $\underline{e}_1, \underline{e}_2, \underline{e}_3$



current config.

$\underline{E}_1, \underline{E}_2, \underline{E}_3$



Simple Shear Deformation.

$$\begin{cases} x_1 = X_1 + X_2\tan\gamma \\ x_2 = X_2. \\ x_3 = X_3 \end{cases}$$

← fixed number $(0, \pi/2)$.

$\underline{x} = x_i\underline{e}_i$

$$[\underline{\underline{F}}] = \begin{bmatrix} 1 & \tan\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underline{\underline{F}} = \underline{e}_1\underline{e}_1 + \tan\gamma\, \underline{e}_1\underline{e}_2 + \underline{e}_2\underline{e}_2 + \underline{e}_3\underline{e}_3.$

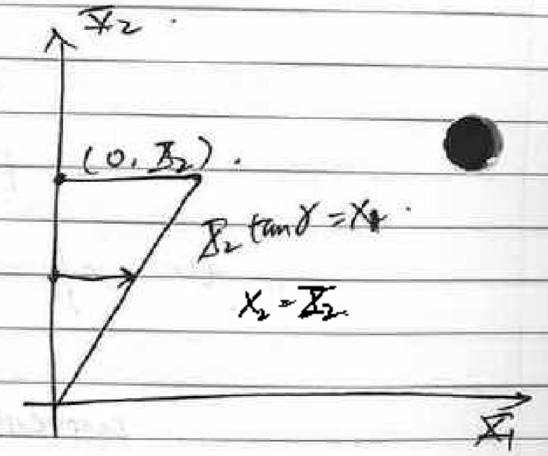
$\det \underline{\underline{F}} = 1.$

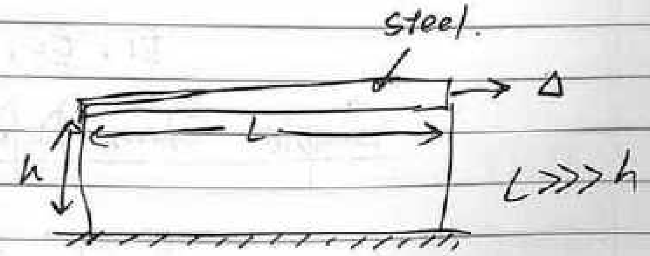
$\hookrightarrow \det(\underline{\underline{R}}\,\underline{\underline{U}}) = \underbrace{\det \underline{\underline{R}}}_{=1} + \det \underline{\underline{U}} \rightarrow \lambda_1\lambda_2\lambda_3.$

$\downarrow$

$\frac{V}{V_0}.$

$\underline{\underline{F}} =$





Office Hour:

$\underline{\underline{F}} = (\delta_{ij} + u_{i,j})\, \underline{e}_i \underline{e}_j$

$\underline{\underline{E}} = \frac{1}{2}\left[u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right].$

$\downarrow$

$,j = \frac{\partial}{\partial X_j}$

Week 4 (5). Mon.

Linear Theory (small deformation).

↳ perturbation theory.

geometry change small.

(gradients, of displacements small, << 1).

$$\underset{\approx}{E} = \frac{1}{2}\Big[\underbrace{\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}}_{\text{leading order terms}} + \underbrace{\frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}}_{\text{higher order terms}}\Big].$$

$\underline{u} = u_k \underline{e}_k$.

$$\approx \underbrace{\frac{1}{2}\Big[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\Big]}_{\text{leading order terms}}$$

$\underline{\underline{C}} - \underline{\underline{I}} \equiv 2\underline{\underline{E}}$.

$\underline{\underline{F}} = \underline{\underline{I}} + \frac{\partial u_i}{\partial X_j}\underline{e}_i\underline{e}_j$ $= \underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}}$.

Small rotation tensor

$$\frac{\partial u_i}{\partial X_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)}_{\text{sym}\ \to\ \varepsilon_{ij}} + \underbrace{\frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right]}_{\text{Anti-sym}\ \to\ \omega_{ij}}$$

$C_{ij} = (\delta_{ij} + \varepsilon_{ij} + \omega_{ij})(\delta_{ji} + \varepsilon_{ji} + \omega_{ji})$
$= \delta_{ij}\delta_{ji} + \delta_{ij}\varepsilon_{ji} + \delta_{ij}\omega_{ji}$
$+ \varepsilon_{ij}\delta_{ji} + \varepsilon_{ij}\varepsilon_{ji} + \varepsilon_{ij}\omega_{ji}$
$+ \omega_{ij}\delta_{ji} + \omega_{ij}\varepsilon_{ij} + \omega_{ij}\omega_{ji}$

$\underline{\underline{C}} = (\underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}})^T(\underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}})$.

$= (\underline{\underline{I}} + \underline{\underline{\varepsilon}} - \underline{\underline{\omega}})(\underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}})$.

$= \underline{\underline{I}} + 2\underline{\underline{\varepsilon}} + H.O.T.$

$\underline{\underline{C}} = \underline{\underline{U}}^2$. (???)

$\underline{\underline{C}} \approx \underline{\underline{I}} + 2\underline{\underline{\varepsilon}}$.

$\underline{\underline{U}} \simeq \underline{\underline{I}} + \underline{\underline{\varepsilon}}$. $\underline{\underline{U}}^2 = \underline{\underline{I}} + 2\underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}}\,\underline{\underline{\varepsilon}}$

$\approx \underline{\underline{I}} + 2\underline{\underline{\varepsilon}}$.

$\underline{\underline{U}}^{-1} \simeq \underline{\underline{I}} - \underline{\underline{\varepsilon}}$.

$\underline{\underline{F}}\,\cancel{A} = \underline{\underline{R}}\,\underline{\underline{U}}$

$\underline{\underline{R}} = \underline{\underline{F}}\,\underline{\underline{U}}^{-1}$. $\underline{\underline{R}} \approx (\underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}})(\underline{\underline{I}} - \underline{\underline{\varepsilon}})$.

$= \underline{\underline{I}} - \cancel{\underline{\underline{\varepsilon}}} + \underline{\underline{\omega}}$.

$\underline{V} = \frac{\partial \underline{x}}{\partial t}\Big|_{\underline{X}\ \text{fix}} = \frac{\partial \underline{u}}{\partial t}\Big|_{\underline{X}}$.

↳ (at a fixed material point).

So

$\underline{x} = \underline{X} + \underline{u}(\underline{X}, t)$.

$\underline{A} = \frac{\partial^2 \underline{x}}{\partial t^2}\Big|_{\underline{X}} = \frac{\partial^2 \underline{u}}{\partial t^2}\Big|_{\underline{X}}.\ \cancel{\frac{\partial u}{\partial t}\Big|_{X}}$

$\underline{V}(\underline{X}, t)$.

mechanics : quantities in spatial descrip.

density

$\rho(\underline{X}, t)$. ↓

the material derivative :

$$f(\underline{x}, t) = f(\underline{\chi}(\underline{X}, t), t).$$

$$\frac{D}{Dt} f \equiv \dot{f} \qquad \text{Fixed } \underline{X}.$$

$$= \frac{\partial f}{\partial x_i} \underbrace{\frac{\partial \chi_i}{\partial t}\bigg|_{\underline{X} = \chi^{-1}(\underline{x}, t)}}_{V_i(\underline{x}, t)} + \frac{\partial f}{\partial t}\bigg|_{\underline{X}}.$$

↳ velocity.

$$= \frac{\partial f}{\partial x_i} V_i(\underline{x}, t) + \frac{\partial f}{\partial t}\bigg|_{\underline{X}} = \nabla_{\underline{x}} f \cdot \underline{V} + \frac{\partial f}{\partial t}\bigg|_{\underline{x}}$$

$$\nabla_{\underline{x}} f = \frac{\partial f}{\partial x_j} \underline{e_j}.$$

$$\nabla_{\underline{X}} g = \frac{\partial g}{\partial X_j} \underline{e_j}.$$

$$\underline{V} = \underline{V}(\underline{X}, t). \equiv \frac{\partial \underline{\chi}}{\partial t}\bigg|_{\underline{X}}.$$

$$\underset{\downarrow}{\underline{v}} = \underline{V}(\underline{\chi}^{-1}(\underline{x}, t), t) \quad \underline{X} = \underline{\chi}^{-1}(\underline{x}, t).$$

In the spatial configuration .

$$\underline{a} = \underline{A}(\underline{\chi}^{-1}(\underline{x}, t), t).$$

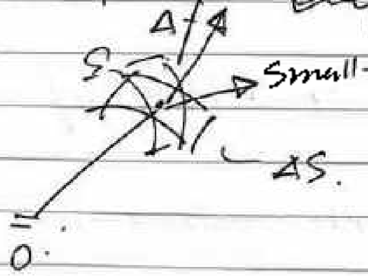
$$\underline{a} = \underline{v} \cdot \nabla_{\underline{x}} \underline{V} + \frac{\partial \underline{V}}{\partial t}\bigg|_{\underline{x}}.$$

Concept of Stress

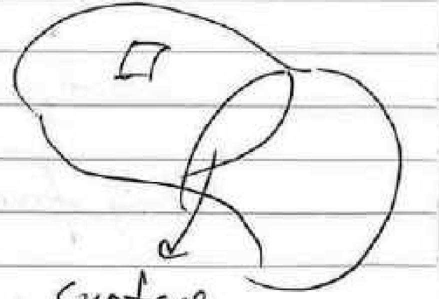
if given displacement field, & ref config.

↳ then we can calculate everything

<u>Assumption</u>. Cauchy's hypothesis.



Small → force almost uniform



Important : orientation → X shape .

$$\underline{t} = \frac{\Delta f}{\Delta S} \qquad \Delta S \to 0$$

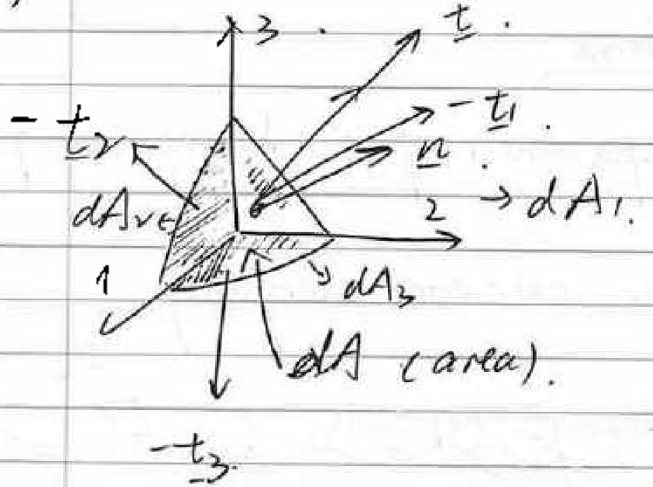
$$= \underline{t}(\underline{n}, \cancel{\underline{x}}, t).$$

$\underline{n}$ → outward unit normal vector in the current configuration.

$\underline{t}$ → traction vector.
(stress).

※ If we know traction in 3D, then we
└ know the stress at this point.

Cauchy's theorem.



Force on pyramid?

Body forces → depends on volume of element.

ρ = mass per unit volume in current configuration.

Body forces per unit volume in

linear momentum Balance.

$$\underline{t}\, dA - \underline{t}_1 dA_1 - t_2 dA_2 - t_3 dA_3 + \rho\, \underline{b}\, dV$$

$$= \cancel{m \frac{dV}{dt}} \quad (\rho\, dV)\underline{a}.$$

~~V~~ is the volume of small pyramid
dv

$dA \gg dV, \qquad \frac{dV}{dA} \to 0.$
(in small pyramid).

$$\underline{t} = \underline{t}_1 \frac{dA_1}{dV} + \underline{t}_2 \frac{dA_2}{dV} + \underline{t}_3 \frac{dA_3}{dV}$$

$$\underline{t} = \cancel{\underline{t}_1 \underline{n}_1 + \underline{t}_2 \underline{n}_2 + \underline{t}_3 \underline{n}_3}$$

$$\underline{t}_1 \underline{e}_1 \cdot \underline{n} + \underline{t}_2 \underline{e}_2 \cdot \underline{n} + \underline{t}_3 \underline{e}_3 \cdot \underline{n}.$$

$\underline{t}_1 = \sigma_1 \underline{e}_1$

$\underline{t}_k = \sigma_k \underline{e}_k \quad , k = 1, 2, 3.$

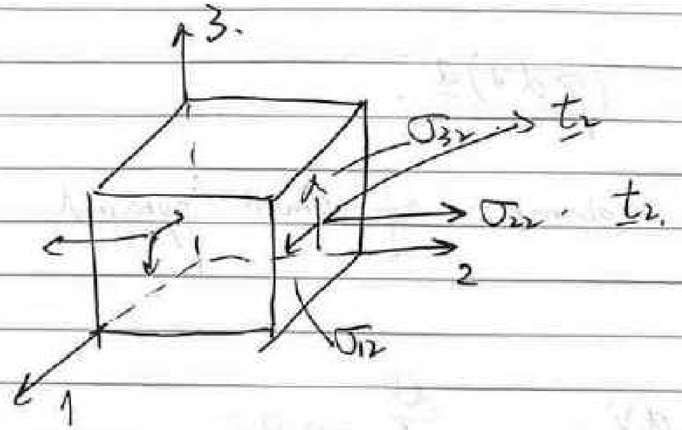
$\underline{t}_j = \sigma_{ij} \underline{e}_i$

$= \sigma_{ij} \underline{e}_i \underline{e}_j \cdot \underline{n}.$

(other text book: $\underline{t} = \underline{\underline{\sigma}}^T \cdot \underline{n}$).

$\underline{\underline{\sigma}}$ = Cauchy or True stress tensor

$t_j = \sigma_{ji} \underline{n}_i. \qquad \underline{t} = \underline{\underline{\sigma}} \cdot \underline{n}.$



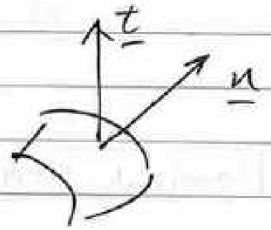
$$\underline{t}_2 = \sigma_{12}\,\underline{e}_1 + \sigma_{21}\,\underline{e}_2 + \sigma_{32}\,\underline{e}_3.$$

Week 4, Wed.

$\underline{\underline{\sigma}}$ True stress tensor in current config.

Cauchy

$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}$ → traction vector.



$$[\underline{t}] = \frac{\text{Force}}{\text{Unit area}}$$

* Equilibrium equation – Deformed configuration
(LMB)
↓
Linear momentum Balance.

Key Results.

$$\underbrace{\nabla_x \cdot \underline{\underline{\sigma}}}_{\text{Divergence}} + \rho \overset{\text{body force}}{\underline{b}} = \rho \cdot \overset{\text{acceleration}}{\underline{a}}$$

spatial.

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho a_i. \quad \dots \quad (\mathcal{N}1)$$

Current config.
⇓
In real-world applications, cuz you didn't don't know current config.

in elastic prob., we use reference config.

In Ref config.

Eq. (N1) become :

$$\nabla_X \cdot \underline{\underline{P}} + \rho_0 \underline{B} = \rho_0 \underline{A}.$$

$$\frac{\partial P_{ij}}{\partial X_j} + \rho_0 B_i = \rho_0 A_i.$$

$\underline{A} = \underline{A}(\underline{X}, t)$.

First Piola Tensor ⇒ Nominal Stress tensor

$\boxed{AMB}$ Angular momentum balance.

$\sigma_{ij} = \sigma_{ji}$.

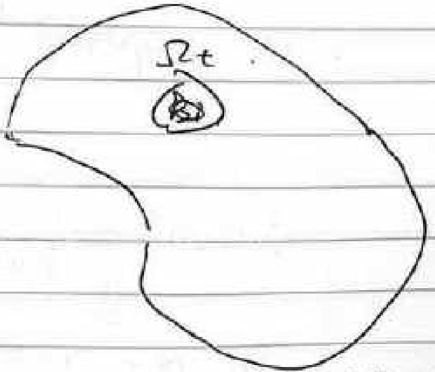
$\rightarrow \underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$.

AMB.

$$\underline{\underline{P}}\,\underline{\underline{F}}^T = \underline{\underline{F}}\,\underline{\underline{P}}^T.$$

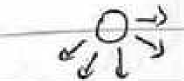
Derivation.

METHOD I: Forces acting on Ω_t.



(Force balance) integral of $\int_{\Omega_t} \rho \underline{b}\, dV$

$$+ \int_{\Omega_t} \underline{\underline{\sigma}} \cdot \underline{n}\, dS \quad \text{(traction)} \qquad (F.1)$$



LMB

(Newton's how). $+ \frac{D}{Dt}\int_{\Omega_t} \rho \underline{v}\, dV$ ☆☆☆ Cannot take inside.

$\rho\, dV = \rho_0\, dV_0$ conservation of mass.

$$\frac{dV}{dV_0} = \frac{\rho_0}{\rho} = \det \underline{\underline{F}}.$$

Jacobian.

Eq. (F1) becomes.

$$\frac{D}{Dt}\int_{\Omega_t} \rho \underline{v}\, dV = \frac{D}{Dt}\int_{\Omega_0} \rho_0 \underline{V}\, dV_0$$

Ω_0 → fixed

So can take $\frac{D}{Dt}$ inside

$$= \int_{\Omega_0} \rho_0 \underline{A}\, dV_0.$$

$$= \int_{\Omega_t} \rho \underline{a}\, dV$$

$$\underbrace{\int_{\partial\Omega_t} \underline{\underline{\sigma}} \cdot \underline{n}\, ds}_{\text{Divergence theorem.}} + \int_{\Omega_t} (\rho \underline{b} - \rho \underline{a})\, dV.$$

⇓

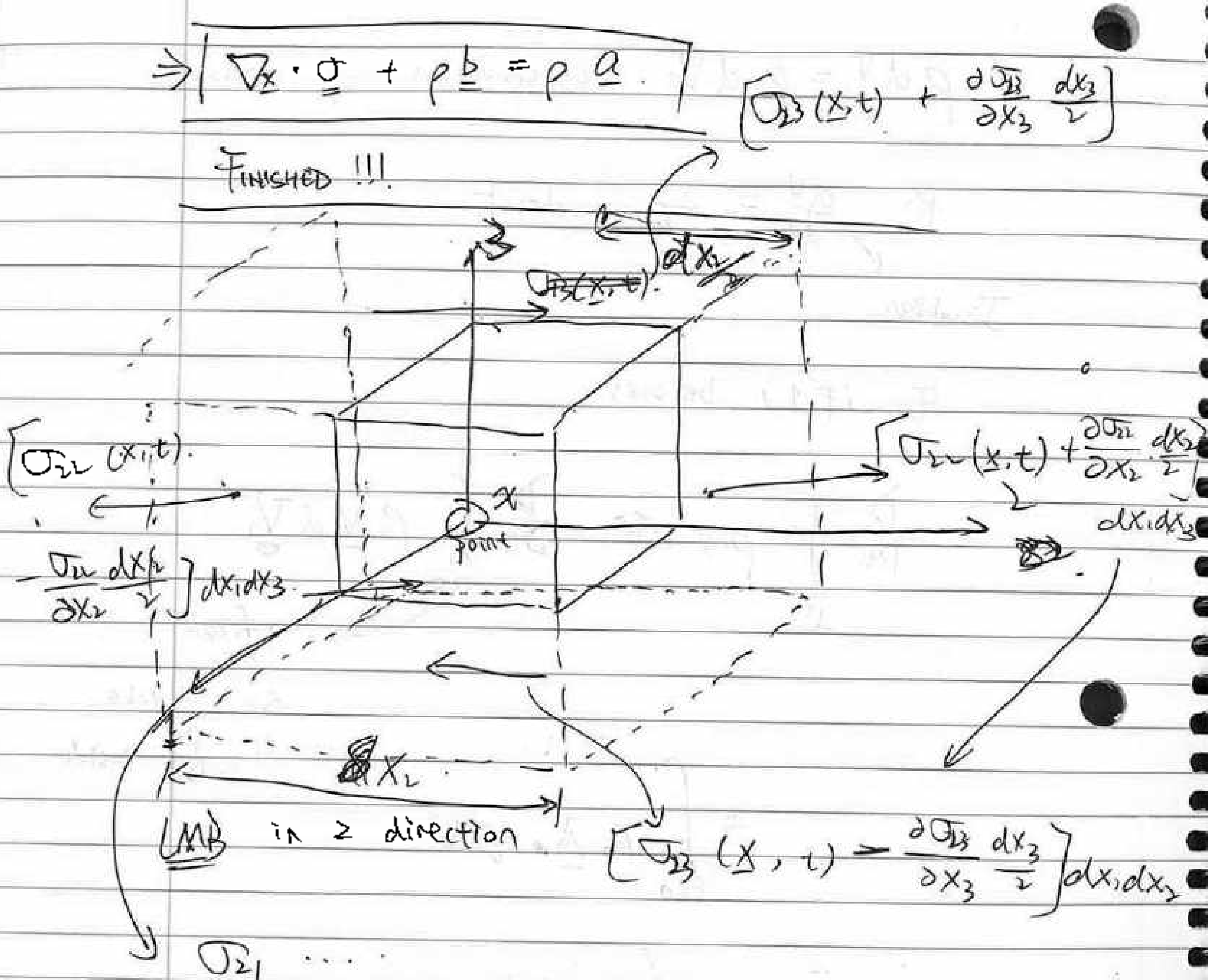
$$\int_{\Omega_t} \nabla_{\underline{x}} \cdot \underline{\underline{\sigma}}\, dV.$$

$$\int_{\Omega_t} [\nabla_{\underline{x}} \cdot \underline{\underline{\sigma}} + \rho \underline{b} - \rho \underline{a}]\, dV = 0$$

This is true for any Ω_t ⇒

$$\Rightarrow \boxed{\nabla_{\underline{x}} \cdot \underline{\underline{\sigma}} + \rho \underline{b} = \rho \underline{a}}$$

FINISHED !!!



LMB in 2 direction

σ_{21} . . .

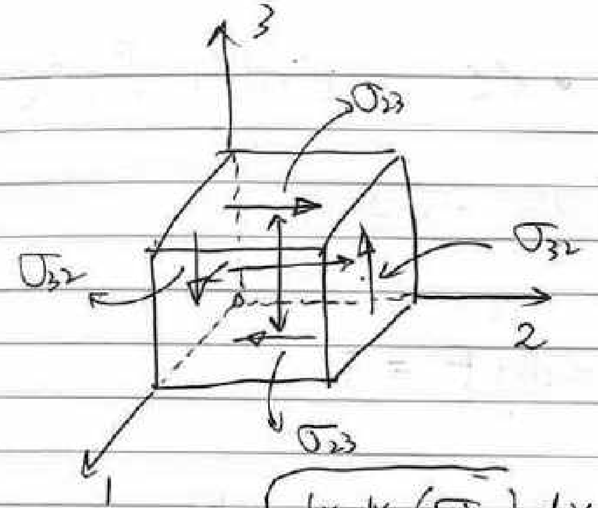
Net force in 1D,

$$\left[\frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \frac{\partial \sigma_{21}}{\partial x_1}\right] dx_1 \cancel{dx_2 dx_3}.$$

$$+ \rho \underline{b}\ dx_1 \cancel{dx_2} dx_3 = \rho \underline{a}\ dx_1 \cancel{dx_2} dx_3.$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho a_i.$$

$\Delta \quad \Delta \quad \Delta \rightarrow$ total force.



just times

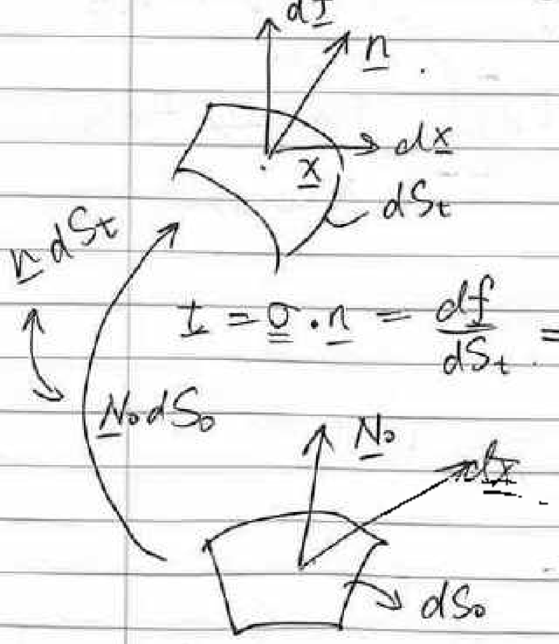
$$\boxed{dx_1 dx_2 (\sigma_{23}}\ dx_3) = (\sigma_{32} \cdot dx_2)\ dx_1 dx_3$$

force

$\sigma_{23} = \sigma_{32}$

$\sigma_{ij} = \sigma_{ji}$

$$\int_{\Omega_0} \rho_0 \underline{B}\, dV_0 + \int_{\partial R_t} \underline{\underline{\sigma}} \cdot \underline{n}\, dS = \int_{\Omega_0} \rho_0 \cdot \underline{A}\, dV_0.$$



$\underline{n}\, dS_t$.

$$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n} = \frac{d\underline{f}}{dS_t} \Rightarrow \underline{t}\, dS_t = d\underline{f}.$$

$$dV = \underline{n}\, dS_t \cdot d\underline{x} = \underline{n} \cdot dS_t \cdot \underline{\underline{F}} \cdot d\underset{\sim}{\underline{X}}.$$

$$dV_0 = \underline{N}\, dS_0 \cdot d\underline{X}.$$

$$\frac{dV}{dV_0} = \det \underline{F} \equiv J.$$

$$\underline{n}\, dS_t \cdot \underline{\underline{F}} \cdot d\underline{X} = J\, \underline{N}\, dS_0 \cdot d\underline{X}.$$

$$d\underline{X} \cdot dS_t\, \underline{\underline{F}}^T \cdot \underline{n} = d\underline{X} \cdot J\, \underline{N}\, dS_0.$$

$d\underline{X}$ is Solitory!

$$dS_t \cdot \underline{\underline{F}}^T \cdot \underline{n} = J \underline{N}\, dS_0.$$

$$\underline{\underline{F}}^T \cdot \underline{n}\, dS_0 = J \underline{N}\, dS_0$$

$$\boxed{\underline{n}\, dS_0 = J \underline{\underline{F}}^T \cdot \underline{N}\, dS_0}$$

Substitute $\int_{\partial\Omega_t} \underline{\sigma} \cdot \underline{n}\, dS$

~~Nansen's~~ Formula

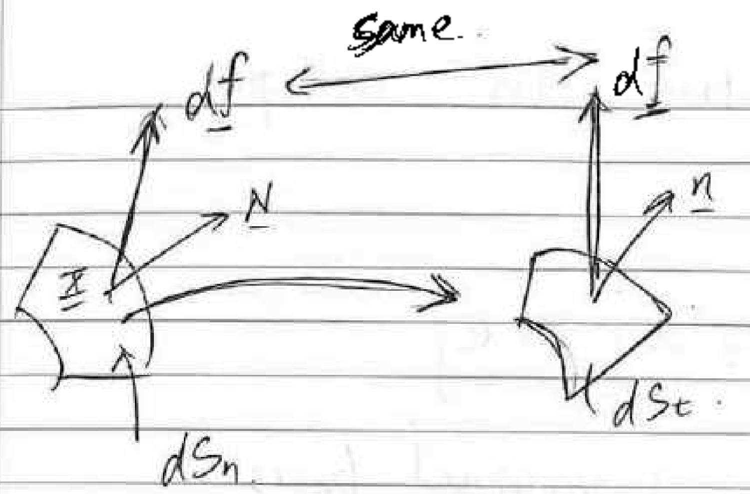
Nansen's

$$\int_{\partial\Omega_0} \underbrace{J \underline{\sigma}\, \underline{\underline{F}}^{-1}}_{} \cdot \underline{N}\, dS_0. \quad \to \underline{P}.$$

By definition $\quad \underline{P} = J \underline{\sigma}\, \underline{\underline{F}}^{-T}.$

Divergence

$$\int_{\partial\Omega_0} \nabla_{\underline{X}} \cdot \underline{P}\, dV_0 \to \nabla_{\underline{X}} \cdot \underline{\underline{P}} + \rho_0 \underline{B} = \rho_0 \underline{A}.$$



$$\underline{t}_\alpha\, dS_0 = d\underline{f} = \underline{t}\, dS_t.$$

~~$\underline{P}\cdot\underline{n}$~~ $\quad \underline{\underline{P}} \cdot \underline{n} = \underline{t}$

↓

not a symmetric tensor

$$\underline{\underline{P}} = J \underline{\underline{\sigma}}\, \underline{\underline{F}}^{-T}$$

↓ symmetric

↳ not symmetric

AMB.

$$\frac{(\underline{\underline{P}}\,\underline{\underline{F}}^T)^T}{J} = \frac{\underline{\underline{P}}\,\underline{\underline{F}}^T}{J} \Rightarrow \boxed{\underline{\underline{F}}\,\underline{\underline{P}}^T = \underline{\underline{P}}\,\underline{\underline{F}}^T}$$

In fluid mech, use current config. at variables.

☆☆☆ Basic balance laws of continuum mechanics.

(Derived in current configuration)

Office hour Fri. 3:30pm.

HW #2.

$$\underset{=}{A} = a_{ij}\, \underline{e}_i\, \underline{e}_j.$$

$\{e_i\}$ original basis.

$\{E_j\}$. New.

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & & \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\text{Symmetry.}}.$$

whole num

↑

λ_i : eigenvalues.

↓ ↓

$\underset{=}{A}$ in new basis: $\underset{=}{A} = \lambda_1 \underline{E}_1 \underline{E}_1 + \lambda_2 \underline{E}_2 \underline{E}_2 + \lambda_3 \underline{E}_3 \underline{E}_3$

↑

$$\rightarrow \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}.$$

eigenvalues: $\lambda_1 = 8$, $\lambda_2 = 6$, $\lambda_3 = 3$.

Original one: $\underset{=}{A} = 6 \underline{e}_1 \underline{e}_1 - 2 \underline{e}_1 \underline{e}_2 - 1 \underline{e}_1 \underline{e}_3 \ldots\ldots$

↳ so, $\underset{=}{A} = 8 \underline{E}_1 \underline{E}_1 + 6 \underline{E}_2 \underline{E}_2 + 3 \underline{E}_3 \underline{E}_3$

↑

what is $\underline{E}_1$?

Original tensor $\underset{=}{A} = a_{ij} \underline{e}_i \underline{e}_j$.

$= \lambda_1 \underline{E}_1 \underline{E}_1 + \lambda_2 \underline{E}_2 \underline{E}_2 + \lambda_3 \underline{E}_3 \underline{E}_3$.

↓

$\underline{E}_1 \cdot \underline{E}_1 = 1$, $\underline{E}_2 \cdot \underline{E}_1 = 0$, $\underline{E}_3 \cdot \underline{E}_1 = 0$

need to normalize to 1

$(a_{ij} \underline{e}_i \underline{e}_j) \cdot \underline{E}_1 = \lambda_1 \underline{E}_1$

$$a_{ij} \underline{e}_i \underbrace{(\underline{e}_j \cdot \underline{E}_1)}_{P_{ij}} = \lambda_1 \underline{E}_1$$

$\underline{E}_1 \cdot \underline{e}_j = P_{ij}$

$\Downarrow$

$\underline{E}_1 = P_{1i} \underline{e}_i$

$a_{ij} P_{ij} \underline{e}_i = \lambda_1 \underline{E}_1 = \lambda_1 P_{1i} \underline{e}_i$

or $[a_{ij} P_{ij} - \lambda_1 P_{1i}]\, \underline{e}_j = \underline{0}$

$\Rightarrow a_{ij} P_{1j} - \lambda P_{1i} = 0 \quad \longrightarrow i = 1, 2, 3$

$A = [a_{ij}]. \quad A\vec{P}_1 - \lambda \vec{P}_1 = 0$

eigen vector of $\underline{A}$ for $\underline{E}$ $\longleftarrow$ $\boxed{\vec{P}_1 = \begin{bmatrix} P_{11} \\ P_{12} \\ P_{13} \end{bmatrix}.}$

or $(A - \lambda I)\vec{P}_1 = \vec{0}$

$\lambda_1 \rightarrow$ eigenvalue of $\underline{\underline{A}}$

$\vec{P}_1$ is a eigen vector for A $[\lambda_1]$.

$$\boxed{\underline{E}_1 = P_{1i}\, \underline{e}_i}$$

$\underline{E}_1 \cdot \underline{E}_1 = 1.$ $\boxed{P_{1i} P_{1i} = 1}$

☆

Same idea applies to E_2

$\underline{E}_2 = P_{2i}\, \underline{e}_i \longleftarrow \lambda_2$

$E_3 \ldots\ldots$

$\underline{\underline{A}} \cdot (\underline{e}_1 + \underline{e}_2) = \ldots$

☆ Get the same no matter which basis ...

$\underline{\underline{A}} = (\underline{e}_1 \underline{e}_2)$. MATRIX.

linear transformation

$\underline{\underline{A}} \cdot \underline{e}_1 =$ first column of $\underline{\underline{A}}$.

$(\underline{e}_1 \underline{e}_2) \cdot \underline{e}_1 = \underline{0}$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\underline{\underline{A}} \cdot \underline{e}_2 = (\underline{e}_1 \underline{e}_2) \cdot \underline{e}_2 = \underline{e}_1 (\underline{e}_2 \cdot \underline{e}_2) = \underline{e}_1.$

$= 1\underline{e}_1 + 0\underline{e}_2 + 0\underline{e}_3$

$\underline{\underline{A}} \cdot \underline{e}_3 = \underline{0}$ $\longrightarrow$ ☆ respect to basis $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$.

$\underline{\underline{A}} = a_{ij}\, \underline{e}_i \underline{e}_j$

$[\underline{e}_1 \underline{e}_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

$$[A] = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} +$$

$$= a_{11} \underline{e}_1 \underline{e}_1 + \cdots$$

$$\underline{V} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3.$$

$$\underline{E}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\underline{V} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

$$\hookrightarrow v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \cdots$$

$$\underline{V} = \underline{e}_1$$

$$\underline{e}_i \underline{e}_j$$

$$\underline{A} = a_{11} \underline{e}_1 \underline{e}_1 + a_{12} \underline{e}_1 \underline{e}_2 + a_{13} \underline{e}_1 \underline{e}_3 + \cdots$$

$$[\underline{A}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

↓ break it down into simple linear trans.

$$\underline{V} = V_1 \underline{e}_1 + V_2 \underline{e}_2 + V_3 \underline{e}_3$$

☆ Eigenvalues → invariants

eigenvectors → be care of the basis !!!

$\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ → $\underline{E}_i = P_{ij} \, \underline{e}_j$

$\{\underline{E}_1, \underline{E}_2, \underline{E}_3\}$.

↑ with respect to this basis.

$\vec{P}_1 = \begin{bmatrix} P_{11} \\ P_{12} \\ P_{13} \end{bmatrix}$ → associated with P_{ij}.

$\underline{\underline{W}}$ = Skew Symmetric

$$\underline{\underline{W}}\cdot\underline{v} = \underset{\uparrow}{\underline{\omega}} \times \underline{v}$$

*** need to tell what $\underline{\omega}$ is

expand:

$$\underline{\underline{W}} = w_{12}\,\underline{e}_1\underline{e}_2 + w_{21}\,\underline{e}_2\underline{e}_1 + \cdots$$

$$w_{11},\ w_{22},\ w_{33},\ = 0.$$

$$\underline{\underline{W}}\cdot\underline{v} = \underline{p} \times \underline{v}.$$

$$w_{21} = -w_{12}$$

$$= w_{12}\,\underline{e}_1\underline{e}_2 - w_{12}\,\underline{e}_2\underline{e}_1 + \cdots$$

$$\underline{\underline{W}}\cdot\underline{v} \rightarrow \underline{v} = v_i\,\underline{e}_i$$

$$\underline{p} \times \underline{v} = \cdots\ v_1 \cdots v_2 \cdots v_3$$

$$p_1 = w_{32}$$

$$p_2 = w_{13}.$$

$$p_3 = w_{21}.$$

$$\triangleright\ \underline{p} = w_{32}\,\underline{e}_1 + w_{13}\,\underline{e}_2 + w_{21}\,\underline{e}_3.$$

Sep 27. Week 5.

PERSONAL REVIEW — So FAR:

→ Cartesian Tensors.

- Review on Notation.

Summation Convention (Indical notation)

Permutation symbol.

Transformation Rule for vectors

▷ Second Order tensors.

▷ Transpose of tensor

Symmetric & Skew-symmetric tensor

Tensor transformation (basis, ...)

Operation of tensors: (products, ...)

Symmetric tensors: Diagonalization.

High order tensor

Trace of second order tensor

~~High order tensor~~

Tensor fields.

Kinematics

Sep 27, Week 5. Mon.

—Review: Last lecture: Balance laws.

True stress tensor ↳ current cooridanate.

$$\rightarrow \quad \underline{\nabla}_x \cdot \overset{\downarrow}{\underline{\underline{\sigma}}} + \rho \underline{b} = \rho \underline{a}$$ invariante form

$\downarrow$

(x config. influence).

$$\underbrace{\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho a_i.}_{\text{3 PDEs.}} \quad i = 1, 2, 3.$$

→ in the current cooridinate

*** ↗.

independent spatial variable are x_i.

AMB $\sigma_{ij} = \sigma_{ji}$, $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$

Balance law in reference configuration, independent variables are $\underline{X}_i$.

$\underline{\underline{P}}$ (Nominal or 1st Piola stress tensor).

$$\underset{\swarrow (\underline{X}).}{\underline{\underline{P}}} = J \underset{\llcorner \rightarrow (\underline{x})}{\underline{\underline{\sigma}}} \underline{\underline{F}}^{-T}. \longrightarrow J = \det \underline{\underline{F}}.$$

true stress.

$$\underline{\underline{\sigma}} = \frac{1}{J} \underline{\underline{P}} \underline{\underline{F}}^T \qquad \hookrightarrow \underline{\underline{\sigma}}(\underline{x} = \underline{\chi}(\underline{X}, t), t).$$

$$\nabla_{\underline{X}} \underline{\underline{P}} + \rho_0 \underline{B}_0 = \rho_0 \underline{A}$$

$$\frac{\partial P_{ij}}{\partial X_j} + \rho_0 B_{0i} = \rho_0 A_i.$$

$\underline{\underline{P}}$ is not always symmetric

So. an.. (AMB) $\underline{\underline{P}}\,\underline{\underline{F}}^T = \underline{\underline{F}}\,\underline{\underline{P}}^T$

Proof.

$$\det \underline{\underline{F}} = J = \frac{dV}{dV_0}, \quad \rho_0 dV_0 = \rho dV$$

$$\frac{\rho_0}{\rho} = \frac{dV}{dV_0} = J = \det \underline{\underline{F}}$$

Material is called immcompassible is $J = 1$, $\forall \underline{X}$

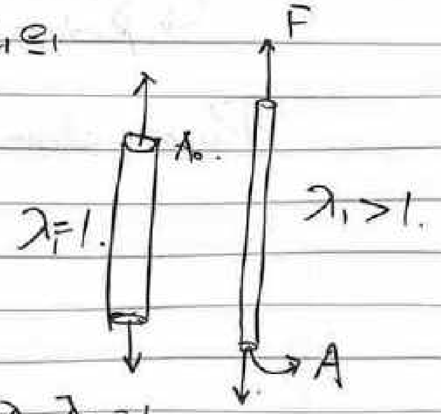
Review Part

$$\underline{\underline{P}} \cdot \underline{N} \, dS_0 = \underline{\underline{\sigma}} \cdot \underline{n} \cdot dS_0.$$

Simple example. $\underline{\underline{\sigma}} = \sigma_{11} \underline{e}_1 \underline{e}_1$

*** $$\sigma_{11} = \frac{F}{A}$$

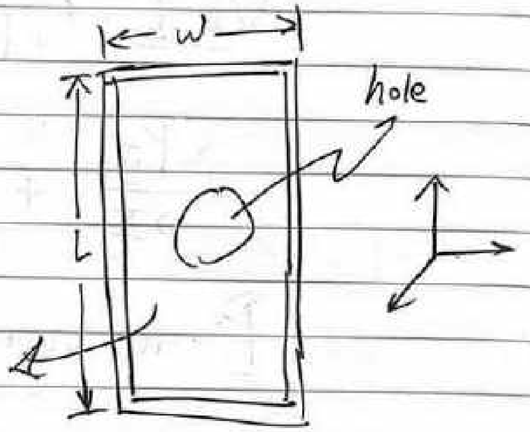
? $$P_0 = \frac{F}{A_0}$$



Incompressible solid. $J = \lambda_1 \lambda_2 \lambda_3 = 1$.

$\lambda = \lambda_2 = \lambda_3$. $\lambda = \frac{1}{\sqrt{\lambda_1}}$.

Plane stress.



$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0, \quad \forall X_3.$

Sym ↦ $\sigma_{31} = \sigma_{32} = 0.$

pressure

None-vanishing Stress state. } hydrostatic pressure (atmosphere). $L, w >> t.$

$\sigma_{11}, \sigma_{12} = \sigma_{21}, \sigma_{22}$ $\quad \sigma_{\alpha,\beta}. \quad \alpha, \beta = 1.2.$

Assumption: $\sigma_{\alpha\beta}(X_1, X_2, t)$. independent of X_3

Equilibrium Eq: $\dfrac{\partial \sigma_{\alpha\beta}}{\partial X_\beta} + \rho b_\alpha = \cancel{\rho}\ 0$

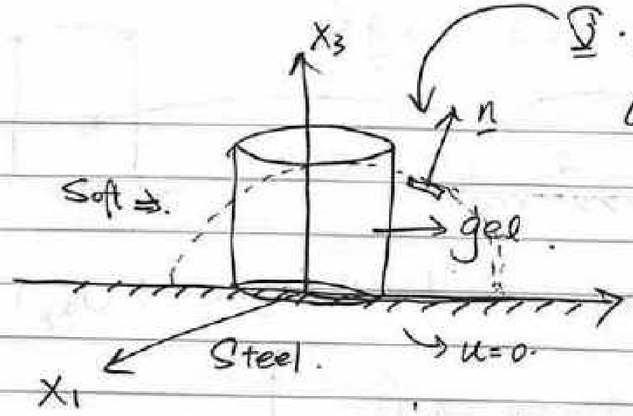
Quasi-Static problem $\underline{A} = 0.$

↳ Reduce to a 2D problem

Pure shear: $\underline{\underline{\sigma}} = \sigma_{12} \underline{e}_1 \underline{e}_2 + \sigma_{21} \underline{e}_2 \underline{e}_1$

$= \tau(\underline{e}_1 \underline{e}_2 + \underline{e}_2 \underline{e}_1).$

Eg.



$\underline{\underline{\Sigma}} \cdot \underline{n} = 0$

(no external traction).

(no disp.)

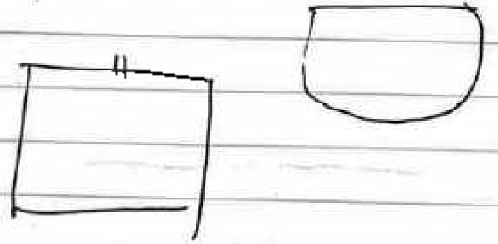
$\hat{u}_3(X_1, X_2, X_3 = 0) = 0.$

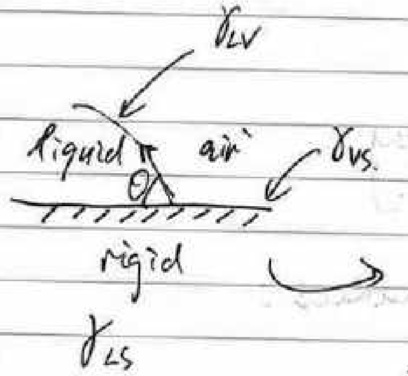
(no shear)

$\tau_{21} = \tau_{12} = 0$, on $X_3 = 0$

$|X_1^2 + X_2^2| = a.$

→ Solve the shape of gel.





↪ Laplace-Young Equation.

Equilibrium.

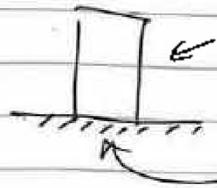
$$\gamma_{LV} \cos\theta - \gamma_{LS} = \gamma_{VS}$$

$\dfrac{\gamma}{G\,a}$ = elastopillary no.

G ↓ shear modulus. a ↓ typical length scale

water leak out → Poroelasticity

EASY WAY TO DO THIS:

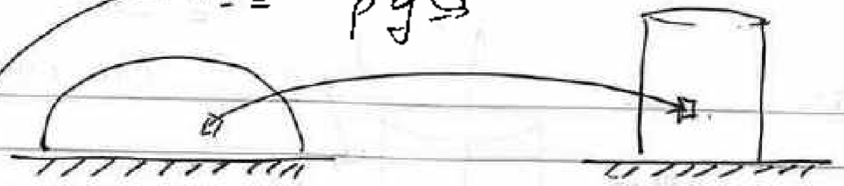


← $\underline{P} \cdot \underline{N} = 0$ $\quad \nabla_{\underline{X}} \underline{P} = -\rho_0 \underline{B}_0$

$= -\rho_0 g \underline{e}_3$

glued. → $u_3 = 0$

$$\nabla_{\underline{x}} \cdot \underline{\underline{\sigma}} = \rho g \underline{e}_3$$



3 equations → 6 unknown. $(\sigma_{\alpha\beta})$.

3 unknowns → $\underline{\hat{u}}$.

9 unknowns.

Nonlinear Elasticity

Continuum Mechanics.

Sep 29, Wed, Week 5.

Constitutive law.

$$\underline{\underline{\sigma}} = \underline{\underline{\Psi}}\left(\underline{\underline{F}}(t'),\ -\infty < t' \le t\right).$$

$\downarrow$

$\underline{\underline{\sigma}}(t)$ $\updownarrow$ $\Updownarrow$ Follow the whole deformation history.

how to obtain the function → Ω.

▽ Hyperelasticity (Green's elasticity).

Elasticity $\underline{\underline{\sigma}}$ depends only on $\underline{\underline{F}}(t)$.

$$\underline{\underline{\sigma}} = \underline{\underline{\Psi}}(\underline{\underline{F}}(t)).$$

↳ response function.

Mathematically,

$$\underline{\underline{P}} = \frac{\partial W(\underline{\underline{F}})}{\partial \underline{\underline{F}}}$$

→ the strain energy density.

← energy per unit volume

$\Uparrow$

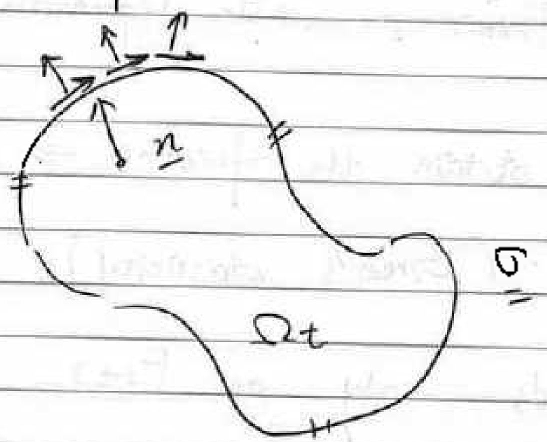
* the definition of hyperelastic materials.

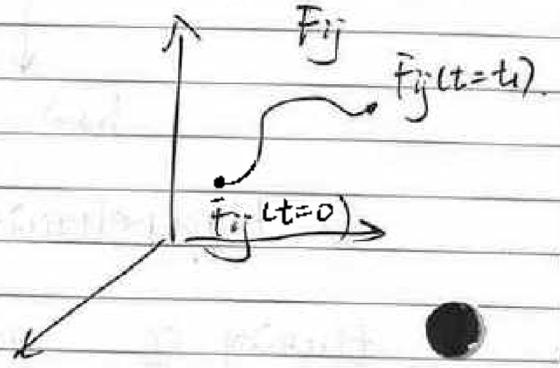
$$P_{ij} = \frac{\partial W}{\partial F_{ij}}.$$

$$dw = \frac{\partial w}{\partial F_{ij}} dF_{ij} = \frac{\partial w}{\partial \underline{\underline{F}}} : d\underline{\underline{F}}$$

$$= P_{ij}\, dF_{ij} = \underline{\underline{P}} : d\underline{\underline{F}}$$ work

$$d\phi = -\underline{t} \cdot d\underline{x}.$$ Potential





external work rate

$$= \int_{\partial \Omega_t} (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{v}\, ds + \int_{\Omega_t} \rho \cdot \underline{b} \cdot \underline{v}\, dV_t$$

in the reference configuration :

(Piola stress

$$\int_{\partial \Omega_0} (\underline{\underline{P}} \cdot \underline{N}) \cdot \underline{V}\, dS_0 + \int_{\Omega_0} \rho_0\, b_0 \cdot V\, dV_0$$

$$\int_{\partial \Omega_0} (\underline{\underline{P}} \cdot \underline{N})\, dS_0 = \int_{\partial \Omega_0} P_{ij} N_j V_i\, dS_0$$

div. Theo.

$$= \int_{\partial \Omega_0} (P_{ij} V_i)_{,j}\, dV_0 \qquad ,j = \frac{\partial}{\partial X_j}$$

$$= \int_{\partial \Omega_0} \underbrace{P_{ij,j}}_{\swarrow \text{LMB}} V_i\, dV_0 + \int_{\partial \Omega_0} P_{ij} V_{i,j}\, dV_0$$

$$P_{ij,j} = -\rho_0 B + \rho_0 A_i$$

$$EWR = \int_{\Omega_0} \rho_0 \underline{A} \cdot \underline{V}\, dV_0 + \underbrace{\int_{\Omega_0} P_{ij} V_{i,j}\, dV_0}$$

$$F_{ij} = \delta_{ij} + u_{i,j} \qquad \int_{\Omega_0} P_{ij} \dot{F}_{ij}\, dV_0$$

$$V_{i,j} = \cancel{\frac{\partial u}{\partial}} \frac{\partial}{\partial X_j}\left[\frac{\partial u_i}{\partial t}\Big|_{X_i} \right]$$

$$= \frac{\partial F_{ij}}{\partial t}\Big|_{\underline{X}}.$$

$$\vee$$

$$\frac{DF_{ij}}{Dt} = \dot{F}_{ij}$$

move the body

kinetic energy, (KE).

$$=\frac{1}{2}\frac{D}{Dt}\left[\int_{\Omega_0} \rho_0\,(\underline{V}\cdot\underline{V})\,dV_0\right]$$

in the reference configuration

$$+\int_{\Omega_0} \underbrace{\underline{\underline{P}} : \dot{\underline{\underline{F}}}}_{\text{work rate}}\, dV_0 .$$

completely general)

deform the body.

there could be dessipation in the process.

☆ hyperelastic

$$\frac{\partial W}{\partial \underline{\underline{F}}} = \underline{\underline{P}}$$

assume $P_{ij} = \frac{\partial W}{\partial F_{ij}}$.

$$\frac{D}{Dt}\int_{\Omega_0} W(\underline{F})\,dV_0 .$$

$$dW = \frac{\partial W}{\partial F_{ij}}\, dF_{ij}$$

$$\cancel{\dot{(dW)} = \frac{\partial W}{\partial F_{ij}}\, \dot{dF_{ij}}}$$

$$\frac{DW}{Dt} = \underbrace{\frac{\partial W}{\partial F_{ij}}}_{P_{ij}}\ \underbrace{\frac{D F_{ij}}{Dt}}_{\dot{F}_{ij}}$$

$t_1 \rightarrow t_2$; have to integrate the rate to time.

Total work from $t_1 \rightarrow t_2$

$$\int_{t_1}^{t_2} EWR\, dt = \int_{t_1}^{t_2} \frac{D}{Dt} \cdot \int_{\Omega_0} \rho\,(\underline{V})^2\, dV_0\, dt .$$

$$+ \int_{t_1}^{t_2} \frac{D}{Dt}\left(\int_{\Omega_0} W(\underline{F})\cdot dV_0\right) dt$$

t_1 : $\underline{F}_1$ $\underline{V}_1$

t_2 : $\underline{F}_2$ $\underline{V}_2$

$\quad\ \ \shortparallel \qquad \shortparallel$

$\quad\ \ \underline{\underline{F}} \qquad \underline{V}$

$$\int_{\Omega_0} W(\underline{\underline{F}}_2)\, dV_0$$

$$- \int_{\Omega_0} W(\underline{F}_1)\, dV_0 = 0$$

Motivation for Hyperelasiticity

↳ No energy loss during loading

Objectivity

- $\underset{\Theta}{W}(\underline{\underline{F}}) = W(\underline{\underline{Q}}\,\underline{\underline{F}})$

$\downarrow$

rigid body rotation.

$\underline{\underline{Q}}$: rotation.

$\uparrow$ (any)

$\underline{\underline{F}} = \cancel{\underline{\underline{Q}}\,\underline{\underline{R}}}\ \underline{\underline{R}}\,\underline{\underline{U}}$

only depends $\downarrow$ on stretch tensor

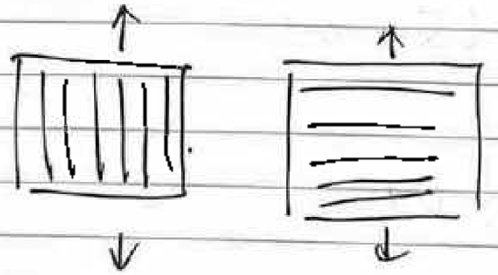
$W(\underline{\underline{F}}) = W(\underline{\underline{R}}^T\underline{\underline{R}}\,\underline{\underline{U}}). = \bar{W}(\underline{\underline{U}}).$

☆ only the stretching parts make a diff.

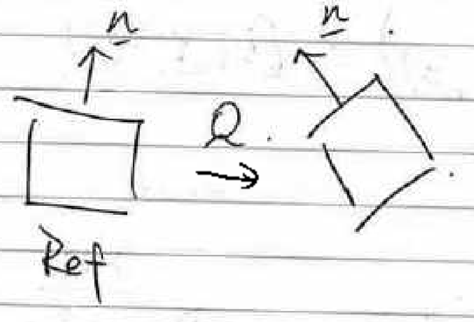
$= \hat{W}(\underline{\underline{C}}). \quad (\because \underline{\underline{U}}^2 = \underline{\underline{C}}).$

$\underline{\underline{P}} \Rightarrow P_{ij} = \frac{\partial W}{\partial F_{ij}} = \frac{\partial \hat{W}}{\partial C_{\cancel{ij}kl}} \frac{\partial C_{kl}}{\partial F_{ij}}$

$\hookrightarrow \quad = 2\underline{\underline{F}} \frac{\partial \hat{W}}{\partial C_{ij}}$ (HW).



Not isotropic.



$W(\underline{\underline{F}}) = W(\underline{\underline{F}}\,\underline{\underline{Q}})$

~~only for isotropic~~

isotropic : true for all $\underline{\underline{Q}}$

depends on the symmetry of the materials.

$$\underline{x} = \underline{X} + \underline{u}(\underline{X}, t)$$

$$\underline{X} = \chi^{-1}(\underline{x}, t)$$

$$\underline{u}(\chi^{-1}(\underline{x}, t)) = \hat{u}(\underline{x}, t)$$

$$\underline{x} - \underline{X} = \underline{u} \qquad \curvearrowright \frac{\partial u_i}{\partial X_j} = \frac{\partial u_i}{\partial x_k} \cdot \frac{\partial x_k}{\partial X_j}$$

$$\frac{\partial u_i}{\partial X_j} = \frac{\partial \hat{u}_i}{\partial x_k} \cdot \frac{\partial x_k}{\partial X_j}.$$

$$\underline{x} - \underline{X} = \underline{u}.$$

$$\underline{X} - \cancel{d}\,\underline{x} = \underline{u}$$

Oct. 5, Mon, Week 6.

$$\det(\underline{\underline{I}} + d\underline{\underline{B}}) = I + \frac{1}{\mathbb{R}}(d\underline{\underline{B}}) + \underline{O}(d\underline{\underline{B}})^2$$

$$d(\det\underline{\underline{A}}) = \det A \cdot (1 + \overset{tr}{\cancel{\det}}(\underline{A}^{-1} d\underline{A})) - \det\underline{\underline{A}}$$

$$= (\det A)\,tr(\underline{A}^{-1} d\underline{A}).$$

$$= (\det A)(\underline{\underline{A}}^{-T} : d\underline{\underline{A}})$$

$$= \frac{\partial \det\underline{\underline{A}}}{\partial \underline{\underline{A}}} : d\underline{\underline{A}}.$$

↳ hold true for all $d\underline{\underline{A}}$.

$$\Rightarrow \frac{\partial \det\underline{\underline{A}}}{\partial \underline{\underline{A}}} = (\det\underline{\underline{A}})\underline{\underline{A}}^{-T}.$$

$$\frac{\partial(\det\underline{\underline{A}}_{ij})}{\partial A_{ij}} = (\det\underline{\underline{A}})A_{jk}^{-1}$$

$$I_1\underline{\underline{I}} - \underline{\underline{C}}$$

$$\hookrightarrow \underline{\underline{P}} = \frac{\partial W}{\partial \underline{\underline{F}}} = 2\underline{\underline{F}}\frac{\partial \hat{W}}{\partial \underline{\underline{C}}} = 2\underline{\underline{F}}\left[\frac{\partial \Phi}{\partial I_1}\frac{\partial I_1}{\partial \underline{\underline{C}}} + \frac{\partial \Phi}{\partial I_2}\frac{\partial I_2}{\partial \underline{\underline{C}}} + \frac{\partial \Phi}{\partial I_3}\frac{\partial I_3}{\partial \underline{\underline{C}}}\right]$$

$$\frac{\partial I_1}{\partial \underline{\underline{C}}} \downarrow \underline{\underline{I}}$$

1st Piola stress.

$$\underline{\underline{P}} = 2\left[\left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2}\right)\underline{\underline{F}} + \frac{\partial \Phi}{\partial I_2}\underline{\underline{F}}\,\underline{\underline{C}} + I_3 \frac{\partial \Phi}{\partial I_3}\underline{\underline{F}}^{-T}\right.$$

$$\underline{\underline{F}}\,\underline{\underline{C}}^{-1} = \underline{\underline{F}}\,(\underline{\underline{F}}^T\underline{\underline{F}})^{-1}$$

$$= \underline{\underline{F}}\,(\underline{\underline{F}}^{-1}\underline{\underline{F}}^{-T}).$$

$$= \underline{\underline{F}}^{-T}$$

Recall

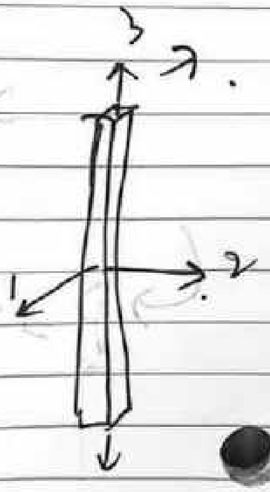
$$\underline{\underline{\sigma}} = \frac{1}{J}\underline{\underline{P}}\,\underline{\underline{F}}^T.$$

implies. $\Longrightarrow$ $J = \det \underline{\underline{F}}$

$$\hookrightarrow \frac{1}{J}\left[\left(\frac{\partial \Phi}{\partial I_1} + I_1 \cdot \frac{\partial \Phi}{\partial I_2}\right)\underline{\underline{F}}\,\underline{\underline{F}}^T - \frac{\partial \Phi}{\partial I_2}(\underline{\underline{F}}\,\underline{\underline{F}}^T)^2\right.$$

$$\left. + I_3 \frac{\partial \Phi}{\partial I_3}\underline{\underline{I}}\right] = \underline{\underline{\sigma}}.$$

Tension Test.



$$\underline{\underline{F}} \rightarrow \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_4 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

$$\underline{\underline{F}} = \lambda\, \underline{e_i}\,\underline{e_i} + \lambda_2\, \underline{e_2}\,\underline{e_2} + \lambda_3\, e_3 e_3$$

$I_1 \neq$

$I_2 \neq$

$$\underline{\underline{F}}^T\underline{\underline{F}} \Rightarrow \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_1^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$$

$$\mathrm{tr}(\underline{\underline{C}}) = \lambda_1^2 + \lambda_1^2 + \lambda_3^2.$$

$$\lambda_1 = \lambda_2 = \lambda.$$

~~P~~ $\det \underline{\underline{C}} = \lambda^4 \lambda_3^2.$

$$\underline{\underline{P}} = \underline{\underline{\sigma}} / \lambda.$$

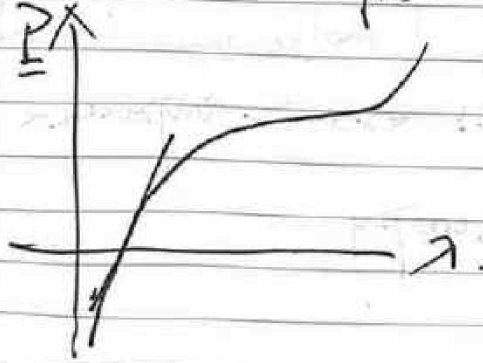
$$P_{33} = \frac{\sigma_{33}}{\lambda}. \qquad \sigma_{ij} = 0,\ j, i \neq 3.$$

$$\lambda \rightarrow I_2, I_1, I_3$$

loading → λ → curve.

fit: ✓

constitutive modeling



$= 1 - a \rightsquigarrow \frac{\delta L}{L_0}$

strain ener. dens. function.

det $\underline{\underline{F}}$

$$\underset{\uparrow}{\Phi} = C_1(I_1 - 3 - 2\log(\overset{\uparrow}{J})) + C_2(\ln J)^2.$$

(C1. C2 are material constants.

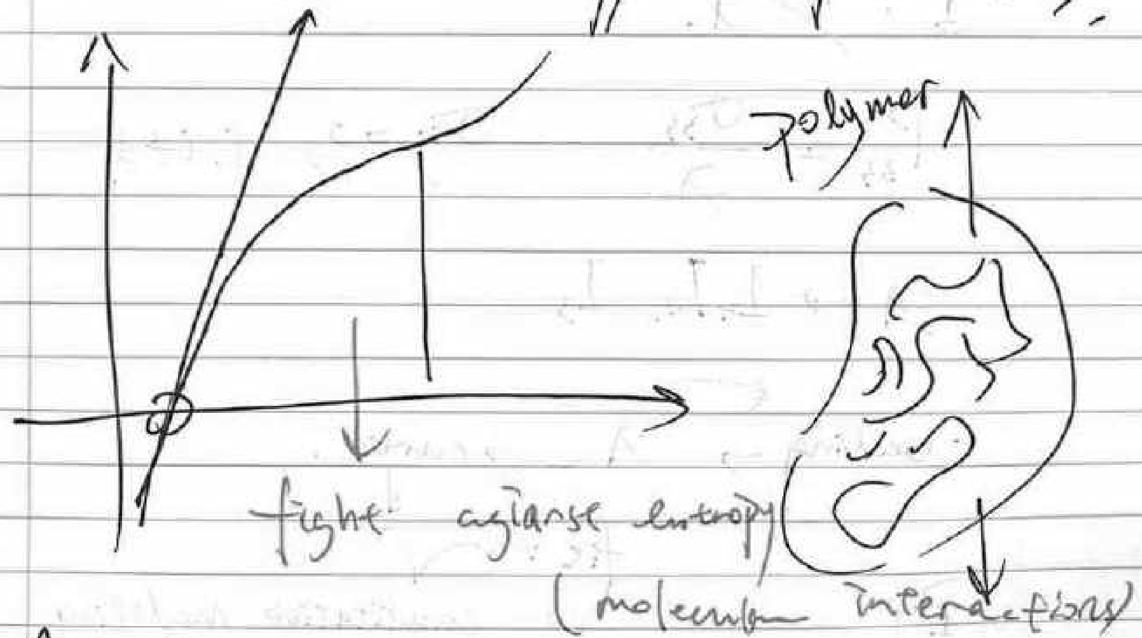
common model

locally linear model.

$$\underline{\underline{\sigma}} = \lambda \underline{\underline{\varepsilon}}.$$

simple tese & simple shear

determine differnce parameters.



~~Am~~ Temperature is ~~major~~ importance

↳ entropy.

Incompressible.

$$\frac{dV}{dV_0} = \overset{J}{\widetilde{\det \underline{\underline{F}}}} - \underline{\underline{I}}.$$

Isotropic deformation.

Define new energy function.

$$W_{new} = W(\underline{\underline{F}}) - \underset{\uparrow}{P}(J-1).$$

Lagrangian multiplier

(impose constraints)

$$\underline{\underline{P}} = \frac{\partial W_{new}}{\partial \underline{\underline{F}}} = \frac{\partial W}{\partial \underline{\underline{F}}} - P \cdot \frac{\partial \det(\underline{\underline{F}})}{\partial \underline{\underline{F}}}.$$

$$= \frac{\partial W}{\partial \underline{\underline{F}}} - P\underbrace{(\det \underline{\underline{F}})}_{1}\underline{\underline{F}}^{-T}.$$

$J=1$

$$\underline{\underline{\sigma}} = \underline{\underline{P}}\,\underline{\underline{F}}^T.$$

$$\boxed{\underline{\underline{\sigma}} = \frac{\partial W}{\partial \underline{\underline{F}}} \underline{\underline{F}}^T - P\underline{\underline{I}}.} \longrightarrow \text{model.}$$

$$I_3 = \det \underline{\underline{C}} = \det(\underline{\underline{F}}^T \underline{\underline{F}}) = \det(\underline{\underline{F}}^T)\det \underline{\underline{F}} = 1.$$

$$\underbrace{\text{Isotropic incompressible}}\ \text{solid}.$$

$$W = \Phi(I_1 I_2).$$

$$\underline{\underline{\sigma}} = \left[\left(\frac{\partial\Phi}{\partial I_1} + I_2\frac{\partial\Phi}{\partial I_2}\right)\underline{\underline{b}} - \frac{\partial\Phi}{\partial I_2}\,\underline{\underline{b}}^2\right] - p\underline{\underline{I}}$$

Corrected Note: (Based on Wiley books).

Review.

Hyperelasticity $\rightarrow$ $\underline{\underline{P}} = \dfrac{\partial W(\underline{\underline{E}})}{\partial \underline{\underline{F}}}$

→ gradient of strain energy density. with respect to $\underline{\underline{F}}$.

→ end exactly where u start.

Objectivity ~ $W(\underline{\underline{F}}) = \hat{w}(\underline{\underline{C}}) = \bar{w}(\underline{\underline{E}})$.

$$\underline{\underline{C}} = \underline{\underline{F}}^T\underline{\underline{F}}$$

Isotropic Material ~ $W(\underline{\underline{F}}) = W(\underline{\underline{F}}\,\underline{\underline{Q}})$.

$\forall$ orthogonal tensor $\underline{\underline{Q}}$

Define $\bar{\underline{\underline{F}}} = \underline{\underline{F}}\,\underline{\underline{Q}}$ ~ if isotropic.

$$W(\bar{F}) = W(\underline{\underline{F}}) \quad \forall \underline{\underline{Q}}.$$

Objectivity ~ $W(\underline{\underline{F}}) = \hat{W}(\underline{\underline{C}}) = W(\underline{\underline{F}}^T\underline{\underline{F}})$.

$$= \hat{W}(\underline{\underline{F}}\,\underline{\underline{Q}})^T(\underline{\underline{F}}\,\underline{\underline{Q}}) = \hat{W}(\underline{\underline{Q}}^T\underline{\underline{C}}\,\underline{\underline{Q}}).$$

$\hat{W}$ is a scalar invariant of Tensor $\underline{\underline{C}}$.

$$\underbrace{\det\left[\underline{\underline{C}} - \lambda\underline{\underline{I}}\right]}_{\text{independent of } \underline{\underline{Q}}.} = (-\lambda)^3 + I_1\lambda^2 - I_2\lambda + I_3,$$

$$\rightarrow \begin{cases} I_1 = \operatorname{tr}\underline{\underline{C}} \\ I_2 = \frac{1}{2}\left[(\operatorname{tr}\underline{\underline{C}})^2 - \operatorname{tr}\underline{\underline{C}}^2\right]. \\ I_3 = \det\underline{\underline{C}}. \end{cases}$$

→ For isotropic material.

$$\hat{W} = \Phi(I_1, I_2, I_3).$$

↖ isotropic.

$$\rightarrow \underline{\underline{P}} = \frac{\partial W}{\partial \underline{\underline{F}}} = 2\underline{\underline{F}}\frac{\partial \hat{W}}{\partial \underline{\underline{C}}}.$$

$$= 2\underline{\underline{F}}\left[\frac{\partial\Phi}{\partial I_1}\underbrace{\boxed{\frac{\partial I_1}{\partial \underline{\underline{C}}}}}_{\underline{\underline{I}}} + \frac{\partial\Phi}{\partial I_2}\underbrace{\boxed{\frac{\partial I_2}{\partial \underline{\underline{C}}}}}_{I_1\underline{\underline{I}} - \underline{\underline{C}}} + \frac{\partial\Phi}{\partial I_3}\underbrace{\boxed{\frac{\partial I_3}{\partial \underline{\underline{C}}}}}_{I_3\underline{\underline{C}}^{-T} = I_3\underline{\underline{C}}^{-1}.}\right]$$

$$I_3 = \det\underline{\underline{C}}.$$

<u>most general</u>: how to find $\dfrac{\partial \det\underline{\underline{A}}}{\partial \underline{\underline{A}}}$.

$$d(\det\underline{\underline{A}}) = \frac{\partial \det\underline{\underline{A}}}{\partial \underline{\underline{A}}} : d\underline{\underline{A}}$$

$$= \frac{\partial(\det\underline{\underline{A}})}{\partial A_{ij}}\, dA_{ij}.$$

$$d(\det\underline{\underline{A}}) = \det(\underline{\underline{A}} + d\underline{\underline{A}}) - \det(\underline{\underline{A}})$$

$$= \det(\underline{\underline{A}}(\underline{\underline{I}} + \underbrace{\underline{\underline{A}}^{-1}d\underline{\underline{A}}}_{d\underline{\underline{B}}}) - \det\underline{\underline{A}}$$

$J=1 \sim \underline{\underline{\sigma}} = \underline{\underline{P}}\,\underline{\underline{F}}^{T} = \frac{\partial W}{\partial \underline{\underline{F}}} \underline{\underline{F}}^{T} - p\underline{\underline{I}}.$

$$d(\det(d\underline{\beta})) = \det \underline{\underline{A}}(1 + tr(\underline{\underline{A}}^{-1} d\underline{\underline{A}})) - \det \underline{\underline{A}}$$

$$= (\det \underline{\underline{A}})\, tr(\underline{\underline{A}}^{-1} d\underline{\underline{A}})$$

$$= (\det \underline{\underline{A}})(\underline{\underline{A}}^{-T} : d\underline{\underline{A}})$$

$$= \frac{\partial \det \underline{\underline{A}}}{\partial \underline{\underline{A}}} : d\underline{\underline{A}}$$

$$\rightarrow \frac{\partial \det \underline{\underline{A}}}{\partial \underline{\underline{A}}} = (\det \underline{\underline{A}})\, \underline{\underline{A}}^{-T}$$

$$\cdots + \frac{\partial \Phi}{\partial I_3}\frac{\partial I_3}{\partial \underline{\underline{C}}} \rightarrow I_3 \underline{\underline{C}}^{-T} = I_3 \underline{\underline{C}}^{-1}$$

$$I_3 = \det \underline{\underline{C}}$$

$$\underline{\underline{P}} = 2\left[\left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2}\right)\underline{\underline{F}} - \frac{\partial \Phi}{\partial I_2}\underline{\underline{F}}\,\underline{\underline{C}} + I_3 \frac{\partial \Phi}{\partial I_3}\underline{\underline{F}}^{-T}\right]$$

$\underline{\underline{\sigma}} = \frac{1}{J}\underline{\underline{P}}\,\underline{\underline{F}}^{T} \rightarrow$ true stress.

Tension Test

For isotropic

$$\underline{\underline{\sigma}} = 2\left[\left(\frac{\partial \Phi}{\partial I_1} + I_1\frac{\partial \Phi}{\partial I_2}\right)b - \frac{\partial \Phi}{\partial I_2} b^2\right] - p\underline{\underline{I}}$$

Oct 7. Wed, Week 6.

Incompressible hyperelasticity.

- Kinematics — quantities deformation.

  $\underline{\underline{C}}, \underline{\underline{E}}, \underline{\underline{U}}$ strain measures.

- Balance laws. — stresses.

  $\underline{\underline{P}}, \underline{\underline{\sigma}}, \ldots$ other stresses measures.

  e.g. 2nd Piola stress.

  Biot stress.

- Constitutive Model.

  Relationship stress – strain.

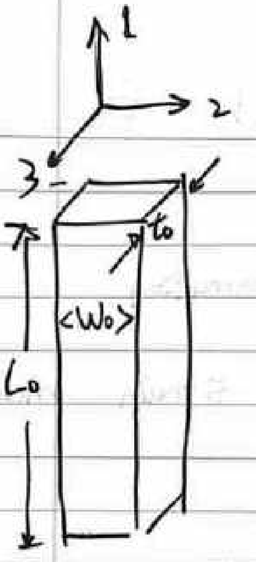
$$\underline{\underline{P}} = -p\underline{\underline{F}}^{-T} + 2\left[\left(\frac{\partial \Phi}{\partial I_1} + I_1\frac{\partial \Phi}{\partial I_2}\right)\underline{\underline{F}} - \frac{\partial \Phi}{\partial I_2}\underline{\underline{F}}\,\underline{\underline{C}}\right] \quad (1)$$

Recall I_1, I_2 are invariants of $\underline{\underline{C}}$.

$I_1 = tr\,\underline{\underline{C}}, \quad I_2 = \frac{1}{2}\left[(tr\,\underline{\underline{C}})^2 - tr(\underline{\underline{C}})^2\right].$

Lagrange multiplier enforce $\det \underline{\underline{F}} = J = 1.$

E.g. Uniaxial Tension or Compression test.



Ref. config.

$L_0 \gg W_0$ and t_0. , Tension test.

$\underline{\underline{P}} \cdot \underline{N}$ on all laterial surface is 0.

$\underline{N} = \underline{e}_2$ or $\underline{e}_3$.

undeformed lateral surfaces. $\leftarrow$ $\boxed{P_{13} = P_{23} = P_{33} = 0, \ P_{12} = P_{22} = P_{32} = 0}$

$\nabla_X \cdot \underline{\underline{P}} = \underline{0}$. $\leftrightarrow$ Balance law.

Simpliest model: $\Phi = \frac{\mu}{2}(I_1 - 3)$

ideal rubber → Neo-Hookean solid.

$= \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$

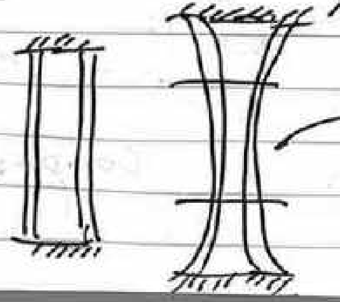
Principal coordinate.

Principal stretches.

Eqn (1) : →

$$\underline{\underline{P}} = -p\underline{\underline{F}}^{-T} + \mu \underline{\underline{F}}.$$

$\leftarrow$ material model. (constitutive).



$u_1 = (\lambda_1 - 1)X_1$.

$u_2 = u_3 = (\lambda_2 - 1)X_2$

$\hookrightarrow (\lambda_2 - 1)X_3$

$\lambda_2 = \lambda_3$

$$\underline{\underline{F}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$\det \underline{\underline{F}} = 1$. $\lambda_1 \lambda_2^2 = 1$. $\rightarrow \lambda_2 = \frac{1}{\sqrt{\lambda_1}}$

incompressibility.

Subs. into constitutive model.

make sure you satisfy boundary conditions.

* Not satisfy balance law → nonequilibrium states.

$$[\underline{\underline{F}}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda_1}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\lambda_1}} \end{bmatrix}.$$

$$[\underline{\underline{F}}^{-T}] = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_1} & 0 \\ 0 & 0 & \sqrt{\lambda_1} \end{bmatrix}$$

$P_{11} = -p/\lambda_1 + \cancel{\mu\lambda_1} \ \mu\lambda_1$.

$P_{22} = -p\sqrt{\lambda_1} + \mu / \sqrt{\lambda_1}$.

$= P_{33}$.

$P_{12} = P_{21} = P_{23} = P_{32}$

$= P_{13} = P_{31} = 0$.

$\lambda_1 = \text{const.} \Rightarrow$ equilibrium equation automatically satisfied.

B.C. are automatically satisfied.

→ Now, determine P.

$$\underline{\underline{BC}} = P_{22} = P_{33} = 0. \Rightarrow P\sqrt{\lambda_1} = \mu/\sqrt{\lambda_1}$$

$$\Rightarrow \boxed{P = \frac{\mu}{\lambda_1}.}$$

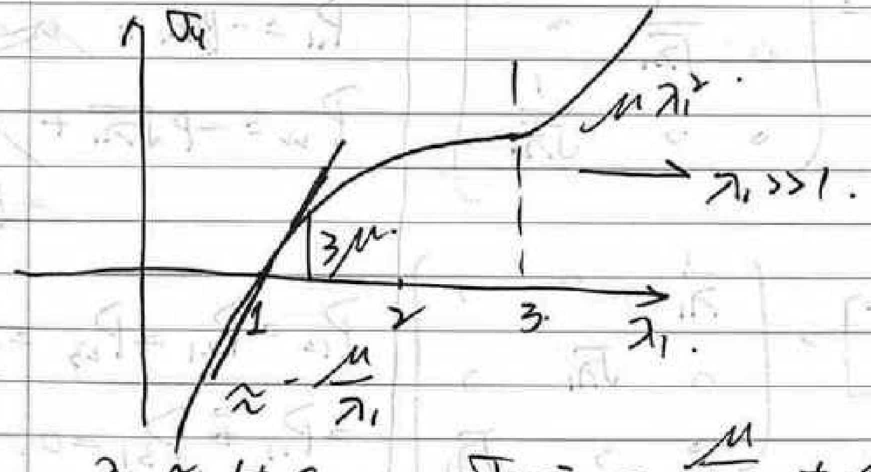
Substitute back to P_{11}. ←

$$P_{11} = -\frac{\mu}{\lambda_1^2} + \mu\lambda_1.$$

$$\underline{\underline{\sigma}} = \underline{\underline{P}}\,\underline{\underline{F}}^T.$$

$$\sigma_{11} = P_{11}\lambda_1$$

$$\sigma_{11} = -\frac{\mu}{\lambda_1} + \mu\lambda_1^2.$$



$\lambda_1 \approx 1+\varepsilon.$

$$\sigma_{11} = -\frac{\mu}{1+\varepsilon} + \mu(1+\varepsilon)^2$$

feasible modules

$$\cong -\mu(1-\varepsilon) + \mu(1+2\varepsilon) = 3\mu\varepsilon$$

$\varepsilon \ll 1$

$\mu = E/3.$

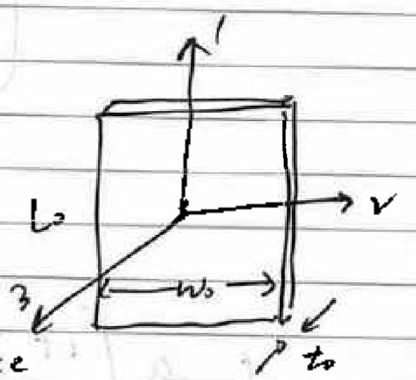
$$\frac{E}{2(1+\nu)} = \mu$$

↳ $\nu = 1/2$.

Poisson ratio.

plane stress deformation

(Surface : traction free).



B.C. : $\underline{\underline{P}} \cdot \underline{e}_3 = 0$, on surface

$t_0 \ll L_0$
$t_0 \ll W_0$.

$u_1(X_1, X_2)$. $u_2(X_1, X_2)$.
$u_3(X_1, X_2)$. (Assumption).

$$P_{13} = P_{23} = P_{33} \equiv 0, \text{ in the region (every where).}$$

↳ plane stress assumption.

$$\underline{\underline{F}} = \begin{bmatrix} 1+\frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & 0 \\ \frac{\partial u_2}{\partial X_1} & 1+\frac{\partial u_2}{\partial X_2} & 0 \\ 0 & 0 & \cancel{\frac{\partial u_3}{\partial X_3}} \rightarrow \text{wrong} \end{bmatrix}$$

$$\begin{matrix} \downarrow & \downarrow \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_1} \\ \| & \| \\ 0 & 0 \end{matrix}$$

$$\lambda_3 \rightarrow \lambda_3(X_1, X_2).$$

$$\boxed{1+\frac{\partial u_3}{\partial X_3}}$$

$$\underline{\underline{P}} = -p\,\underline{\underline{F}}^{-T} + \mu\,\underline{\underline{F}}.$$

$$[\underline{\underline{P}}] = -p\underbrace{\begin{bmatrix} 1+\frac{\partial u_2}{\partial X_3} & -\frac{\partial u_2}{\partial X_1} & 0 \\ -\frac{\partial u_1}{\partial X_2} & 1+\frac{\partial u_1}{\partial X_1} & 0 \\ 0 & 0 & \frac{1}{\lambda_3} \end{bmatrix}}_{[\underline{\underline{F}}^{-T}]}.$$

$$+\mu\begin{bmatrix} 1+\frac{\partial u_1}{\partial X_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad \Phi = \frac{\mu}{2}(I_1 - 3).$$

$$P_{33} = -\frac{p}{\lambda_3(X_1, X_2)} + \mu\lambda_3(X_1, X_2) = 0$$

equilibrium

$$\boxed{\begin{aligned} &\frac{\partial P_{11}}{\partial X_1} + \frac{\partial P_{12}}{\partial X_2} + \underset{0}{\cancel{\frac{\partial P_{13}}{\partial X_3}}} = 0 \\ &\frac{\partial P_{21}}{\partial X_1} + \frac{\partial P_{22}}{\partial X_2} = 0. \end{aligned}}$$

$$P_{11} = \mu\lambda_3^3\cdot\left(1+\frac{\partial u_2}{\partial X_2}\right) + \mu\left(1+\frac{\partial u_1}{\partial x_1}\right).$$

Oct, 13, 2021. Wed.

REVIEW: · plane stress : incompressible neoHookean Solid

$\underline{\underline{F}}$ for plane stress.

$$[\underline{\underline{F}}] = \begin{bmatrix} \overbrace{\begin{matrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{matrix}}^{[\underline{\underline{F}}_{in}]} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \lambda_3 \end{bmatrix} \qquad {}_{,\alpha} = \frac{\partial}{\partial X_\alpha}$$

$$x_\alpha = X_\alpha + u_\alpha(X_1, X_2) \qquad \alpha = 1, 2.$$

Independent of X_3.

λ_3 is the out-of-plane stretch ratio, $\lambda_3(X_1, X_2)$.

$$\underline{\underline{F}}_{in} = x_{\alpha,\beta}\,\underline{e}_\alpha\,\underline{e}_\beta$$

Neo-Hookean

$$\underline{\underline{P}} = -p\,\underline{\underline{F}}^{-T} + \mu\,\underline{\underline{F}}$$

for ~~im~~ incompressibility

$$J = \det\underline{\underline{F}} = 1 = (\overset{(x_{1,1}x_{2,2} - x_{1,2}x_{2,1})}{\det\underline{\underline{F}}_{in}})\lambda_3 = 1$$

$$\Rightarrow \det\underline{\underline{F}}_{in} = \frac{1}{\lambda_3}$$

$$[\underline{\underline{F}}^{-T}] = \begin{bmatrix} x_{2,2}\lambda_3 & -x_{2,1}\lambda_3 & 0 \\ -x_{1,2}\lambda_3 & x_{1,1}\lambda_3 & 0 \\ 0 & 0 & 1/\lambda_3 \end{bmatrix}.$$

$\textcircled{P}_{11}$ **Next step:**

$$\begin{cases} P_{11} = -p\, x_{2,2}\lambda_3 + \mu x_{1,1}. \\ P_{22} = -p\lambda_3 x_{2,1} + \mu x_{1,2}. \\ P_{21} = p\lambda_3 x_{1,2} + \mu x_{2,1}. \\ P_{22} = -p\lambda_2 . x_{1,1} + \mu x_{2,2}. \end{cases}$$

$P_{13} = P_{23} = P_{31} = P_{32} = 0$, consistent with the plane stress assump.

$\underline{\underline{P_{33} = 0.}}$ $= -p\frac{1}{\lambda_3} + \mu\lambda_3 = 0.$

$\hookrightarrow p = \mu\lambda_3^2.$

Substitute

use LMB: (ignore body forces.) & acceleration.

$$P_{11,1} + P_{12,2} = 0.$$

$$-(p\lambda_3^2 x_{2,2})_{,1} + \mu x_{1,11} + (\mu\lambda_3^3 x_{1,2})_{,2}$$

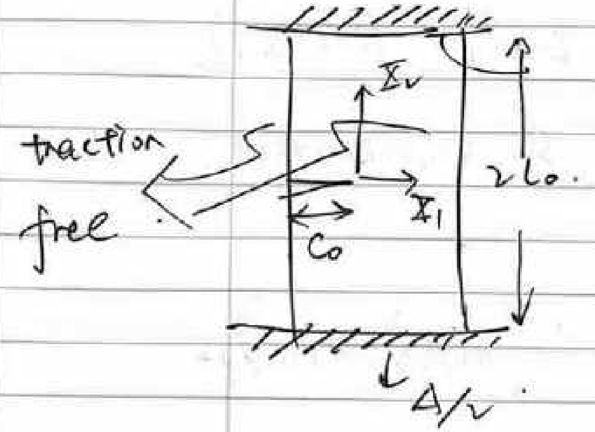
$$+ \mu x_{1,22} = 0.$$

$$0 = P_{21,1} + P_{22,2} = (\mu\lambda_3^3 + x_{1,2})_{,21} + \mu x_{2,11}$$

$$-(\mu\lambda_3^3 x_{1,1})_{,12} + \mu x_{2,22} = 0.$$

$$\begin{cases} \mu\nabla_X^2 x_1 + \mu((\lambda_3^3 x_{1,2})_{,2} - (\lambda_3^3 x_{2,2})_{,1}] = 0 \\ \Uparrow \\ \mu\nabla_X^2 x_2 + \mu[\qquad\qquad] = 0 \\ \lambda_3 = \dfrac{1}{x_{1,1}x_{2,2} - x_{2,1}x_{1,2}} \end{cases}$$

coupled PDEs for unknowns x_1, x_2.



BCs. $X_2 = \pm L_0$

$u_1 = 0$. $u_2 = \frac{\pm\Delta}{2}$

on lateral sides:

$X_1 = -C_0$ $P_{21} = P_{11} = 0$

$X_1 = W_0 - C_0$ $P_{21} = P_{11} = 0$

on Crack forces. $-C_0 < X_1 < 0$, $X_2 = 0^{\pm}$

$P_{12} = P_{22} = 0$

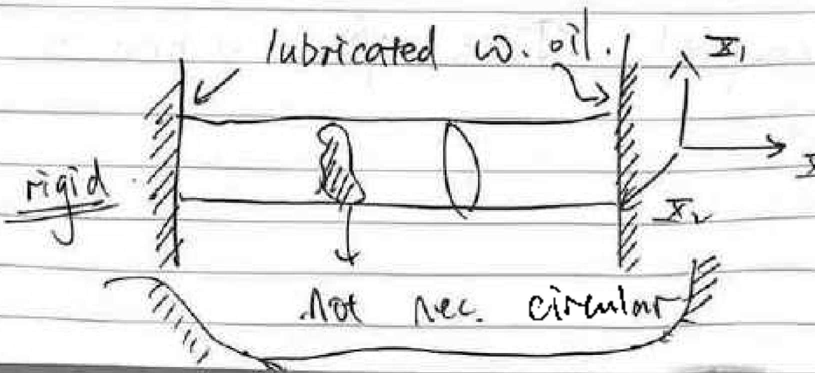
Plane strain

Assumption:

$$\left\{\begin{aligned} u_1 &= u_1(X_1, X_2) \\ u_2 &= u_2(X_1, X_2) \\ u_3 &= 0 \end{aligned}\right. \overset{\text{imply}}{\Longrightarrow} \underline{\underline{F}} = \begin{bmatrix} x_{1,1} & x_{1,2} & 0 \\ x_{2,1} & x_{2,2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\det \underline{\underline{F}} = 1 \rightarrow$ to determine $\underline{\underline{P}}$

$\nearrow P_{33}$, σ_{33} is not zero.



Linear Elasticity

$$\underline{\underline{\varepsilon}} = \frac{u_{i,j} + u_{j,i}}{2} \qquad ,_i = \frac{\partial}{\partial X_i}$$

$\rightarrow$ kinematics.

(one simple strain measure, only).

$\rightarrow$ small strain tensor.

$$\frac{\partial \sigma_{ij}}{\partial X_j} = -\rho_0 B_i \qquad \text{Equilibrium}$$

All you need, is constitutive model.

large deformation $\xrightarrow{\text{linearize}}$ constitutive model.

$$\frac{\partial W(\underline{\underline{E}})}{\partial \underline{\underline{E}}} = \underline{\underline{\sigma}}$$

$\rightarrow$ Small for all in linear stage ($\hat{w} = \bar{w} = w$).

$$\boxed{\sigma_{ij} = \underbrace{\overset{3^4}{K_{ijkl}}}_{\text{independent of strain}} \varepsilon_{kl}} \qquad (\text{Expect})$$

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}$$

$\rightarrow$ quadratic function of strain tensor (try) $W = \frac{1}{2} \cdot K_{ijkl} \varepsilon_{ij} \varepsilon_{kl}$

$$\sigma_{rs} \leftarrow \frac{\partial W}{\partial \varepsilon_{rs}} = \frac{1}{2} \Big[\underbrace{K_{ijkl} \delta_{ir} \delta_{js}}_{K_{rskl}} \varepsilon_{kl} + \underbrace{K_{ijkl}}_{K_{ijrs}} \varepsilon_{ij} \delta_{kr} \delta_{ls} \Big].$$

$$\sigma_{rs} = \frac{1}{2}\left[K_{rskl} \varepsilon_{kl} + K_{ijrs} \varepsilon_{ij} \right].$$

$$\cancel{\sigma}_{ij} = \frac{1}{2}\left[K_{rskl} \varepsilon_{kl} + K_{klrs} \varepsilon_{kl} \right]$$

$$\sigma_{ij} = \frac{1}{2}\left[K_{ijkl} \varepsilon_{kl} + K_{klij} \varepsilon_{kl} \right].$$

$$\sigma_{ij} = \frac{1}{2} \underbrace{\left[K_{ijkl} + K_{klij} \right]}_{\hat{K}} \varepsilon_{kl}$$

$\hat{K}_{ijkl} = \hat{K}_{klij}$ symmetric in kl, ij

$\hookrightarrow$ 81 component.

Symmetry of $\sigma_{ij} \Rightarrow K_{ijkl} = K_{jikl}$

Symmetry of $\varepsilon_{kl} \Rightarrow K_{ijkl} = K_{ijlk}$.

$9 \times 9 \longrightarrow 6 \times 6$

↓

36 independent components.

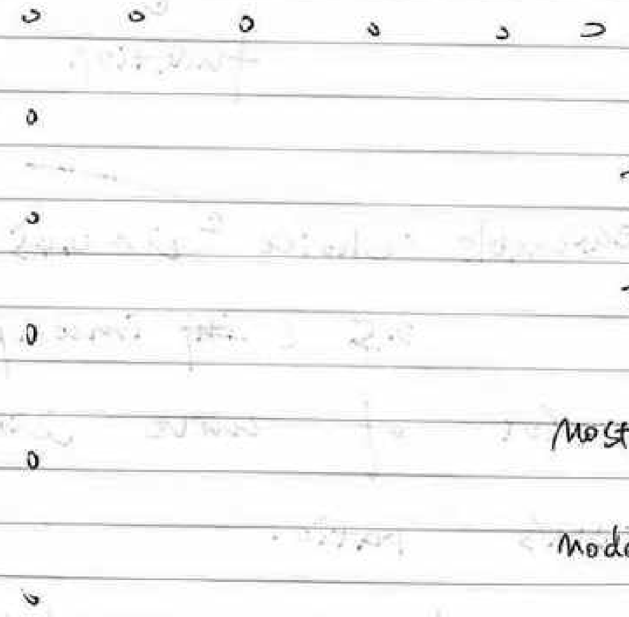
The existence of W.

↓

$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}$

↓

implies $K_{ijkl} = K_{klij}$.



⇓

~~18~~ 21 ind. comp.

↓

Most common model for elastic.

Oct. 15, office hours.

$C_1\ C_2,\ \propto$ shear modulus }
Poisson's ratio. }

letting $\lambda \to 1$
at very small → agrees with Hooke's law.

Plot the curve.

normalize the stress for shear modulus

other terms $\propto$ ratio of C_1, C_2
↓
function only of the
Poisson's ratio

reasonable choice $\nu = 0.45$
0.5 (~~inj~~ incompressible).

a lot of curve with different Poisson's ratio.

* normalized shear modulus G_1.

Piola & Cauchy normalized by

You can normal the stress by G_1.

λ_1, λ_3

~~$\lambda_3^2 - 1 + \frac{C_2}{C_1}\ln(\lambda_1^2\lambda_3) = 0$~~

$$\lambda^2 - 1 + \boxed{\frac{C_2}{C_1}}\ln(\lambda_3^3\lambda^2) = 0$$

$f(\nu)$

incompressible $\frac{C_2}{C_1} \to$ huge

↳ G_1.

$\underbrace{\lambda_3\lambda^2 = 1}$. ← Lambert function

1st order expansion John Hutchinston

$-\lambda$

$-0.45\lambda_3 + 0.45 = \lambda_1 - 1$

$1.45 - 0.45\lambda_3 = \cancel{\lambda_1}\ 0.$

Oct. 18th, 2021, Mon.

Review.

Linear Elasticity.

$$W = \frac{1}{2} K_{ijkl} \cdot \varepsilon_{ij} \varepsilon_{kl}.$$

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}$$

K_{ijkl} has 21 independent constants.

$$K_{ijkl} = K_{jikl} = K_{ijlk} = K_{klij}.$$

Anisotropic

$$\sigma_{ij} = K_{ijkl} \varepsilon_{kl}.$$

$$W = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} = \frac{1}{2} \underline{\underline{\sigma}} \cdot \underline{\underline{\varepsilon}}$$

Isotropy solids.

$$K_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

λ, μ are constants.

General form of isotropic 4th order Tensor

"Introduction to Cartesian Tensors"

"Jim Knowles.

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + \mu [\varepsilon_{ij} + \varepsilon_{ji}].$$

$$= 2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk} \delta_{ij}.$$

Generalized Hooke's LAW.

μ, λ are called Lame constants.

$$\sigma_{kk} = 2\mu \varepsilon_{kk} + 3\lambda \varepsilon_{kk}.$$

$$\sigma_{kk} = (2\mu + 3\lambda) \varepsilon_{kk}.$$

$$2\mu \varepsilon_{ij} + \lambda \frac{\sigma_{kk}}{(2\mu+3\lambda)} \delta_{ij} = \sigma_{ij}.$$

$$\varepsilon_{ij} = \frac{\sigma_{ij}}{2\mu} - \frac{\lambda \sigma_{kk}}{(2\mu+3\lambda)} \cdot \frac{1}{2\mu} \cdot \delta_{ij}$$

$$\varepsilon_{ij} = \frac{(1+\mu)}{E} \sigma_{ij} - \frac{\nu \sigma_{kk}}{E} \delta_{ij}.$$

~~μ~~ ν – Poisson's ratio.

E – Young's Modulus

$$\frac{1}{2\mu} = \frac{1+\nu}{E} \Rightarrow \mu = \frac{E}{2(1+\nu)}$$

Shear Modulus.

$$\frac{\nu}{E} = \frac{\lambda}{(2\mu+3\lambda)2\mu}.$$

Tension test

$\sigma_{11} = \sigma$. $\sigma_{ij} = 0$. $(i,j \neq 1)$.

$$\varepsilon_{11} = \frac{\sigma_{11}}{E}.$$

↳ tension modulus.

$$\varepsilon_{22} = \varepsilon_{33} = -\frac{\mu}{E}\sigma_{11}.$$

$$-\frac{\varepsilon_{22}}{\varepsilon_{11}} = \mu,$$

Poisson's ratio ≥ 0.

There are negative Poisson's ratio material but anisotropic.

Apply a pure hydrostatic tension.

$$\varepsilon_{ij} \leftarrow \varepsilon_{kk} = -\frac{(1+\nu)}{E} p\,\delta_{ij} + \frac{3p\nu}{E}\delta_{ij}.$$

if $\sigma_{ij} = -p\,\delta_{ij}$.

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = -\underbrace{\frac{(1+2\nu)}{E}}_{1/K} P.$$

Bulk Modulus $= -{}^{1}\!/_{K} P \rightarrow -P/K$.

$$K = \cancel{\frac{E}{1-2\nu}}\ \frac{E}{1-2\nu}.$$

$\nu \rightarrow \frac{1}{2}$, $K \rightarrow \infty$.

$\Downarrow$

$\mathrm{tr}\,\underline{\underline{\varepsilon}} = 0$.   **incompressible solid**

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}.$$

$$\sigma_{ij} = 2\mu\cancel{\varepsilon_{ij}} \rightarrow \sigma_{ij} = 2\mu\varepsilon_{ij} - P\delta_{ij}$$

★★★. **Office hour.**

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

General form relation for linear Elasticity

① $\varepsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2}$

② $\sigma_{ij,j} = -\rho_0 B_i$.

② $\sigma_{ij} = \sigma_{ji}$

③ $\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$

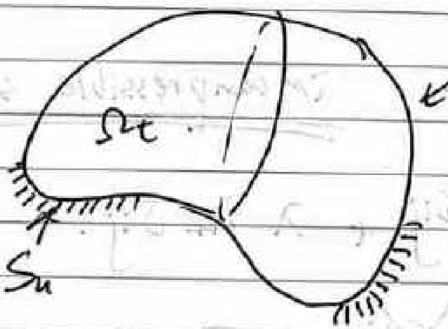
- Substitute ① into ③ to express strains in terms of displacements.
- Substitute stress into ② to obtain.

$$G\nabla^2\underline{u} + (\lambda + G)\nabla(\nabla\cdot\underline{u}) = -\rho_0\underline{B}$$

↳ Navier's equation (3 PDEs)

Subjeet Navier's Eq. to BCs

Typical e.g.

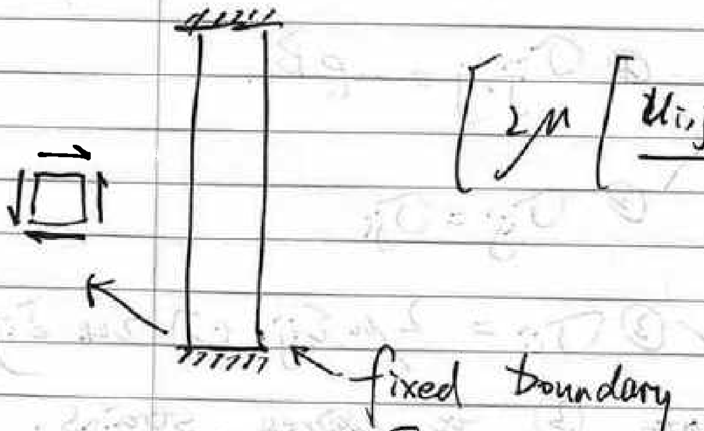


On. traction is prescribed.

mixed BCs.

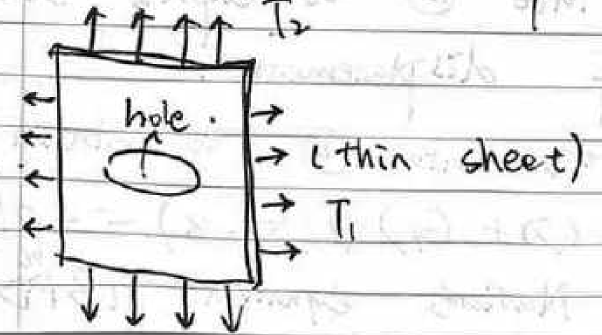
Displacement prescribed.

$$\begin{cases} \sigma_{ij} n_i = T_n(\underline{X}). & \underline{X} \in S_T. \\ S_u \ u_i = f(\underline{X}), & \underline{X} \in S_v \end{cases}$$

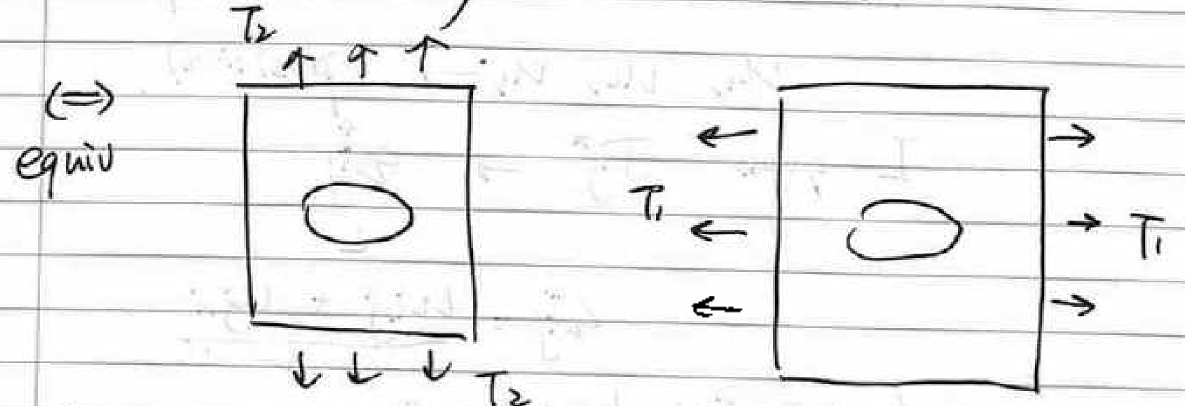


$$\left[2\mu \left[\frac{u_{i,j} + u_{j,i}}{2} \right] + \lambda u_{k,k} \delta_{ij} \right] n_j$$

$$= T_i(\underline{X}).$$



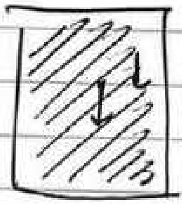
In Linear Elasticity.



Build up complex solutions from simple ones.

$$\underbrace{\sigma_{ij,j} = -\rho_0 B_i}$$

Suppose we guess a solution for ⊛ that also satisfies the Boundary Conditions (BC).
(traction BC).



$$\underline{B} = \underset{\sim}{g} \underline{e}_3$$

If we guess : σ^*_{ij} ...

compute displacement.

$$\varepsilon^*_{ij} = \frac{(1+\nu)}{E} \sigma^*_{ij} - \frac{\nu}{E} \sigma^*_{kk} \delta_{ij}$$

$$\downarrow$$

Integrate strain $\underline{\varepsilon}^*$ to get displacement field

There are three unknown disp.

$u_1, u_2, u_3 \rightarrow$ (position)

I guess $\sigma_{ij}^* \rightarrow \underset{\downarrow}{\varepsilon_{ij}^*}$

$$\varepsilon_{ij}^* = \frac{u_{i,j} + u_{j,i}}{2}$$

6 equations here.

6 equations, & 3 unknowns.

the solutions may not exists, if e.,
not unique.

Plane strain.

↳ linear elas.

$u_3 = 0$, $\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = u_{1,1}$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = u_{2,2}.$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{u_{1,2} + u_{2,1}}{2}.$$

$$\varepsilon_{ij}\ (\text{rest}) = 0.$$

$u_3 = 0$, u_1, u_2 depends on x_1, x_2 only.

You can show that

$$-2\varepsilon_{12,12} + \varepsilon_{11,22} + \varepsilon_{22,11} = 0$$

$$\frac{\partial^2 (\ )}{\partial x_1 \partial x_2} = (\ )_{,12}$$

↳ compatibility equation for plane strain.

: puts a constraint on the strain.

$$\varepsilon_{ij} = \frac{\sigma_{ij}(1+\nu)}{E} - \frac{\nu \sigma_{kk} \delta_{ij}}{E}.$$

↳ you will find this:

$$\underset{\downarrow}{\nabla^2(\sigma_{11} + \sigma_{22})} = \frac{1}{(1-\nu)} \nabla \cdot (\rho_0 b).$$

compatibility equation for stress.

Wed., Oct. 20, 2021. Week 9 (?)

Linear Elasticity

6 Eqns. $\varepsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2}$ — kinematics.

3 Eqns. $\sigma_{ij,j} = -\rho_0 B_i$ — Balance laws.

6 Eqns. $\varepsilon_{ij} = \frac{(1+\nu)\sigma_{ij}}{E} - \frac{\nu \sigma_{kk} \delta_{ij}}{E}$

Constitutive model.

15 Eqns.

unknowns: ε_{ij}, u_i, σ_{ij} 15 unknowns.

Navier Eqns. (Displacement formulation).

$$G\nabla^2 \underline{u} + (\lambda + G)\nabla(\nabla \cdot \underline{u}) = -\rho_0 \underline{B}$$

3 Eqns, & 3 unknowns. u_1, u_2, u_3.

independent variables, X_i.
positions.

dependent is u_i.

Most useful when body is subject to BCs.

Antiplane shear deformation.

$u_\alpha \equiv 0$. $\alpha = 1, 2$. No in-plane disp

$u_3 = u(X_1, X_2)$. independent of X_3.

$$\Downarrow$$

$\varepsilon_{\alpha\beta} = 0$, $\alpha = 1, 2$.

$$\varepsilon_{33} = \frac{\partial u_3}{\partial X_3} = 0.$$

Only non-vanishing strain are.

engineering strain

$$\varepsilon_{13} = \varepsilon_{31} = \frac{1}{2}\frac{\partial u}{\partial X_1} = \frac{1}{2}\gamma_1$$

$$\varepsilon_{23} = \varepsilon_{32} = \frac{1}{2}\frac{\partial u}{\partial X_2} = \frac{1}{2}\gamma_2$$

Constitutive model.

$\sigma_{\alpha\beta} = 0$. in-plane stress

$\sigma_{33} = 0$

$$\sigma_{13} = \sigma_{31} = G\gamma_1.$$

$$\sigma_{23} = \sigma_{32} = G\gamma_2$$

Equilibrium Eqns are identically satisfied in 1 & 2 directions ($B_1 = B_2 = 0$)

$$\frac{\partial \sigma_{31}}{\partial X_1} + \frac{\partial \sigma_{32}}{\partial X_2} = 0$$

$$\Updownarrow$$

$$\frac{\partial \tau_1}{\partial X_1} + \frac{\partial \tau_2}{\partial X_2} = 0 \iff \nabla_X \cdot \vec{\tau} = 0. \quad ①$$

no body force.

$$\underline{\tau} = \tau_1 \underline{e}_1 + \tau_2 \underline{e}_2.$$

γ_1 & γ_2 must satisfy the fact that.

$$\gamma_1 = \frac{\partial w}{\partial X_1}, \quad \gamma_2 = \frac{\partial w}{\partial X_2} \Rightarrow \frac{\partial \gamma_1}{\partial X_2} = \frac{\partial \gamma_2}{\partial X_1}.$$

Stress compatibility

the eqn. int.

$$\Rightarrow \frac{\partial \tau_1}{\partial X_2} = \frac{\partial \tau_2}{\partial X_1}. \quad ②$$

Introduce a stress function ϕ.

$$\tau_1 = \frac{\partial \phi}{\partial \cancel{\phi} X_2} \quad \tau_2 = -\frac{\partial \phi}{\partial X_1}. \quad ③$$

Subs. ② into ①, we see that

$$② \downarrow ③$$

① is satisfied automatically.

$$\frac{\partial \tau_1}{\partial X_1} = \frac{\partial^2 \phi}{\partial X_1 \partial X_2} \quad \frac{\partial \tau_2}{\partial X_2} = -\frac{\partial^2 \phi}{\partial X_2 \partial X_1}$$

Substitute ③ into ②.

$$\frac{\partial^2 \phi}{\partial X_2^2} + \frac{\partial^2 \phi}{\partial X_1^2} = 0 \quad \text{or} \quad \nabla_X^2 \phi = 0.$$

Laplace Eqn in 2D.

Stress function approach.

Disp. Formulation.

$$\left.\begin{aligned} \tau_1 &= G \frac{\partial w}{\partial X_1}. \\ \tau_2 &= G \frac{\partial w}{\partial X_2}. \end{aligned}\right\rangle \nabla_X^2 w = 0.$$

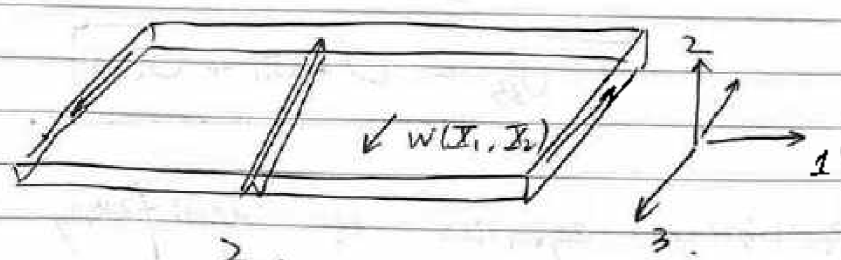
Substitute into ①.

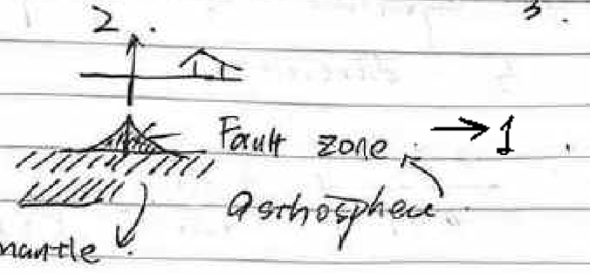
simpliest form of Navier's Equation.

$$\phi + i \cdot w = f(z).$$

$$\phi + iGw$$

☆ Anti-plane Shear.





Reminder: Plane strain

$u_\alpha(X_1, X_2)$, $\alpha = 1, 2$.

$u_3 \equiv 0$.

$\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$ (all others strain components $= 0$).

Compatibility.

$$\boxed{\frac{\partial^2 \varepsilon_{11}}{\partial X_2^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial X_1 \partial X_2} + \frac{\partial^2 \varepsilon_{22}}{\partial X_1^2} = 0}$$

Note

$\frac{\partial \sigma_{3i}}{\partial X_i} = 0$, is automatically satisfied.

$\sigma_{31} = \sigma_{32} = 0$, σ_{33} is independent of X_3.

$$\varepsilon_{33} = 0 \Rightarrow \frac{\sigma_{33}}{E} - \frac{\nu(\sigma_{11} + \sigma_{22})}{E} = 0$$

$$\boxed{\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})}.$$

Equilibrium equation is identically satisfied in 3 direction..

Therefore: $\frac{\partial \sigma_{11}}{\partial X_1} + \frac{\partial \sigma_{12}}{\partial X_2} = \rho_0 B_1$

$$\frac{\partial \sigma_{21}}{\partial X_1} + \frac{\partial \sigma_{22}}{\partial X_2} = \rho_0 B_2$$

Equilibrium Eqns (LMB). → (4a, b)

Assuming $\underline{B} = \underline{0}$.

Airy stress function, ϕ.

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial X_2^2} \qquad \sigma_{12} = -\frac{\partial^2 \phi}{\partial X_1 \partial X_2}.$$

$$\sigma_{22} = \frac{\partial^2 \Phi}{\partial X_1^2}. \qquad (5)$$

Substitute (5) into (4a, b).

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \frac{\nu(\sigma_{22} + \sigma_{33})}{E}.$$

$$\varepsilon_{22} = \frac{\sigma_{22}}{E} - \frac{\nu(\sigma_{11} + \sigma_{33})}{E}$$

$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$.

$$\cancel{\varepsilon_{33} = \frac{\sigma_{33}}{E} - \frac{\nu(\sigma_{11} + \sigma_{22})}{E}}. \quad \frac{\sigma_{12}}{2G} \to \tau.$$

⇓

Simplified plane strain constitutive model

$$\varepsilon_{11} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{11} - \nu\sigma_{22}\right].$$

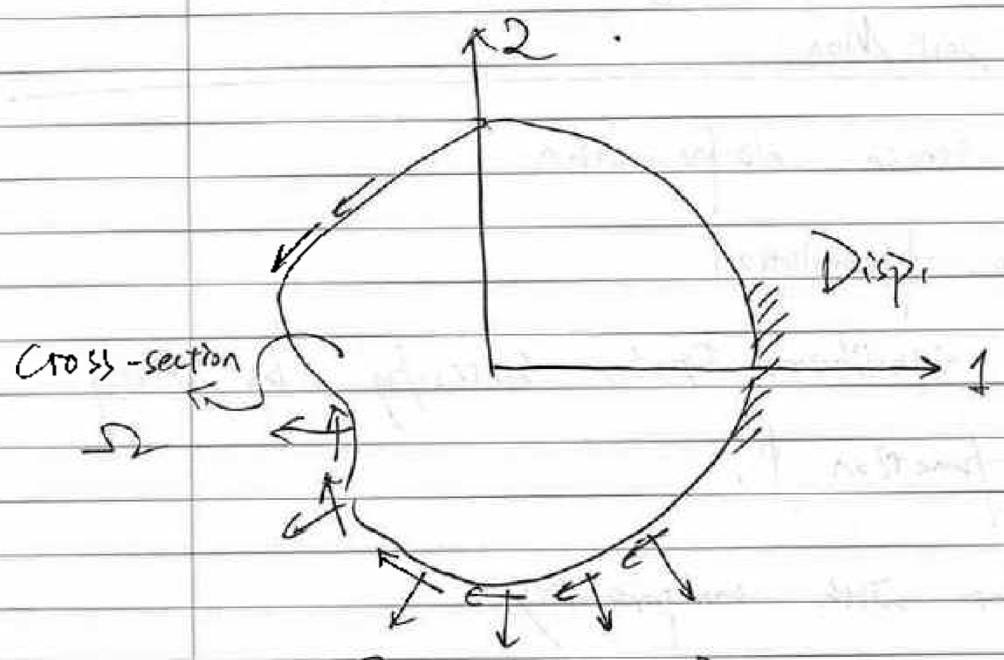
$$\gamma = \frac{\tau}{G} \qquad \varepsilon_{22} = \frac{(1+\nu)}{E}\left[(1-\nu)\sigma_{22} - \nu\sigma_{11}\right]$$

Non-zero Inplane stress fields are

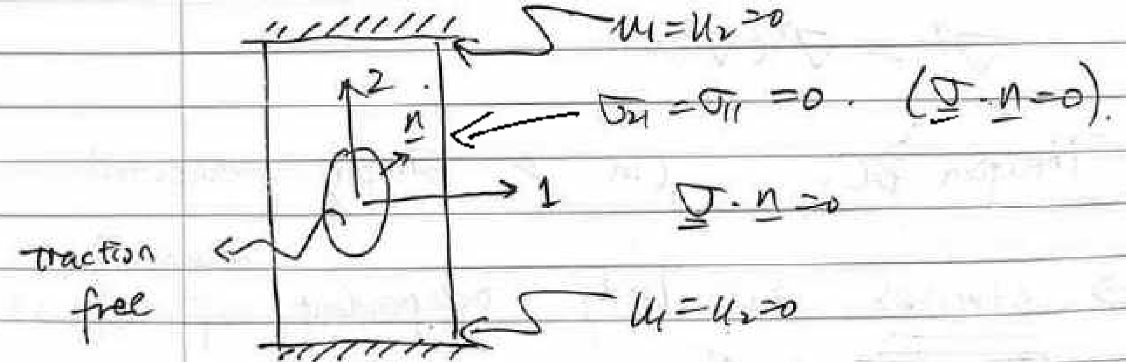
$\sigma_{11}, \sigma_{12}, \sigma_{22}$.

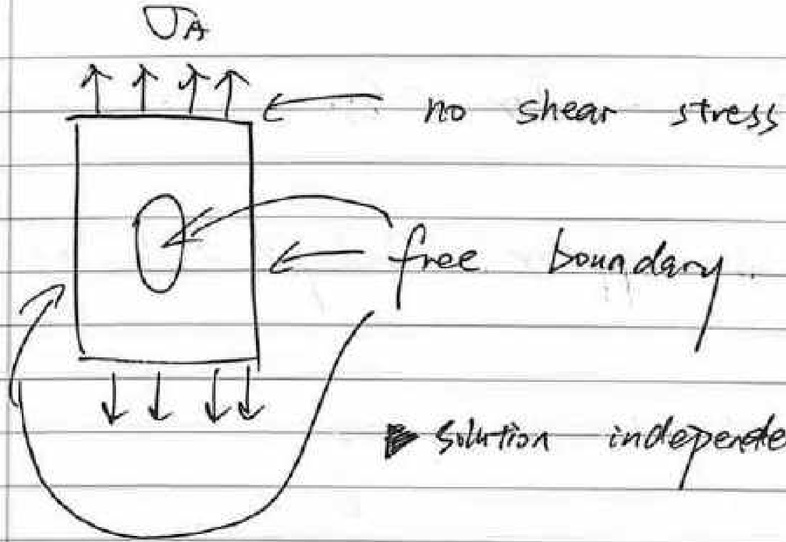
Non-zero Out-of-plane stress.

$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$.



S_T : Traction Boundary Conditions,



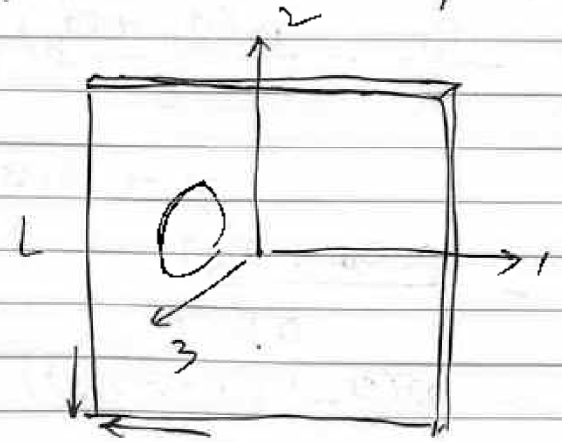


▶ Solution independent of material properties.

Stresses ∝ stress function.

$$\begin{cases} \sigma_{11} = \dfrac{\partial^2 \phi}{\partial x_1^2} \\ \sigma_{22} = \dfrac{\partial^2 \phi}{\partial x_2^2} \\ \sigma_{12} = -\dfrac{\partial^2 \phi}{\partial x_1 \partial x_2}. \end{cases}$$

Plane stress (finite deformation).



$t \ll L$. and other in-plane dimensions.

$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$. → three non zero stresses
↓
σ_{11} σ_{22}, σ_{12}.

$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{32} = \varepsilon_{31} \cong 0$.

$\varepsilon_{\alpha\beta}$ is approx. independent of x_3.

$\frac{\Delta V}{V} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 0$

$\downarrow$

incompressible.

Constitutive model

Plane stress $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$.

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \frac{\nu(\sigma_{22} + \overset{0}{\overset{\shortparallel}{\sigma_{33}}})}{E} = \frac{\sigma_{11}}{E} - \frac{\nu\sigma_{22}}{E}$$

$$\varepsilon_{12} = \frac{\sigma_{12}}{2G}.$$

$$\cancel{8}\ \varepsilon_{22} = \frac{\sigma_{22}}{E} - \frac{\nu(\sigma_{11} + \overset{0}{\overset{\shortparallel}{\sigma_{33}}})}{E} = \frac{\sigma_{22}}{E} - \frac{\nu\sigma_{11}}{E}$$

Plane strain

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \frac{\nu(\sigma_{22} + \overset{\cancel{0} \rightarrow \nu(\sigma_{11}+\sigma_{22})}{\overset{\shortparallel}{\sigma_{33}}})}{\cancel{8}E}.$$

$$= \frac{\sigma_{11}}{E} - \frac{\nu(\sigma_{22} + \nu(\sigma_{11}+\sigma_{22}))}{E}.$$

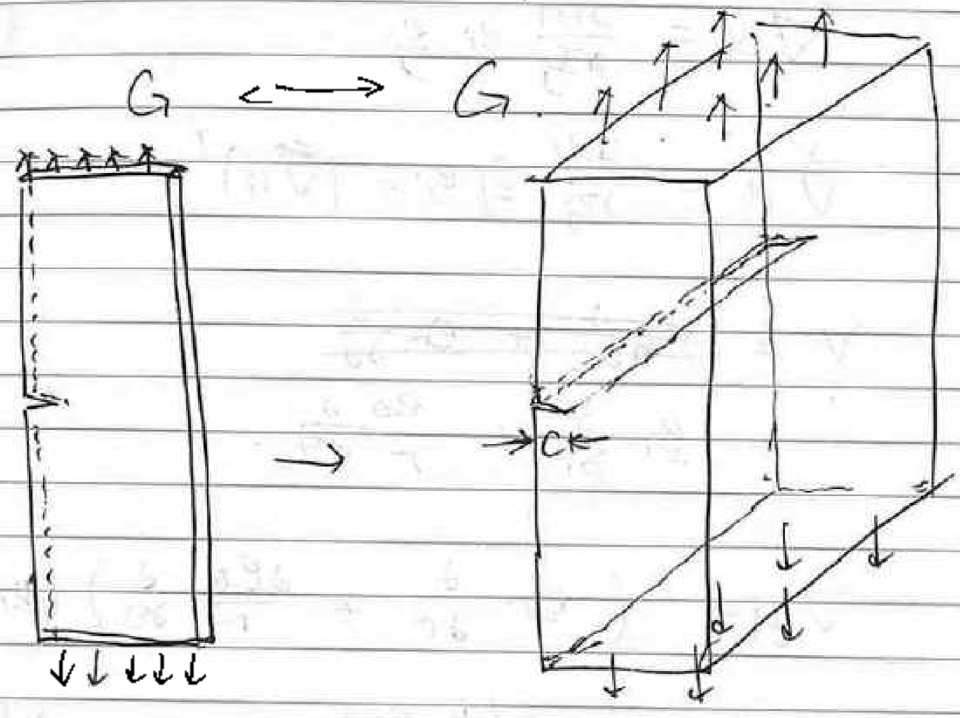
$$= \frac{(1-\nu^2)\sigma_{11}}{E} - \frac{\nu(1+\nu)\sigma_{22}}{E}$$

Compatibility & Equilibrium.

$\nabla^4 \phi \overset{\Downarrow}{=} 0$. (No Body force).

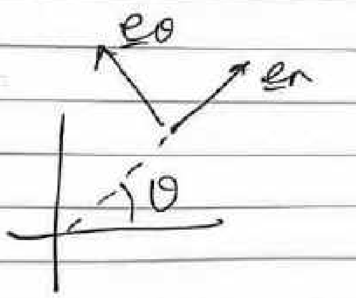
ν. $\longrightarrow$ $\frac{\nu}{1-\nu}$

plane stress. plane strain.



$$\underline{\varepsilon} = \frac{\nabla \underline{u} + (\nabla \underline{u})^T}{2}.$$

$$\varepsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2}$$



$\underline{u} = u_r \underline{e_r} + u_\theta \underline{e_\theta}$.

$$\begin{cases} \underline{e_r} = \cos\theta\, \underline{e_1} + \sin\theta\, \underline{e_2} \\ \underline{e_\theta} = -\sin\theta\, \underline{e_1} + \cos\theta\, \underline{e_2}. \end{cases}$$

In Cartesian coordinates

$$\nabla \underline{u} = \frac{\partial u_i}{\partial x_j} \underline{e}_i \underline{e}_j$$

$$\overleftarrow{\nabla} \underline{u} = \frac{\partial u_i}{\partial x_j} \underline{e}_i \underline{e}_j$$

$$\overrightarrow{\nabla} \underline{u} = \frac{\partial u_i}{\partial x_j} \underline{e}_j \underline{e}_i = (\overleftarrow{\nabla} \underline{u})^T$$

$$\nabla = \cancel{\underline{e}_\theta \frac{\partial}{\partial r} + \underline{e}_r \frac{\partial}{\partial \theta}}.$$

$$\underline{e}_r \frac{\partial}{\partial r} + \frac{\underline{e}_\theta}{r} \frac{\partial}{\partial \theta}.$$

$$\nabla \cdot \underline{u} = \left(\underline{e}_r \frac{\partial}{\partial r} + \frac{\partial \underline{e}_\theta}{r} \frac{\partial}{\partial \theta} \right) (u_r \underline{e}_r + u_\theta \underline{e}_\theta)$$

$$= \underline{e}_r \cdot \frac{\partial (u_r \underline{e}_r)}{\partial r} + \underline{e}_r \cdot \frac{\partial (u_\theta \underline{e}_\theta)}{\partial r}$$

$$= \underline{e}_r \cdot \frac{\partial u_r}{\partial r} \cdot \underline{e}_r + \underline{e}_r \left(\frac{\partial u_\theta}{\partial r} \underline{e}_\theta \right).$$

$$+ \frac{\underline{e}_\theta}{r} \left(\frac{\partial u_r}{\partial \theta} \underline{e}_r + u_r \underbrace{\frac{\partial \underline{e}_r}{\partial \theta}}_{\underline{e}_\theta} \right).$$

$$+ \frac{\underline{e}_\theta}{r} \frac{\partial u_\theta}{\partial \theta} \underline{e}_\theta + \frac{\underline{e}_\theta}{r} u_\theta (\underline{e}_r)$$

$$\frac{\partial \underline{e}_\theta}{\partial \theta} = - \cos\theta \, \underline{e}_1 - \sin\theta \, \underline{e}_2 = - \underline{e}_r$$

$$(\sim - \frac{u_\theta}{r}) \underline{e}_\theta \underline{e}_\theta$$

$$\nabla \underline{u} = \frac{\partial u_r}{\partial r} \underline{e}_r \underline{e}_r + \frac{\partial u_\theta}{\partial r} \underline{e}_\theta \underline{e}_\theta + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \underline{e}_\theta \underline{e}_r,$$

$$+ \left[\frac{u_r}{r} \cdot \underbrace{\underline{e}_\theta \underline{e}_\theta}_{\varepsilon_{rr}} + \frac{1}{r} \cdot \frac{\partial u_\theta}{\partial \theta} \right] \underline{e}_\theta \underline{e}_\theta$$

$$\underline{\underline{\varepsilon}} = \overset{\varepsilon_{rr}}{\frac{\partial u_r}{\partial r}} \underline{e}_r \underline{e}_r + \overset{\varepsilon_{\theta\theta}}{\left[\frac{u_\theta}{r} + \frac{1}{r} \cdot \frac{\partial u_\theta}{\partial \theta} \right]} \underline{e}_\theta \underline{e}_\theta$$

$$+ \frac{1}{2} \underbrace{\left[\frac{\partial u_\theta}{\partial r} - \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} - \frac{u_\theta}{r} \right]}_{\varepsilon_{\theta r}} \underline{e}_\theta \underline{e}_r + \frac{1}{2} [\quad] \underline{e}_r \underline{e}_\theta$$

$\sigma_{\alpha\beta}$ (Cartesian) $\{ \underline{e}_1, \underline{e}_2 \}$.

$\downarrow$

σ_{rr}, $\sigma_{r\theta}$, $\sigma_{\theta\theta}$ | polar coordinates.

$\{ \underline{e}_r, \underline{e}_\theta \}$.

Oct. 27. 2021. Wed

★★★ { Anti-plane shear. / plane strain / plane stress } stress functions.

$\Downarrow$

$\nabla^2 \phi = 0$ harmonic

$\nabla^4 \phi = 0$ biharmonic

$\underline{u} = u_r e_r + u_\theta \underline{e_\theta}$

$\underline{e_r} = \cos\theta \ \underline{e_1} + \sin\theta \ \underline{e_2}$

$\underline{e_\theta} = -\sin\theta \ \underline{e_1} + \cos\theta \ \underline{e_2}$.

$\underline{\underline{\sigma}} = \sigma_{ij} \underline{e_i} \underline{e_j} = \sigma_{rr} \underline{e_r} \underline{e_r} + \cdots\cdots$

$\nabla \cdot \underline{\underline{\sigma}}$ → Easy in Cartesian Coordinate

plane strain or plane stress

$$\begin{cases} \dfrac{\partial\sigma_{rr}}{\partial r} + \dfrac{1}{r}\cdot\dfrac{\partial\sigma_{r\theta}}{\partial\theta} + \dfrac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 . \\ \dfrac{1}{r}\cdot\dfrac{\partial\sigma_{\theta\theta}}{\partial\theta} + \dfrac{\partial\sigma_{r\theta}}{\partial r} + \dfrac{2\sigma_{r\theta}}{\partial r} = 0 \end{cases}$$

DERIVE THIS ONE !!!!

Third eqn. automatically satisfied.

$$\sigma_{rr} = \frac{\phi_{,r}}{r} + \frac{\phi_{,\theta\theta}}{r} .$$

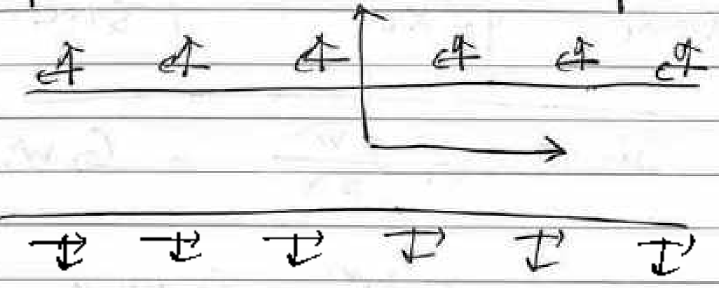
$$\sigma_{\theta\theta} = \phi_{,rr} \qquad \sigma_{r\theta} = -(\phi_{,\theta}/r)_{,r}$$

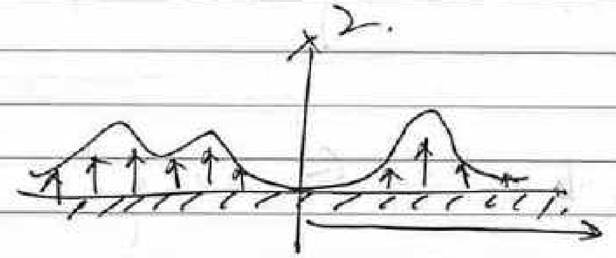
$$\nabla^4\phi = 0 = \overbrace{\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial r^2}\right)}^{\nabla^2} \underbrace{\left(\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial\theta^2}\right)}_{\nabla^2}\phi = 0$$

Technique of solution

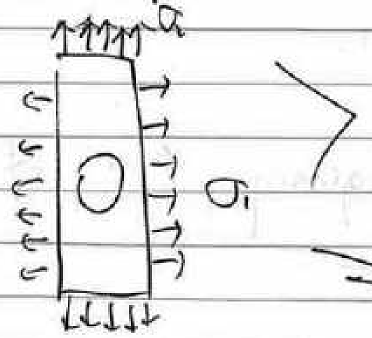
① Fourier transform.

[Strip or half space problem]





② superposition. (simple idea X technique)



Sum of two Solutions.

⟹ X work for nonlinear elasticity

↳ traction free.

③ saperation of variables.

↳ works for simple geometry.

Complex variable method

function theory.

(Antiplane shear).

$\nabla^2\phi = 0.$ and $\nabla^2 w = 0.$

$$\boxed{\phi_{,x}{}^{-\phi} = \sigma_{23}, \quad \phi_{,y} = \sigma_{32}} \quad ①$$

$x = x_1$, $y = x_2$ (stress function).

$$\boxed{\begin{aligned}\sigma_{13} &= G\frac{\partial w}{\partial x} = G w_{,x} \\ \sigma_{23} &= G\frac{\partial w}{\partial y} = G w_{,y}\end{aligned}} \quad ②$$

①. ②.

$$\Rightarrow \phi_{,x} = G w_{,y} \quad ③.$$

$$\phi_{,y} = -G w_{,x}$$

Define a complex function.

$$f(z) \equiv \underset{\text{real part of } f.}{\phi} + \underbrace{iGw}_{\text{imaginary part of } f}$$

③ is a rotation between real part of f and its Imaginary parts.

③ is called the $\underbrace{\text{Cauchy - Riemann}}_{CR.}$ Eqns.

$h(z) = u + iv.$

$$\begin{bmatrix} u_{,x} = v_{,y} \\ u_{,y} = -v_{,x}. \end{bmatrix}$$

Any fn with Real & Imaginary parts that satisfies the CR Eqns is called an analytic fcn in D

→ in a Domain D.

$$\begin{cases} \phi_{,xx} = G w_{,xy}. \\ \phi_{,xy} = -G w_{,xy}. \end{cases} \rightarrow \nabla^2\phi = 0$$

$\cos x$ ← Replace x by z. $= \cos z$.

$\| \quad\quad |$

$\frac{e^x + e^{-x}}{2} \quad\quad \cos z = \frac{e^z + e^{-z}}{2}$

$$e^z = e^{x+iy} = e^x e^{iy}$$

$$= e^x[\cos y + i\sin y]$$

$$= e^x\cos y + i e^x \sin y.$$

$$\frac{(e^x\cos y + i e^x\sin y) + e^{-x}(\cos y - i\sin y)}{2}$$

$$= \frac{(e^x + e^{-x})\cos y + i(e^x - e^{-x})\sin y}{2}$$

$$= \underbrace{\cosh x \cos y}_{u} + i\underbrace{\sinh x \sin y}_{v} = \cos z.$$

$$\begin{cases} u_{,x} = v_{,y} \\ u_{,y} = -v_{,x}. \end{cases} \checkmark \rightarrow CR$$

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \rightarrow a_n = \frac{1}{n!} f^{n}(z_0) = \frac{\partial^n f}{\partial z^n}\Big|_{z_0}$$

↖ analytic solution.

$$f(z) = \sum_{i=0}^{\infty} a_n n (z - z_0)^{n+1}$$

$$f'(z) = \phi_{,x} + i G_d \omega_{,x}$$

$$= \frac{\phi_{,y}}{i} + \frac{i G \omega_{,y}}{i}$$

$$= -i\phi_{,y} + G\omega_{,y}$$

$$\boxed{\begin{gathered} \phi = Re[\bar{z}\varphi(z) + \chi(z)] \\ \uparrow \qquad \uparrow \\ \text{Analytic} \\ \bar{z} \equiv x - iy \end{gathered}}$$

$\Leftarrow$ take this methods.

displacements :

$$2G(u_1 + iu_2) = \chi\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}$$

$$\psi(z) = \chi'(z) = \frac{d\chi}{dz}.$$

$\chi = 3 - 4\nu$. plane strain.

$= \frac{3-\nu}{1+\nu}$ stress.

$$\sigma_{11} + \sigma_{22} = 2[\varphi'(z) + \overline{\varphi'(z)}].$$

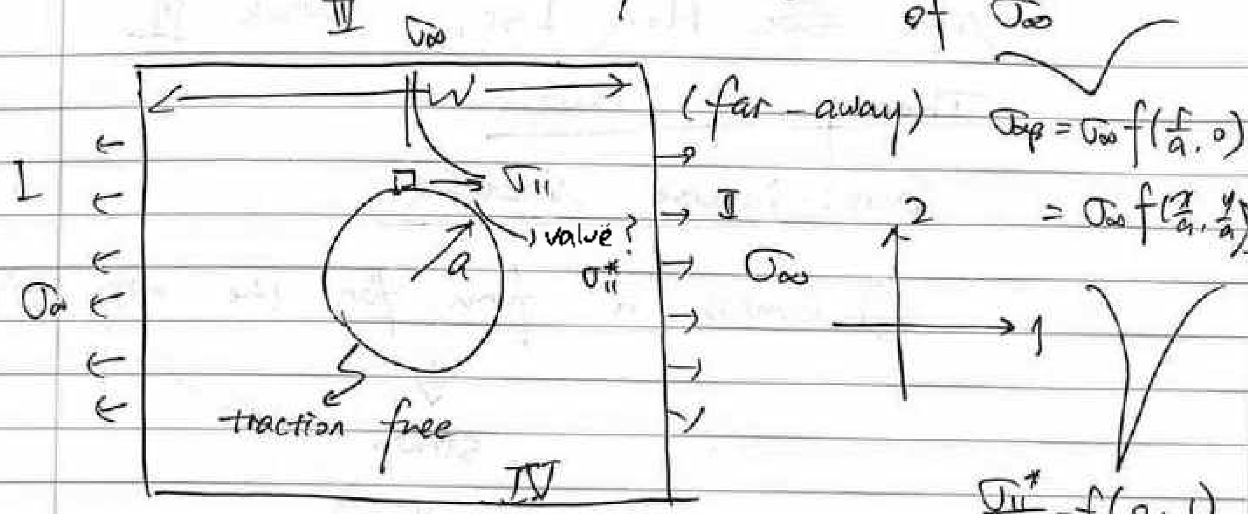
$$\sigma_{22} + i\sigma_{12} = \varphi'(z) + \overline{\varphi'(z)} + \bar{z}\varphi''(z) + \psi'(z).$$

$$\frac{dz^n}{dz} = nz^{n-1}$$

☆ Any rule with differentiations can be applied to complex variable theory.

(doesn't depend on the hole size).

$$\frac{\sigma_{11}^*}{\sigma_\infty} = \text{const. independent of } \sigma_\infty$$



$$\sigma_{\alpha\beta} = \sigma_\infty f(\frac{r}{a}, 0)$$

$$= \sigma_\infty f(\frac{x}{a}, \frac{y}{a})$$

$$\frac{\sigma_{11}^*}{\sigma_\infty} = f(0, 1) = 3$$

(infinite). $\rightarrow w \gg a$.

$\underline{\underline{BC}}$: $\sigma_{11} = \sigma_\infty$, as $|x| \rightarrow \infty$

I & II. $\sigma_{21} = 0$, as $|x| \rightarrow \infty$

III & IV : $\sigma_{22} = 0$. as $|y| \rightarrow \infty$

$\sigma_{12} = 0$, as $|y| \rightarrow \infty$.

$\underline{BC}$ on hole : $\underline{\sigma} \cdot \underline{n} = 0$ on hole.

$$\underline{n} = \underline{e_r} = \cos\theta\, \underline{e_1} + \sin\theta\, \underline{e_2}.$$

$$\sigma_{\alpha\beta}\, \underline{n}_\beta = 0 \rightarrow r = a = \sqrt{x^2 + y^2}$$

$$\sigma_{11}\cos\theta + \sigma_{12}\sin\theta = 0.$$

$$\sigma_{21}\cos\theta + \sigma_{22}\sin\theta = 0$$

$|\theta| \leq \pi$.

∵ linear problem

1: σ_{ij} independent of G, ν.
$\hookrightarrow \sigma_{\alpha\beta}$

$\rightarrow \sigma_{\alpha\beta}$. proportional to σ_∞

Mon. ~~Oct.~~ Nov. 1st, Week 11

Theory of Torsion

Semi-inverse Method

① Guess a form for the disp. field

⇓

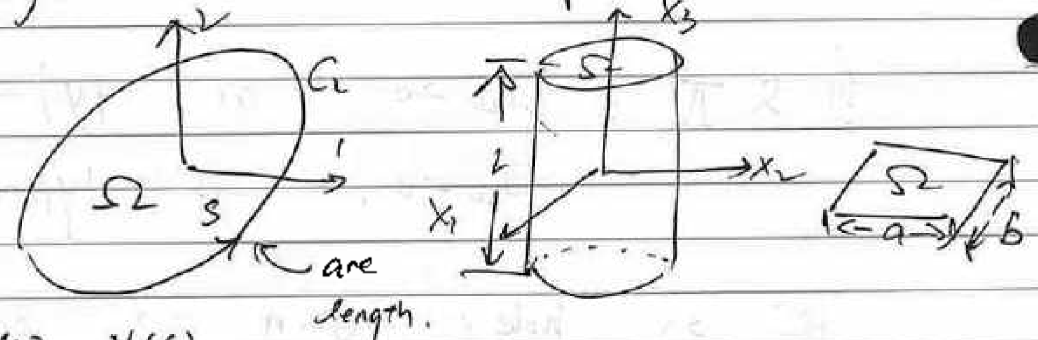
strain

⇓

stress → satisfy equilibrium.

check the BCs are satisfied.

* Cylinder with a uniform cross-section.



$X_1(s)$, $X_2(s)$

$\underbrace{\qquad}$ parameterize by

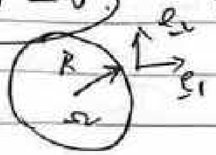
$\underline{u}$ is the displacement field.

$\underbrace{u_1 = -\alpha X_2 X_3}$ $\quad \underbrace{u_2 = \alpha X_1 X_3}$ $\quad u_3 = w(X_1 X_2)$

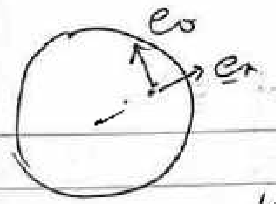
α : a constant. walking function.

To. motivate this, look at a special case

Assume → $\boxed{w = 0}$. Bar is circular.



$\underline{e}_2$, $\underline{e}_1$ $\quad \underline{u} = u_1 \underline{e}_1 + u_2 \underline{e}_2 = u_r \underline{e}_r + u_\theta \underline{e}_\theta$



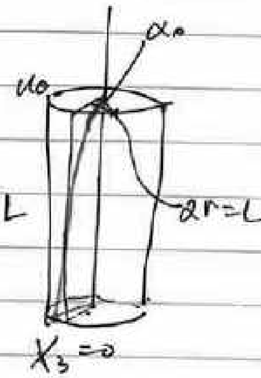
$\underline{e}_r = \underline{e}_1 \cos\theta + \underline{e}_2 \sin\theta$. ①

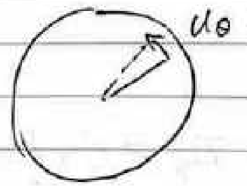
$\underline{e}_\theta = -\underline{e}_1 \sin\theta + \cos\theta \, \underline{e}_2$.

$u_r = \underline{u} \cdot \underline{e}_r \qquad u_\theta = \underline{u} \cdot \underline{e}_\theta$ ②

$u_1 = -\alpha \sin\theta X_3 \qquad u_2 = +\alpha r \cos\theta X_3$

$u_r = 0, \quad u_\theta = \alpha r X_3$. ③





on surface : $u_\theta = \alpha R X_3$.
$(r = R)$

$u_\theta = \alpha r L = r\theta_0$

$rL = r\theta_0$. $\quad \alpha = \frac{r\theta_0}{L}$ > the unit of trace per unit length

Strain tensor in cylindrical coordinate

the strain due to < $\varepsilon_{rr} = \varepsilon_{\theta\theta} = \varepsilon_{r\theta} = \varepsilon_{r3} = \varepsilon_{33} = 0$

$\varepsilon_{\theta 3} = \frac{1}{2} \cdot \alpha r$

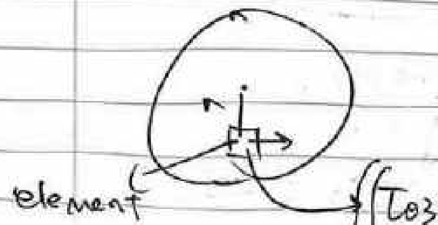
$L ? = R\theta_0 \qquad ? = \frac{R\theta_0}{L} = R\alpha$

$\tau_{\theta 3} = \frac{G}{2} \alpha r$ ← stress !!

↳ only non-trivial stress component in polar coordinate.

∘ traction free BCs automatically satisfied

on surface of cylinder



$\iint \tau_{\theta 3} \cdot r \, dA = M$.

(Recall the walking function of z)

$\varepsilon_{11}=0, \quad \varepsilon_{22}=0, \quad \varepsilon_{12}=0, \quad \varepsilon_{33}=0.$

$$\varepsilon_{13}=\frac{1}{2}\left[-\alpha x_2+\frac{\partial w}{\partial x_1}\right]$$

$$\varepsilon_{23}=\frac{1}{2}\left[\alpha x_1+\frac{\partial w}{\partial x_2}\right]$$

$$\Rightarrow \begin{cases} \tau_{\alpha\beta}=0, & \alpha=1,2. \\ \tau_{33}=0 \end{cases}$$

$$\tau_{13}=G[-\alpha x_2+w_{,1}] \qquad \tau_{23}=G[\alpha x_1+w_{,2}]$$

Enforce equilibrium,

Equilibrium in 1, 2 directions are satisfied automatically. → No Body Force.

in 3 direction. $\boxed{w \text{ is harmonic.}}$

equilibrium is satisfied.

$$\frac{\partial \tau_{13}}{\partial x_1}+\frac{\partial \tau_{23}}{\partial x_2}=0. \Rightarrow$$ Traction free BCs on the side of the Bar

$$\underbrace{\frac{\partial^2 w}{\partial x_1^2}+\frac{\partial^2 w}{\partial x_2^2}}_{\nabla^2 w=0.}=0$$ in some sense, anti-plane shear.

$\tau_{ij} n_j=0$ (traction free).

$$\underline{t}=\frac{dx_1}{ds}\underline{e}_1+\frac{dx_2}{ds}\underline{e}_2$$

$$\underline{n}=\frac{dx_2}{ds}\underline{e}_1-\frac{dx_1}{ds}\underline{e}_2$$

$$\tau_{ij} n_j=0$$

$$\Updownarrow$$

$$\boxed{\tau_{31}\underline{n}_1+\tau_{32}\underline{n}_2=0} \quad (\text{BCs}).$$

$$\boxed{\tau_{31}\cdot\frac{dx_2}{ds}-\tau_{32}\frac{dx_1}{ds}=0}$$

$$G[-\alpha x_2+w_{,1}]\frac{dx_2}{ds}-G[\alpha x_1+w_{,2}]\frac{dx_1}{ds}=0.$$

(we can cancel the G).

$$\underbrace{w_{,1}\frac{dx_2}{ds}-w_{,2}\frac{dx_1}{ds}}_{\text{grad}}=\alpha\left[x_1\frac{dx_1}{ds}+x_2\frac{dx_2}{ds}\right].$$

$$\text{grad}: \quad \underbrace{\nabla w\cdot\underline{n}}_{\frac{dw}{dn}}=\frac{\alpha}{2}\left[\frac{d(x_1^2+x_2^2)}{ds}\right]$$

Traction free BCs

guarantee this *

$$\frac{dw}{dn}=\frac{\alpha}{2}\frac{d(x_1^2+x_2^2)}{ds}. \quad > \text{BCs for Laplace}$$

$$\nabla^2 w=0.$$ ↖ Noimen BCs.

$$\oint_C \frac{dw}{dn}\,ds=0$$ (existence of solution) condition to be satisfied

$\Downarrow$

automatically satisfied. → $\frac{dw}{dn}$.

$$M=\alpha G\underbrace{\iint_A[x_\alpha x_\alpha+x_1 w_{,2}-x_2 w_{,1}]\,dx_1 dx_2}_{K:\ \text{torsional stiffness}}$$

$$\boxed{M = k\alpha}$$

w is harmonic

$\Downarrow$

w is the real / Im part of an analytical function.

$f(z) = w + i\phi$. $i = \sqrt{-1}$.

$z = x_1 + i x_2$

w, φ are related by CR Eqns.

$$\begin{cases} w_{,1} = \phi_{,2} \\ w_{,2} = -\phi_{,1} \end{cases}$$

$\sigma_{13} = G[-\alpha x_2 + w_{,1}] = G[-\alpha x_2 + \phi_{,2}]$

$\sigma_{23} = G[\alpha x_1 + w_{,2}] = G[\alpha x_1 - \phi_{,1}]$

Note: w, φ, is harmonic, so φ also satisfy $\nabla^2\phi = 0$.

Check. Equilibrium Eqn is automatically satisfied

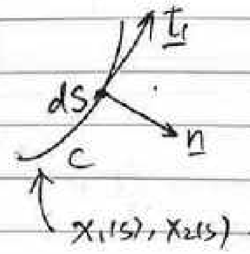
$\sigma_{13} n_1 + \sigma_{23} n_2 = 0 \Rightarrow$ BCs.

$\sigma_{13} \frac{dx_2}{ds} - \sigma_{23} \cdot \frac{dx_1}{ds} = 0$.

Wed, Nov. 3rd, Week 12

REVIEW. Displacement. $\rightarrow \nabla^2 w = 0$

TRACTION FREE BCs: $\frac{dw}{dn} = x_2 n_1 - x_1 n_2$



$= x_2 \frac{dx_2}{ds} + x_1 \frac{dx_1}{ds} = \frac{1}{2} \frac{d(x_1^2 + x_2^2)}{ds}$.

$\underline{t} = \frac{dx_1}{ds} \underline{e}_1 + \frac{dx_2}{ds} \underline{e}_2$

$\underline{n} = \frac{dx_2}{ds} \underline{e}_1 - \frac{dx_1}{ds} \underline{e}_2$.

$$\begin{cases} \sigma_{13} = \frac{G}{2}\alpha\left[-x_2 + w_{,1}\right] \\ \sigma_{23} = \frac{G}{2}\alpha\left[x_1 + w_{,2}\right]. \end{cases}$$

$$\begin{cases} u_1 = -\alpha x_2 x_3 \\ u_2 = \alpha x_1 x_3 \\ u_3 = \alpha w(x_1, \cancel{x_2}). \end{cases}$$

A different approach.

$f(z) = w + i\phi \leftarrow$ conjugate harmonic function to w.

$w_{,1} = \phi_{,2}$ & $w_{,2} = -\phi_{,1}$ $\underline{CR}$

$\nabla^2\phi = 0$

$\sigma_{13} = \frac{G}{2}\alpha\left[-x_2 + \phi_{,2}\right]$, $\sigma_{23} = \frac{G}{2}\alpha\left[x_1 - \phi_{,1}\right]$

T.F. BCs $\sigma_{31} n_1 + \sigma_{32} n_2 = 0 \Rightarrow \left[-x_2 + \phi_{,2}\right] n_1 + \left[x_1 - \phi_{,1}\right] n_2 = 0$.

$$\underbrace{-x_2 \frac{dx_2}{ds} + x_1\left(-\frac{dx_1}{ds}\right)}_{\frac{d(x_1^2 + x_2^2)}{ds}}$$

$$-\phi_{,1} n_2 + \phi_{,2} n_1 = 0.$$

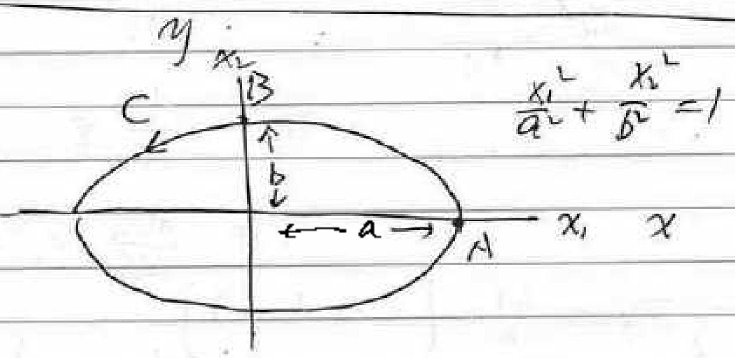
$$\underbrace{\phi_{,1} t_1 \quad \phi_{,2} t_2}_{\nabla\phi \cdot \underline{t} = \frac{d\phi}{ds}}$$

$$\Rightarrow \frac{d\phi}{ds} = \frac{d(x_1^2+x_2^2)}{ds} \Rightarrow \phi = \frac{x_1^2+x_2^2}{2} + \text{const} \Rightarrow \text{Curve } C$$

Set const = 0.

$$BC: \boxed{\begin{array}{l} \phi = \frac{x_1^2+x_2^2}{2} \quad \text{on } C \\ \nabla^2\phi = 0 \end{array}}$$

Example



$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

$f(z) = w + i\phi.$

$$f(z) = i(\underset{\text{const.}}{C^2})z^2 \rightarrow z = x+iy \Downarrow z^2 = x^2 - y^2 + 2ixy.$$

$\vee$

$ic^2z^2 + ik^2 \rightarrow$ some other const.

$C\ k\ k$ are real numbers.

hopefully satisfy the boundary conditions.

$$\hookrightarrow = \underbrace{ic^2(x^2-y^2) + ik^2}_{\text{imaginery part.}} \underbrace{-2c^2xy}_{\text{real part}}.$$

$$\underline{BC} \text{ on } C: c^2(x^2-y^2) - k^2 = \frac{x^2+y^2}{2}$$

$$k^2 = x^2\overbrace{\left[\tfrac{1}{2} - c^2\right]}^{\frac{1}{a^2}} + y^2\overbrace{\left[\tfrac{1}{2}+c^2\right]}^{\frac{1}{b^2}}$$

$$c^2 = \frac{1}{2}\frac{a^2-b^2}{a^2+b^2} \qquad k^2 = \frac{a^2b^2}{a^2+b^2}$$

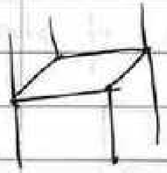
that means $\rightarrow \phi \Downarrow \nabla$

$$\sigma_{13} = G\alpha[-x_2 + \phi_{,2}]$$
$$\sigma_{23} = G\alpha[x_1 - \phi_{,1}]$$

$$\begin{cases} \sigma_{13} = \dfrac{-2G\alpha a^2 y}{a^2+b^2} \\ \sigma_{23} = \dfrac{2G\alpha b^2 x}{a^2+b^2} \end{cases}$$

$w = -2c^2 xy$ Also know waping function.

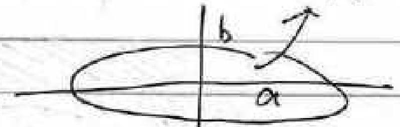
Note $c=0$, $a=b$, circle



$$M = \iint_y (\sigma_{32}x - \sigma_{12}y)\,dA$$
$$= \frac{G\pi\alpha b^3a^3}{a^2+b^2} = k\alpha.$$

$$\hookrightarrow k = \frac{G(\pi ab)a^2b^2}{(a^2+b^2)}$$

$A = \pi ab$



how to calculate waping function / stresses in torsion.

Prandtl stress function approach:

$$\nabla^2\phi = 0$$
$$\phi = \frac{1}{2}(x^2+y^2), \text{ on } C.$$

Define a function: $\Phi = \phi - \frac{1}{2}(x^2+y^2)$

on the BCs, $\Phi = 0$. on C

$\nabla^2\Phi = -2. \rightarrow$ Poisson's equation

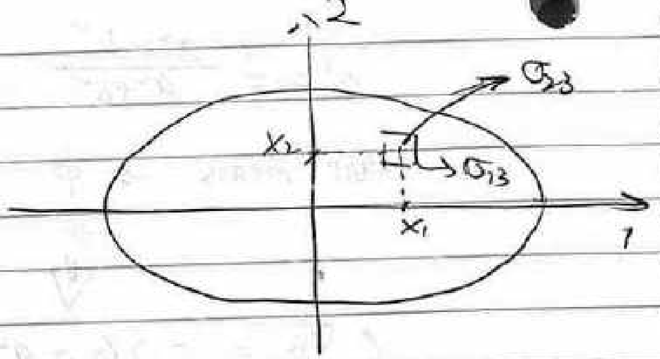
$$\phi = \Phi + \frac{1}{2}(x^2+y^2)$$

~~$\sigma_{13} = \frac{G}{2}\alpha$~~

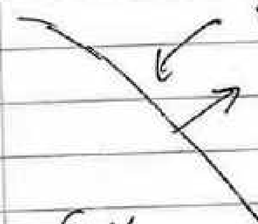
$$\begin{cases} \sigma_{13} = G\alpha\,\Phi_{,2} \\ \sigma_{23} = -G\alpha\,\Phi_{,1} \end{cases}$$

Calculate the moment:

$M = \alpha 2G \iint_A \Phi \, dA$.



Constant Φ curve.



$\Phi = c$

the gradient of Φ. normal to surface.

$\nabla \Phi = \Phi_{,1} \underline{e}_1 + \Phi_{,2} \underline{e}_2$

$\underline{\tau} = G\alpha$

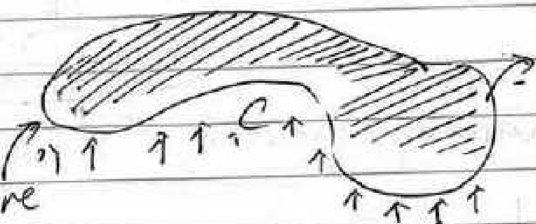
$[\Phi_{,2} \underline{e}_1 - \Phi_{,1} \underline{e}_2] \Rightarrow \underline{\tau} \cdot \nabla \Phi = 0$.

The constant line are the direction of shear vector

lines of shear stresses.

$\nabla \cdot \Phi = \Phi_{,1}^2 + \Phi_{,2}^2 \underset{\sim}{\propto} [|\Phi| G\alpha]$

Prandtl soap film analogy

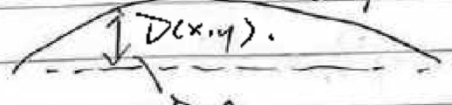


T → surface tension.

($\approx$ const)

↳ Force / length.

↙ film deformed.



$\nabla^2 D = -\frac{P}{T}$ ← pressure.

Small membrane deflection formula.

define new const.

$\frac{1}{2} d = \frac{DT}{P}$

$D = \frac{dP}{2T}$

$\nabla^2 \phi = -2$

Similar !!

also $D = 0$ on C

$\nabla^2 d = -2 \Rightarrow d = 0$ on C

$c_1 = \frac{\mu}{2} \quad c_2 = \frac{\Lambda}{2} \quad E, \nu$

$\sigma = \frac{1}{J}\left(c_1(\lambda_1^2 - 1) + c_2 \ln(J)\right) = 0$

$\lambda_i = 1 + \varepsilon_{ii} \quad \frac{1}{\lambda_i} = 1 - \varepsilon_{ii}$

~~λ^2 [illegible] 0~~ $\lambda_i^2 = 1 + 2\varepsilon_{ii} + \cancel{\varepsilon_{ii}^2}$

$J = \det F = \lambda_1^2 \lambda_3 = 1 + 2\varepsilon_{11} + \varepsilon_{33}$

$\ln(1 + \varepsilon_{ii}) \approx \varepsilon_{ii}$

$c_1 >$

Solid Mechanics

Nov. 8, Mon, Week 12.

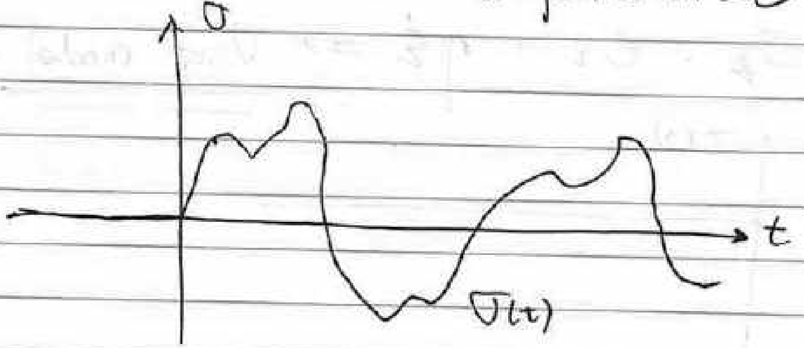
Linear viscoelasticity

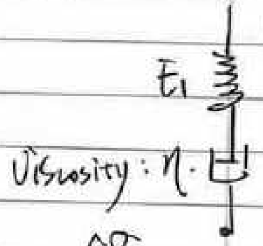
ideal model: Deformation $\not\leftrightarrow$ stress history.

real material: Deformation = D (history).

- Cartoon Models

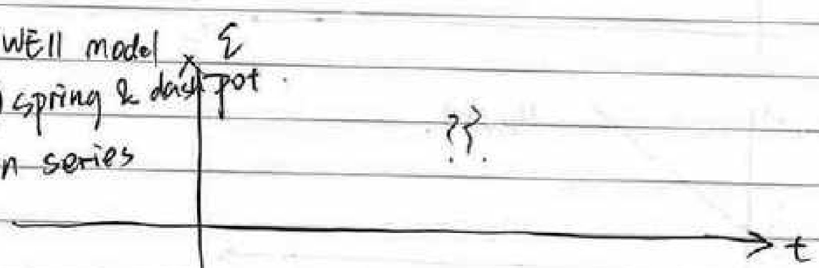
- uniaxial tension test.



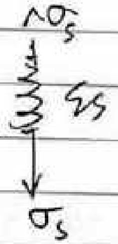


- MAXWELL model

spring & dashpot in series



→ long run : simple fluid.
(suddenly apply a stress)

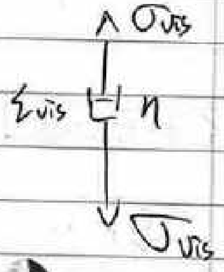


$\sigma_s = E\varepsilon_s$.

$$\varepsilon = \frac{\sigma}{E}$$

$$\dot{\varepsilon}_{vis} = \frac{\sigma}{\eta}$$

same since they are in series.

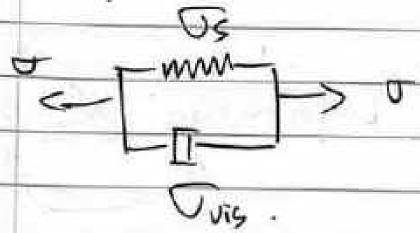


$$\dot{\varepsilon}_{vis} = \frac{\sigma_{vis}}{\eta}.$$

$$\boxed{\underbrace{\dot{\varepsilon}}_{\dot{\varepsilon}_s + \dot{\varepsilon}_{vis}} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\varepsilon}}$$

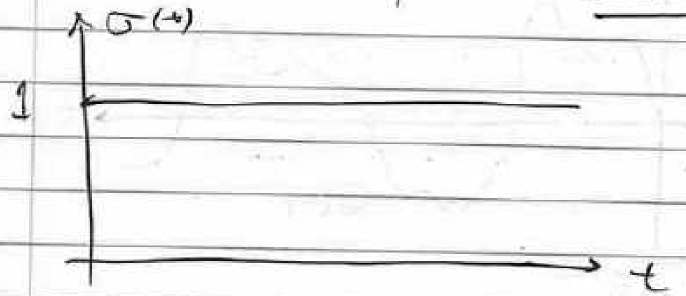
⇒ ODE in time

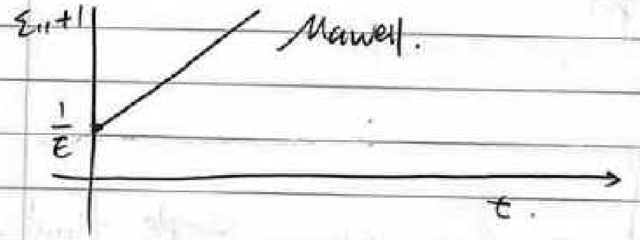
Voigt model.

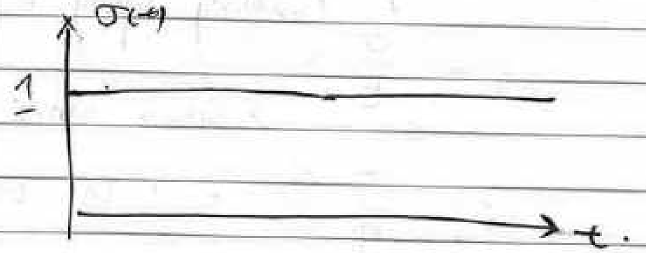


$\sigma = \sigma_s + \sigma_{vis}$.

- $\sigma_{\cancel{k}} = E\varepsilon + \eta\dot{\varepsilon}$ => Voigt model.

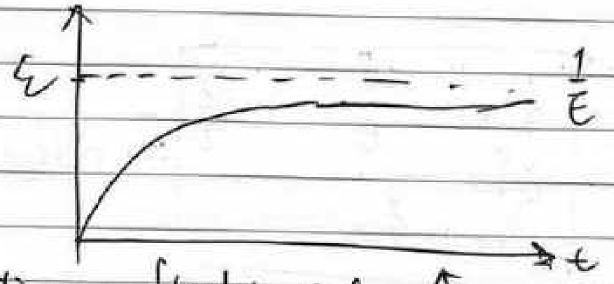






Solve a ODE to get this curve.

$t > 0$, => $\eta\dot{\varepsilon} + E\varepsilon = 1$

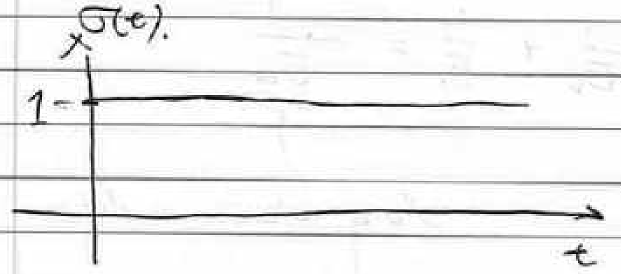


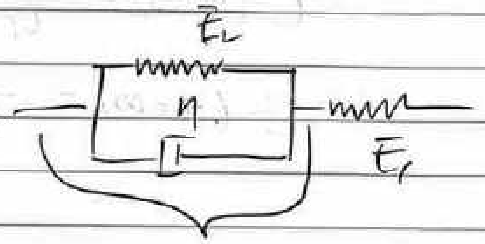
$$\varepsilon = A e^{-\frac{E}{\eta}t} + \frac{1}{E}$$

$$= \frac{1}{E}\left[1 - e^{-\frac{E}{\eta}t}\right].$$

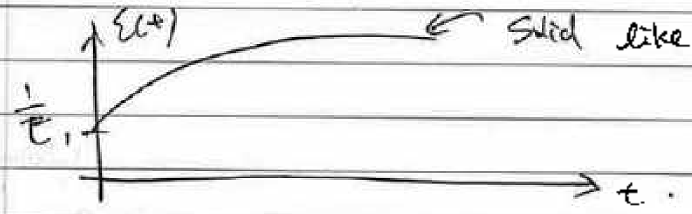
Short time: fluid; & long time: solid

- Standard model





Voigt element.



Solid like behavior

$$\frac{E_1+E_2}{E_1E_2} = \frac{1}{E_2} + \frac{1}{E_1}$$

Solid behavior for both long & short time

$$\underbrace{\dot{\sigma} + \frac{E_1+E_2}{2}\sigma = E_1\dot{\varepsilon} + \frac{E_2E_1}{\eta}\varepsilon}_{\text{3 parameters to determine}}.$$

Linear ODE

- Concept of Creep function.

Creep function $C(t)$ strain history due to a unit stress $\sigma = H(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0. \end{cases}$

$C(t) = \left(\frac{1}{E} + \frac{t}{2}\right) H(t)$. <=> Maxwell

$C(t) = \frac{1}{E}\left(1 - e^{-\frac{Et}{\eta}}\right) H(t)$. <=> Voigt.

$C(t) = \underbrace{\frac{E_1+E_2}{E_1E_2}}_{1/E_\infty} - \frac{1}{E_2} e^{-E_2t/\eta}$

$\frac{E_2}{\eta}$ units: $\frac{1}{[t]}$

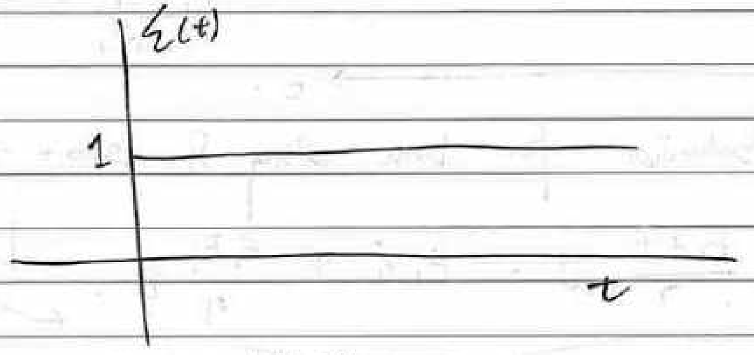
t_c <=> Creep relaxation time

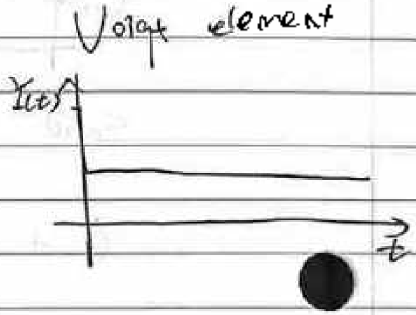
$C(t=0) = \frac{1}{E_1}$ ← short time modulus.

$C(t=\infty) = \frac{1}{E_2} + \frac{1}{E_1} = \frac{1}{E_\infty}$

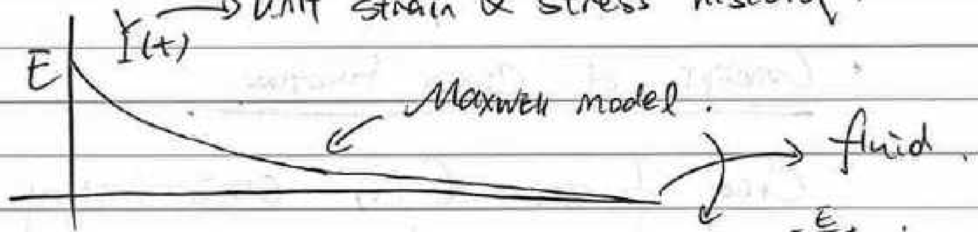
↑ long time modulus.

$\frac{E_\infty}{E_1} = 10^{-3}$.



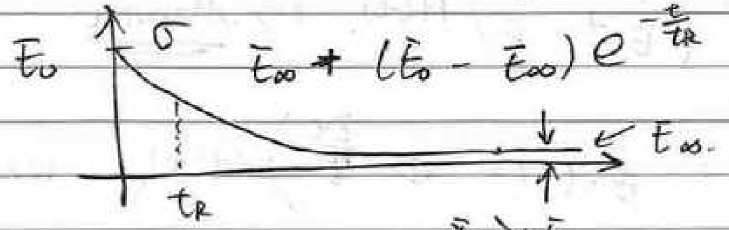

Voigt element

$\varepsilon(t) = H(t)$.



$Y(t) = E e^{-\frac{E}{\eta}t} H(t)$.

Für standard solid:



$E_\infty + (E_0 - E_\infty) e^{-\frac{t}{t_R}}$

$t_R = \cancel{\frac{E_1 + E_2}{\eta}}$, $E_0 = E_1$.

☆ $\hookrightarrow \frac{\eta}{E_1 + E_2}$ (typo)

Boltzmann superposition principle

① Assume system is linear.

② Assume Causuality.

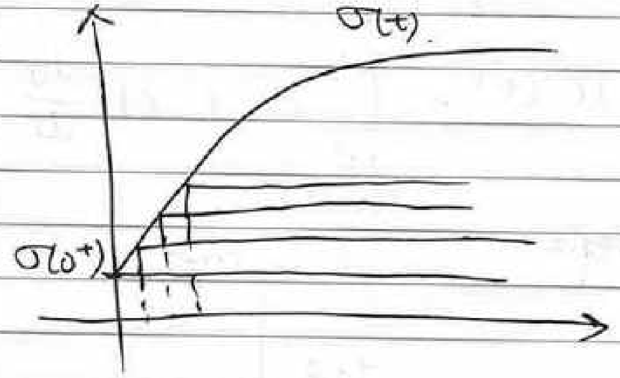
③ Non-Aging.

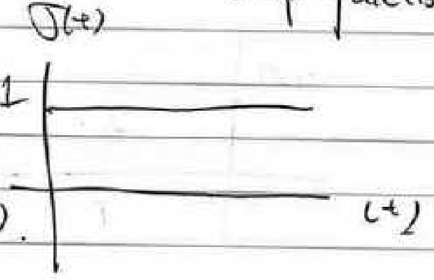
Creep.

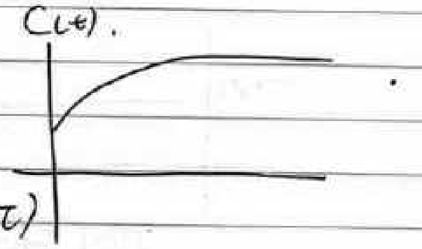
↓

Given C(t)

Response of system due to a unit stress step function.







$\sigma(t) = \sigma(0^+) H(t) + \sigma(\Delta t) H(t - \Delta\tau) + \sigma'(\Delta\tau)\Delta\tau H(t+\Delta\tau)$.

$\varepsilon(t) = \sigma(0^+) C(t) + \sigma'(0)\Delta\tau C(t - \Delta\tau) + \sigma'(\Delta\tau)\Delta\tau C(t - 2\Delta\tau) H(t-2\Delta\tau) + \ldots\ldots$

$$= \sigma(0^+) C(t) + \int_{0^+}^{t} \sigma'(\tau) C(t-\tau)\, d\tau.$$

Nov. 10, Wed, Week 12.

Review: Boltzmann superposition –

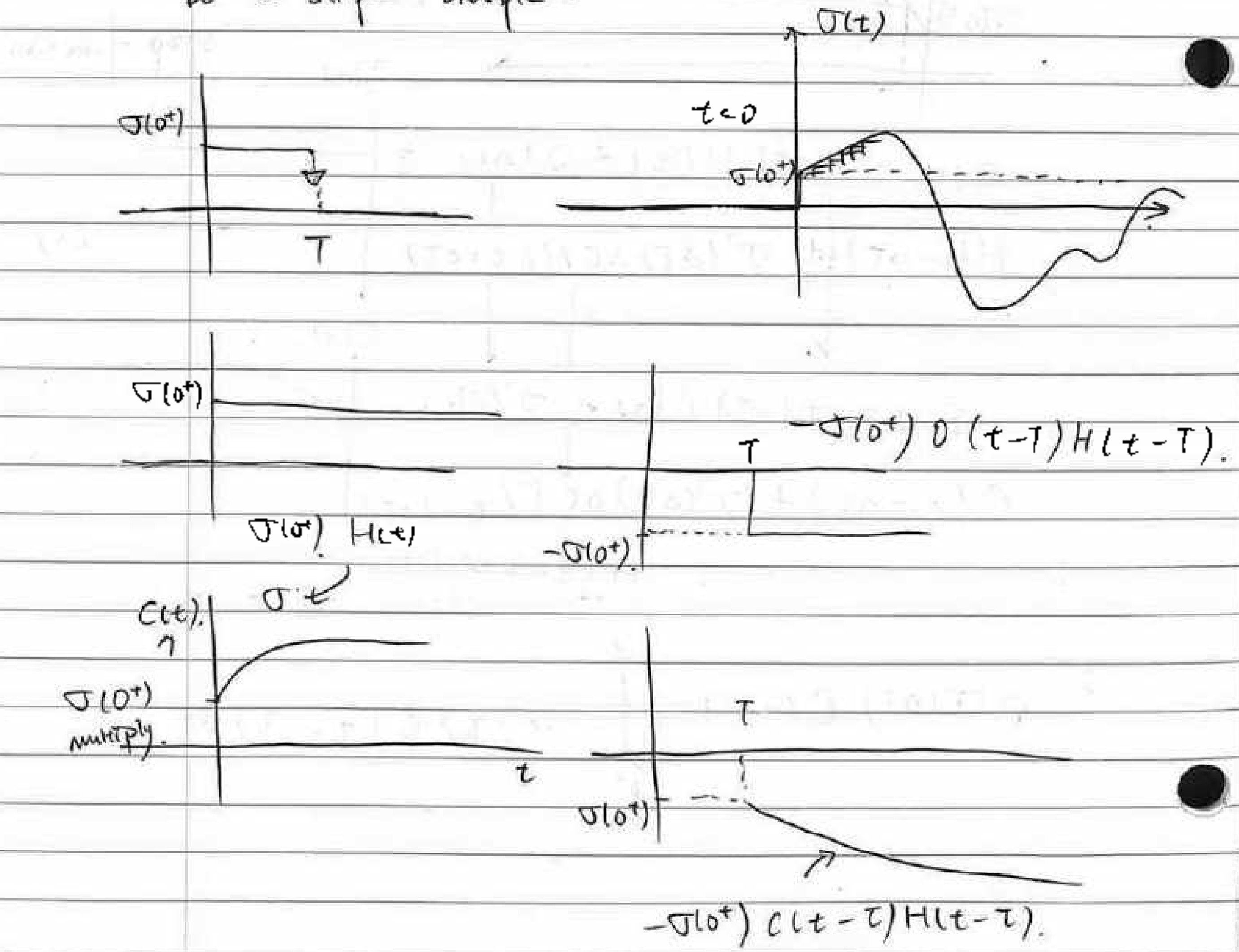
Any stress history can be broken down into sum of step functions.

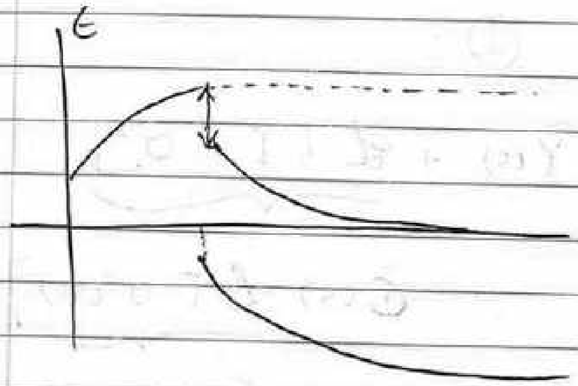
$\downarrow$ Each step ft. $H(t-\tau)$

$C(t-\tau)$.

Real Linear viscoelastic model for uniaxial tension $\Rightarrow$ $\varepsilon(t) = \sigma(0^+)C(t) + \int_{0^+}^{t} C(t-\tau)\frac{d\sigma}{d\tau}\,d\tau$

Do a simple example





[Convolution product

$f(t), g(t).$ define for $t \in (0, \infty)$.

$$\longrightarrow f*g = \int_0^t f(t-\tau)\,g(\tau)\,d\tau = \int_0^t g(t-\tau)\,f(\tau)\,d\tau.$$

$$① \quad \boxed{\varepsilon = \sigma(0^+)C(t) + C*\sigma'} \qquad \sigma' \equiv \frac{d\sigma}{dt}$$

The Laplace transform of a function f defined in zero to infinity.

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt.$$

$\Downarrow$

$\tilde{f}(s)$ $\rightarrow$ is a function of s.

$\hookrightarrow$ s in the transform variable.

$\hookrightarrow$ S is in general a complex variable

Properties.

$$\mathcal{L}[f'(t)] = s\hat{f}(s) - f(0^+).$$

$$\mathcal{L}[f*g] = \tilde{f}(s)\,\tilde{g}(s) \leftarrow \text{convolution Theorem.}$$

Laplace transform ① :

$$\tilde{\varepsilon}(s) = \sigma(0^+)\,\tilde{Y}(s) + \underbrace{\mathcal{L}[Y * \sigma']}_{\tilde{\mathbb{C}}(s)\,\underbrace{\mathcal{L}[\sigma'(t)]}_{[s\tilde{\sigma}(s) - \sigma(0^+)]}}$$

$$\boxed{\tilde{\varepsilon}(s) = s\,\tilde{\mathbb{C}}(s)\,\tilde{\sigma}(s)} \quad ②$$

clear & linear. →

transform domain.

$$\updownarrow$$

Messy $\Rightarrow \varepsilon(t) = \sigma(0^+)\,\mathbb{C}(t) + \int_{0^+}^{\tau} \mathbb{C}(t-\tau)\,\frac{d\sigma}{d\tau}\,d\tau.$

Do the same thing with Relaxation function $Y(t)$.

Apply a strain history. $\varepsilon(t)$ ← given.

$$\sigma(t) = \varepsilon(0^+)\,Y(t) + \int_{0^+}^{t} Y(t-\tau)\,\frac{d\varepsilon}{d\tau}\,d\tau.$$

$$\boxed{\tilde{\sigma}(s) = s\,\tilde{Y}(s)\,\tilde{\varepsilon}(s)} \quad ③$$

on the transform plane, — it's trivial.

Combine ② & ③

$$\hat{\sigma}(s) = s\,\hat{Y}(s).\,s\,\tilde{C}(s)\,\hat{\sigma}(s).$$

$$\rightarrow \boxed{s\,\tilde{Y}(s)\cdot\tilde{C}(s) = 1}$$ ← related.

$$\tilde{Y}(s) = \frac{1}{s^2\,\tilde{c}(s)}.$$

$$Y(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2\,\tilde{C}(s)}\right].$$

$$\hookrightarrow \tilde{Y}(s).\,\hat{C}(s) = \frac{1}{s^2}$$

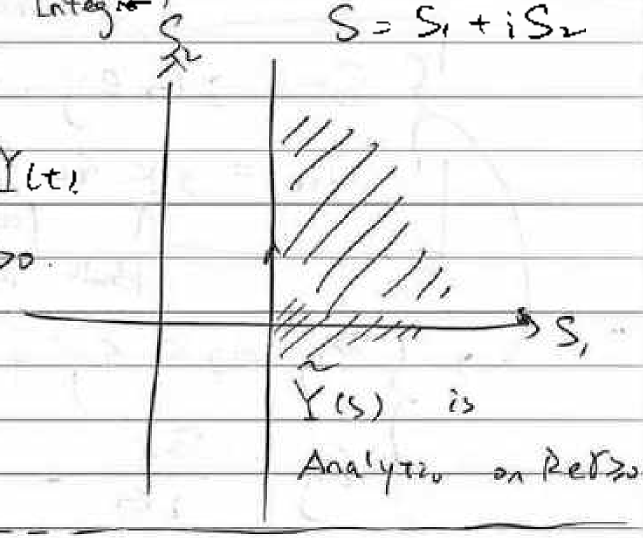
$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t.$$

$$\int_0^t Y(t-\tau)\,C(\tau).\,d\tau = t. \qquad t \geq 0.$$

(Bromwich Integral)

$$\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \tilde{Y}(s).\,e^{st}\,ds = Y(t)$$

$t > 0.$



Isotropic Linear viscoelastic solid.

$$\sigma_{ij} = 2G\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\,\delta_{ij}.$$

MEMORIZE THIS!!!
☆☆

OR.

$$\varepsilon_{ij} = \frac{\sigma_{ij}}{2G} \overset{-}{\cancel{+}} \frac{\nu}{E}\,\sigma_{kk}\,\delta_{ij} \qquad E = \frac{G}{2(1+\nu)}.$$

deviatoric stress tensor.

$$S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}.$$

Strain deviator tensor

volumetric part

in literature, $\frac{1}{3}\sigma_{kk} = \sigma$; $\varepsilon_{kk} = e$

change of volume.

e_{ij}

shear Modulus. (relaxation modulus).

$$\boxed{\begin{cases} S_{ij} = 2G\, e_{ij}. \\ \sigma_{kk} = 3K\, \varepsilon_{kk}. \end{cases}}$$

(relaxation) Bulk Modulus.

exactly the SAME

$$\begin{cases} \sigma_{ij} = 2G\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\delta_{ij} \\ \varepsilon_{ij} = \frac{\sigma_{ij}}{2G} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}. \end{cases}$$

$$S_{ij} = e_{ij}(0^+)\,Y_1(t) + \int_{0^+}^{t} Y_1(t-\tau).\frac{\partial e_{ij}}{\partial \tau}\cdot d\tau$$

$Y_1(t) \longleftrightarrow 2G.$ $\sigma_{kk} = \varepsilon_{kk}(0^+)\,Y_2(t) + \int_{0^+}^{t} Y_2(t-\tau)\frac{\partial \varepsilon_{kk}}{\partial \tau}d\tau$

$Y_2 \longleftrightarrow 3K$

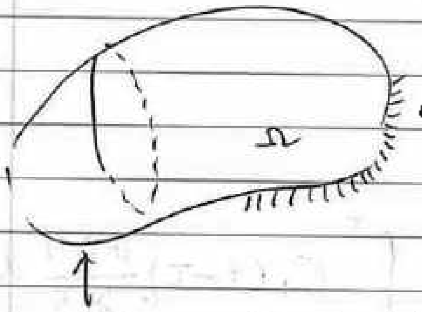
Consider $\mathcal{L}$

$$\boxed{\begin{aligned} \tilde{S}_{ij} &= s\,\tilde{Y}_1(s)\,\tilde{e}_{ij} \\ \tilde{\sigma}_{kk} &= s\,\tilde{Y}_s(s)\,\tilde{\varepsilon}_{kk} \end{aligned}}$$

Linear Elasticity $\begin{cases} \sigma_{ij,j} = 0. & \text{no body Force.} \\ \varepsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2}. & \text{kinematics.} \end{cases}$

Constitutive model. $\begin{cases} S_{ij} = e_{ij}(0^+).Y_1(t) + Y_1 * e'_{ij} \\ \sigma_{kk} = \varepsilon_{kk}(0^+)\,Y_2(t) + Y_2 * \varepsilon'_{kk} \end{cases}$

General Problem



$\partial\Omega_u$ given

$\underset{\sim}{u}(\underset{\sim}{x}, t) = \underset{\sim}{f}(\underset{\sim}{x}, t).$

$\partial\Omega_T$. $\sigma_{ij}n_j = T_i(\underline{x}, t).$ given

involve time.

Linear Elasticity ⇓ correspondence principle

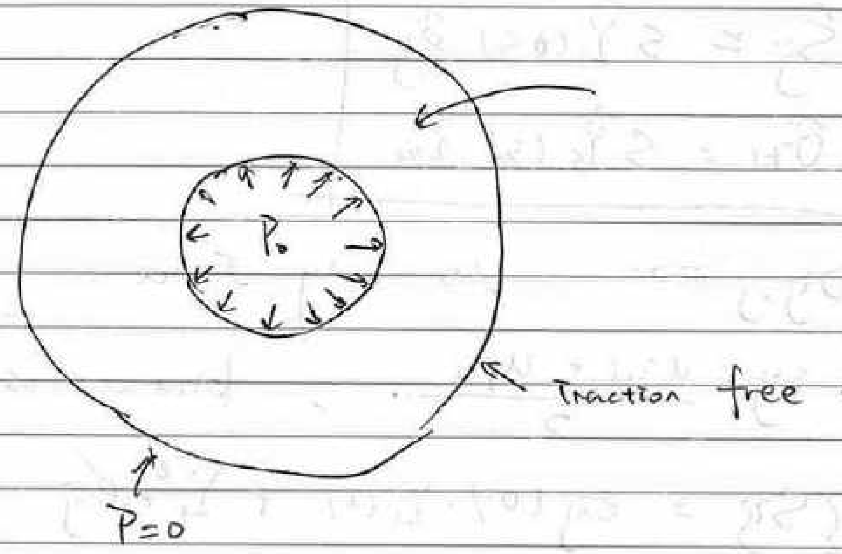
$$\begin{cases} \tilde{\sigma}_{ij,j} = 0. \\ \tilde{\varepsilon}_{ij} = \frac{\tilde{u}_{i,j} + \tilde{u}_{j,i}}{2} \\ \tilde{S}_{ij} = \underbrace{s\,\tilde{Y}_1(s)}_{2G}.\,\tilde{e}_{ij}(s) \end{cases}$$

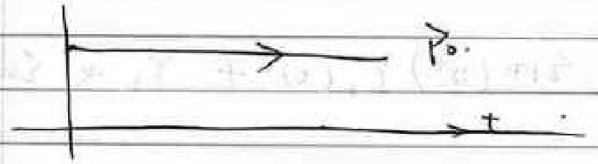
&

$$\tilde{\sigma}_{cc} = \overbrace{s\,\tilde{Y}_2(s)}^{3K}\,\tilde{\varepsilon}_{kk}(s)$$

$$\underline{\underline{BC}}\begin{cases} \tilde{\underline{u}}(\underline{x}, s) = \underset{\sim}{f}(\underline{x}, s). \\ \tilde{\sigma}_{ij}n_j = \tilde{T}_i(\underline{x}, s) \end{cases}$$

SIMPLE Example



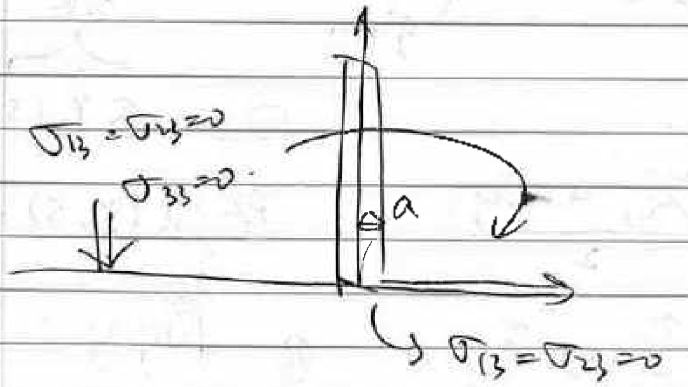


BCs. $\begin{cases} \sigma_{rr}(r=a,\ t>0) = P_0. \\ \sigma_{rr}(r=b,\ t\geq 0) = 0 \end{cases}$

$$sY(s).C_1(s)\,s = 1$$

$$\hat{e}_{ij}(s) = sC_1(s)\tilde{S}_{ij}$$

$$e_{ij} = C_1(t).\,\tilde{S}_{ij}(0^+) + \int_{0^+}^{t} C_1(t-\tau).\,\underset{\overset{\parallel}{0}}{\frac{\partial S_{ij}}{\partial \tau}}\,d\tau$$



$$u_i = \Delta\, g(\tfrac{r}{a}, 0).$$

$$u_3(x_1, x_2, x_3=0) = -\Delta$$

$$(x_1^2+x_2^2)<1.$$

$$\sigma_{ij} = \frac{G\Delta}{a}\, f(\tfrac{r}{a}, 0).$$

Office Hour.

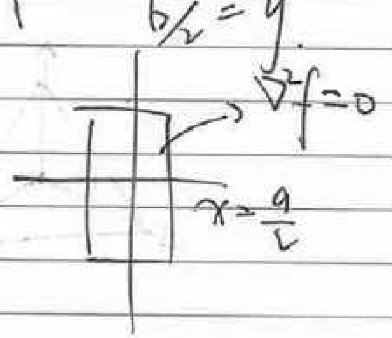
$$\psi \rightarrow \phi.$$

$$\nabla^2\phi = 0. \qquad \phi = \frac{x^2+y^2}{2}$$

formulate

$f \rightarrow$ BCs to simple.

$$f\Big|_{BCs} = 0.$$



$f \rightarrow$ harmonic

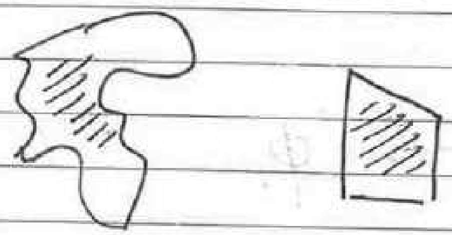
↳ saparation of variables

Differential EQs.

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \Rightarrow \frac{\partial^2\phi}{\partial x^2} = -\frac{\partial^2\phi}{\partial y^2}.$$

$$f = \frac{\partial^2\phi}{\partial x^2} + 1.$$

Example :

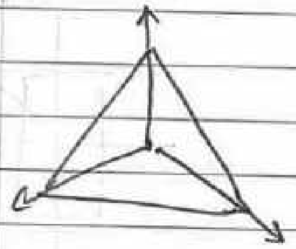


PRANDTL : Stress function

Three common formulations for solving torsion : w, ϕ, Φ_p.

Solution for Laplacian

Harmonic Analysis.



November 15, 2021. Week 13. Mon.

REVIEW : Correspondence principle.

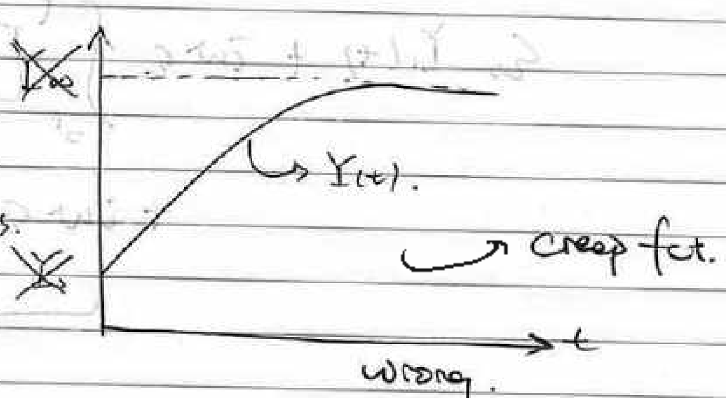
$$Y(t) = Y_\infty + (Y_0 - Y_\infty)\sum_{j=1}^{n} a_{ij}\, e^{-t/t_j}.$$

$Y_\infty = Y(t=\infty)$ Long time Modulus.

$Y_0 = Y(t=0)$. Instantaneous Modulus.

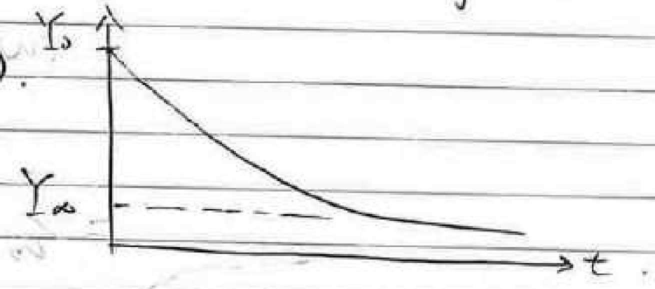
$$\sum_{j=1}^{n} a_j = 1$$

t_j = Relaxation times.

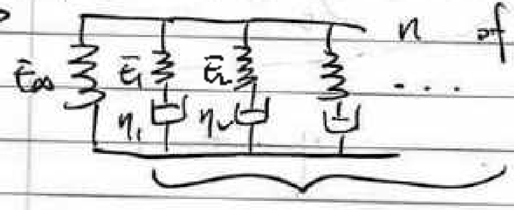


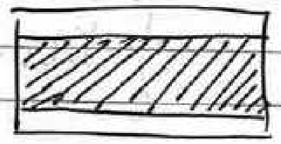
Power law model.

$$Y(t) = Y_\infty + (Y_0 - Y_\infty)\cdot \frac{1}{1+(t/t_0)^\alpha}$$



Corresponds to



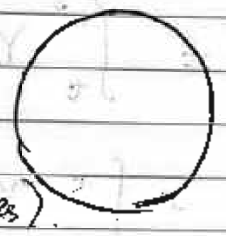


Rheology.

Shear strain : $\varepsilon = \varepsilon_0 e^{i\omega t}$.

* what is the response.

(long time / steady state res)



Shear stress $\sigma(t) = \underbrace{\varepsilon(0^+)}_{\varepsilon_0} Y_1(t) + \int_{0^+}^{t} (Y_1(t) - \tau)\frac{d\varepsilon}{d\tau} d\tau$

$\varepsilon_0 i\omega e^{i\omega\tau}$.

$$= \varepsilon_0 Y_1(t) + i\omega\varepsilon_0 \int_0^t Y_1(t-\tau) e^{i\omega\tau} d\tau.$$

$$\varepsilon_0 Y_1(t) + i\omega\varepsilon_0 \int_{0^+}^{t} [Y_1(t-\tau) - Y_1(t)] e^{i\omega\tau} d\tau$$

$$+ \underbrace{i\omega\varepsilon_0 Y_1(t) \int_{0^+}^{t} e^{i\omega\tau} d\tau}.$$

$$i\omega\varepsilon_0 Y_1(t) \cdot \frac{e^{i\omega\tau}}{i\omega} \Big|_{0^+}^{t}$$

$$= \varepsilon_0 Y_1(t) e^{i\omega t} - \varepsilon_0 Y_1(t)$$

$$= \varepsilon_0 Y_1(t) . e^{i\omega t} + i\omega\varepsilon_0 \int_0^t [Y_1(t-\tau) - Y_1(t)] e^{i\omega\tau} d\tau$$

$$\int_0^t Y_1(t-\tau) e^{i\omega\tau} d\tau. \qquad \eta = t - \tau$$

$$\tau = t - \eta.$$

$$= \int_t^0 Y_1(\eta) . e^{i\omega(t-\eta)} (-d\eta)$$

$$= \int_0^t Y_1(\eta) e^{i\omega(t-\eta)} d\eta = e^{i\omega t} \int_0^t Y_1(\eta) e^{-i\omega\eta} d\eta$$

$$\int_0^t Y_1(t) \cdot e^{i\omega\tau} d\tau.$$

$$= Y_1(t) . e^{i\omega t} \int_0^t e^{i\omega\eta} d\eta.$$

$$\sigma(t) = \varepsilon_0 Y_1(t) . e^{i\omega t} + i\omega e^{i\omega t} \int_0^t [Y_1(\eta) - Y_1(t)] e^{i\omega\eta} d\eta$$

$$= e^{i\omega t} \Big[\underbrace{\varepsilon_0 Y_1(t) + i\omega \int_0^t [Y_1(\eta) - Y_1(t)] e^{-i\omega\eta} d\eta}\Big]$$

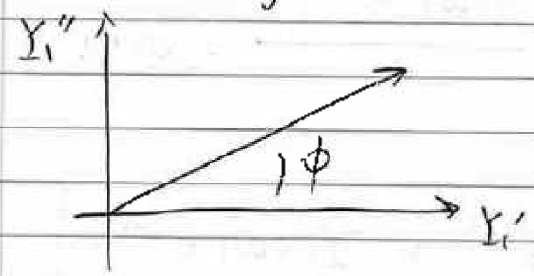
Converge to $\xrightarrow{t\to\infty}$ $\underbrace{\varepsilon_0 Y_1(\infty) + i\omega \int_0^\infty [Y_1(\eta) - Y_1(\infty)] e^{-i\omega\eta} d\eta}_{\hat{Y}_1(\omega)}$.

Complex Modulus

$$\sigma(t \to \infty) = \varepsilon_0 e^{i\omega t} \hat{Y}_1(\omega).$$

→ $\cos\omega t + i\sin\omega t$.

$$\hat{Y}_1 = \underbrace{\hat{Y}_1'(\omega)}_{\text{Storage modulus}} + i \underbrace{\hat{Y}_1''(\omega)}_{\text{lost modulus}}.$$



$$\tan\phi = \frac{\hat{Y}_1'(\omega)}{\hat{Y}_1''(\omega)}$$

→ loss tangent.

$$\hat{Y}_1'(\omega) = Y_1(\infty) + \omega \int_0^{\infty} \left[Y_1(\eta) - Y_1(\infty)\right] \sin(\omega\eta)\, d\eta.$$

$$\hat{Y}_1''(\omega) = \omega \int_0^{\infty} \left[Y_1(\eta) - Y_1(\infty)\right] \cos(\omega\eta)\, d\eta.$$

$\hat{Y}_1'(\omega) = \hat{Y}_1'(-\omega)$ → even fct. of ω

$\hat{Y}_1''(\omega) =$ odd fct. of ω.

Change of energy in a cycle.

$$W = \int_{\text{cycle}} \sigma \, d\varepsilon$$

we apply $\varepsilon = \varepsilon_0 \left[\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right]$

$$= \varepsilon_0 \cos(\omega t).$$

$$\varepsilon_0 \frac{e^{i\omega t}}{2} \rightarrow \varepsilon_0 \hat{Y}_1(\omega) e^{i\omega t} = \sigma(t).$$

$$\varepsilon_0 \frac{e^{-i\omega t}}{2} \rightarrow \underline{\frac{\varepsilon_0 \hat{Y}_1(-\omega)\, e^{-i\omega t}}{2}} = \sigma(t).$$

$$\hat{Y}_1(-\omega) = \hat{Y}_1(\omega)$$

$$\sigma = \frac{\varepsilon_0}{2}\left[\hat{Y}_1(\omega)\, e^{i\omega t} + \overline{\hat{Y}_1(\omega)\, e^{i\omega t}}\right]$$

$$= \varepsilon_0 \left[\hat{Y}_1'(\omega) \cos\omega t - \hat{Y}_1''(\omega) \sin\omega t\right].$$

$$\therefore\ e^{i\omega t} \hat{Y}_1(\omega) = \left[\hat{Y}_1'(\omega) + i\hat{Y}_1''(\omega)\right]\left[\cos\omega t + i\sin\omega t\right].$$

$$\mathrm{Re}\,(\quad) = \hat{Y}_1'(\omega) \cos\omega t - \hat{Y}_1''(\omega) \sin\omega t$$

$$-d\varepsilon = \omega \varepsilon_0 \sin\omega t \, dt$$

$$-\omega \varepsilon_0^2 \int_{\text{cycle}} \left[\hat{Y}_1'(\omega) \cos\omega t - \hat{Y}_1''(\omega) \sin\omega t\right] \sin\omega t \, dt$$

work = W in a cycle

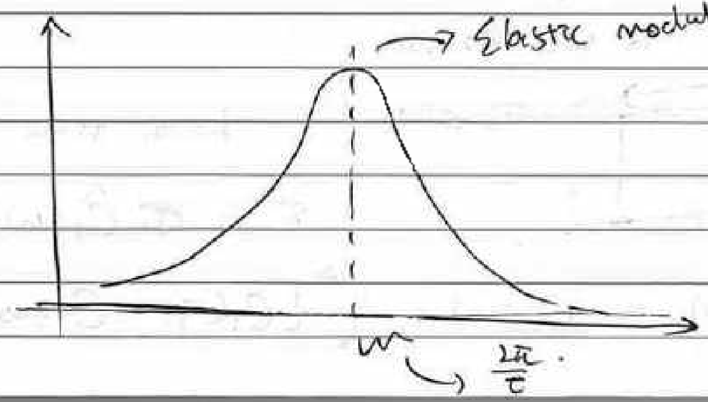
we already show: $\varepsilon_0 e^{i\omega t} \rightarrow \varepsilon_0 \hat{Y}_1(\omega) e^{i\omega t}$.

1st integral

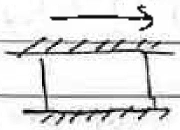
$$\hat{Y}_1'(\omega) \int_{\text{cycle}} \underbrace{\cos\omega t \, d\sin\omega t}_{\frac{d\cos^2\omega t}{2\omega} = 0}$$

$$\omega \varepsilon_0^2 \hat{Y}_1''(\omega) \underbrace{\int_{\text{cycle}} \sin^2(\omega t)\, d\,t\omega}_{\text{Some number}}$$

if you plot the lost modulus – → Elastic modulus @ this freq.



Nov. 17, 2021. Wed., Week 13.



Long time stress.

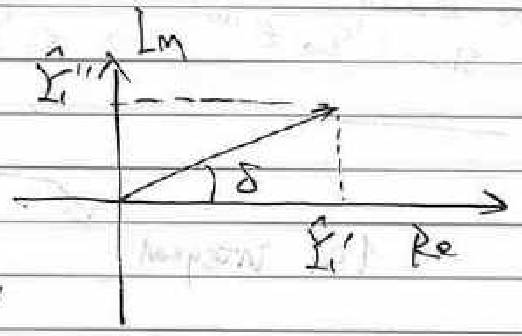
$$\sigma = \varepsilon_0 \underbrace{\hat{Y}_1(\omega)}_{\text{complex modulus}} e^{i\omega t}$$

$$\hat{Y}_1(\omega) = \underbrace{\hat{Y}_1'(\omega)}_{\text{Storage modulus}} + \underset{\sqrt{-1}}{i}\,\underbrace{\hat{Y}_1''(\omega)}_{\text{loss modulus}}.$$

loss modulus → odd function.

Storage modulus → Even function.

Energy loss per cycle $= \pi \varepsilon_0^2 \left| \hat{Y}_1''(\omega) \right|$



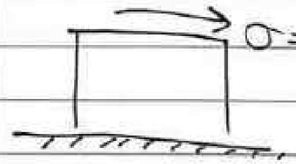
loss tangent.

$$\hat{Y}_1 = |\hat{Y}_1| e^{i\delta}$$

$$\underbrace{\tan\delta}_{\text{loss tangent.}} = \frac{\hat{Y}_1''}{\hat{Y}_1'}$$

$$\underbrace{\hat{Y}_1(\omega)}_{\text{complex modulus.}} = \underbrace{Y_1(\infty)}_{\text{relaxation function}} + i\omega \int_0^\infty [Y_1(\eta) - Y_1(\infty)] e^{-i\omega\eta} d\eta$$

in time domain

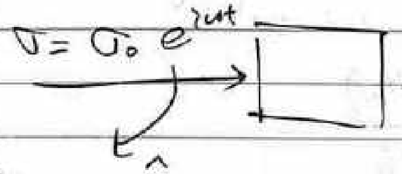


creep modulus.

long time shear strain.

$$\varepsilon = \sigma_0 \hat{C}_1(\omega) e^{i\omega t}.$$

$$\hat{C}_1(\omega) = C_1(\infty) + \int_0^\infty [C_1(\eta) - C_1(\infty)] e^{-i\eta\omega} d\eta$$



$$\longrightarrow \sigma_0 \hat{C}_1(\omega) e^{i\omega t}.$$

$$\hookrightarrow \varepsilon_0 \underbrace{\hat{Y}_1(\omega) \hat{C}_1(\omega)}_{1} e^{i\omega t}$$

$$\sigma_0 = \varepsilon_0 \hat{Y}_1(\omega)$$

$$\hat{Y}_1(\omega) = \frac{1}{\Sigma(\omega)}.$$

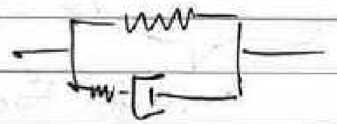
$$\int_0^t C_1(t-\tau) Y_1(\tau) d\tau = t.$$

$$s^2 \cancel{s} \tilde{C}_1 \tilde{Y}_1 = 1$$

↳ Laplace transform

$$\int_0^\infty e^{-st} C_1(t) dt = \tilde{C}_1(s)$$

Standard model

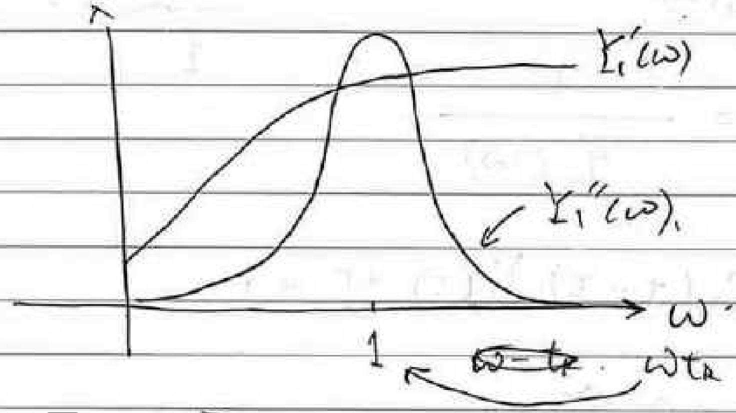


$$Y_1(t) = Y_\infty + (Y_0 - Y_\infty) e^{-t/t_R}$$

$Y_\infty \equiv Y_1(t \to \infty)$ → long time shear modulus

$Y_0 = Y_1(t \to 0)$ → Short time

$$\begin{cases} \hat{Y}_1(\omega) = Y_\infty + \dfrac{i\omega(Y_0 - Y_\infty)}{t_R^{-2} + \omega^2} \\ \hat{Y}_1(\omega) = Y_\infty + \dfrac{\omega^2 t_R^2 (Y_0 - Y_\infty)}{1 + \omega^2 t_R^2} \\ \hat{Y}_1''(\omega) = \dfrac{\omega t_R (Y_0 - Y_\infty)}{1 + \omega^2 t_R^2} \end{cases}$$



$[\omega] \sim \frac{1}{[t]}$

▷ Time Temp. Superposition

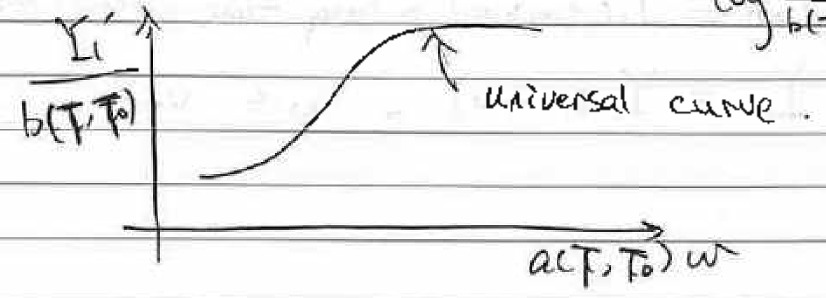
ω : 10^{-2} Radium/s + o 10^{2} rad/s

$$\begin{cases} Y_1'(\omega, T) = b(T, T_0)\,\phi_1(a(T,T_0)\,\omega) \\ Y_2'(\omega, T) = \underbrace{b(T, T_0)}\,\phi_2(\underbrace{a(T, T_0)}\,\omega) \end{cases}$$

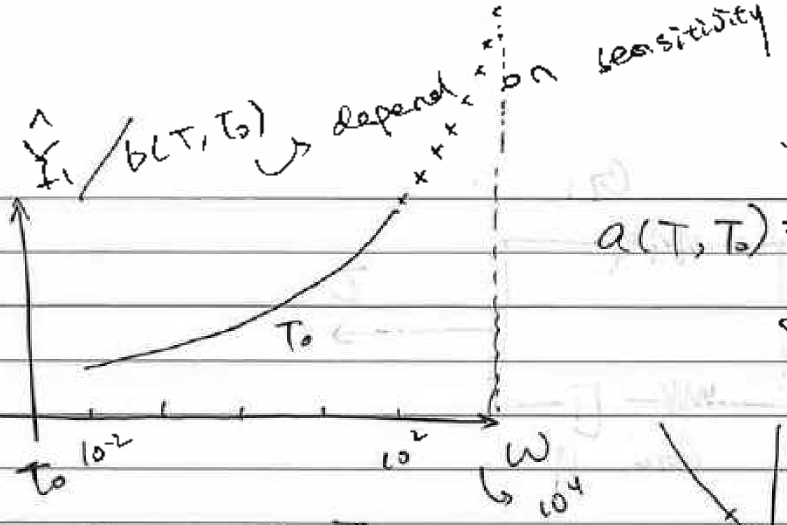
Ref. Temp.

Shift Factors.

$$\frac{Y_1'}{b(t, t_0)} = \phi_1(a(t, t_0)\,\omega)$$

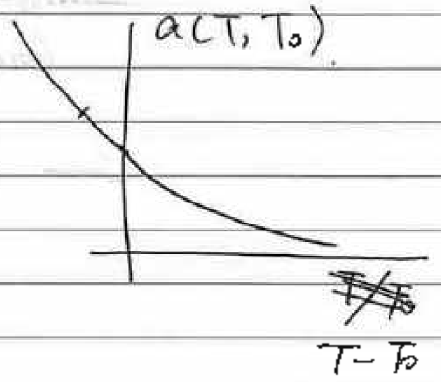


$$\log \frac{\hat{Y}_1}{b(T,T_0)} = \log \hat{Y} - \log b(T - T_0)$$



$$a(T,T_0) = \begin{cases} > 1, & T < T_0 \\ 1, & T = T_0 \\ < 1, & T > T_0 \end{cases}$$

$$\log[a(T,T_0)\,\omega] = \log a(T,T_0) + \log(\omega)$$



$$\hat{Y}_1(\omega) = Y(\infty) + i\omega \int_0^\infty [Y_1(\eta) - Y_1(\omega)]\, e^{i\omega\eta}\, d\eta$$

complex modulus.

↓ Gwein.

↑ time domain (relaxation fct.)

WLF - shift factor.

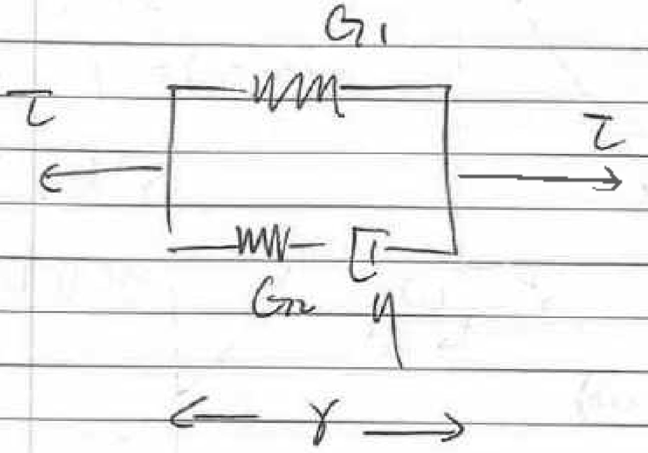
$$a(T,T_0) = \frac{\cancel{C_1(T)T_0} \quad -C_1(T_0)(T - T_0)}{C_2(T_0) + (T - T_0)}$$

Normally, T_0 → glass transition temp. of polymer

Y_1' & Y_2' are not independent to each other

if u know one, you know the other.

$$\frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{\hat{Y}_1'(r) - Y_1(\infty)}{r\omega}$$



↳ shear strain.
Engineering

usually, $\gamma = 2\varepsilon$.

$$2G\varepsilon = \tau.$$

$$\Downarrow$$

$$\gamma,$$

$$\tau = G\gamma,$$

$$e_{ij} = sY_1^*$$

$$s_{ij} = sY_1^* e_{ij}$$

$$\sigma_{kk} = sY_2^* \varepsilon_{kk}$$

$$E = 2G(1+\nu).$$

Linear Elasticity.

$G, \nu.$

Linear Viscoelasticity.

$G, \nu, E, K.$

↑ bulk (relaxation).

OH: superposition precipie.

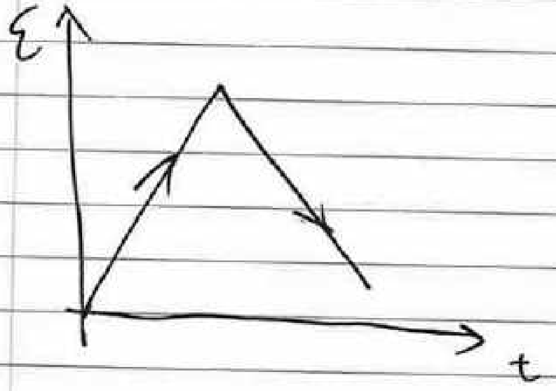
want to stress in tension test.

$$\sigma(t) = \overset{\varepsilon}{\cancel{\sigma}}(0^+) . \Upsilon(t) + \int_{0^+}^{t} \underbrace{\Upsilon(t-\tau)}_{\to 2a} \frac{d\varepsilon}{d\tau} d\tau$$

$$= \varepsilon(0^+) \Upsilon(t) + \Upsilon * \underbrace{\frac{d\varepsilon}{d\tau}}_{\text{strain history}}$$

$\Downarrow$

Stress history

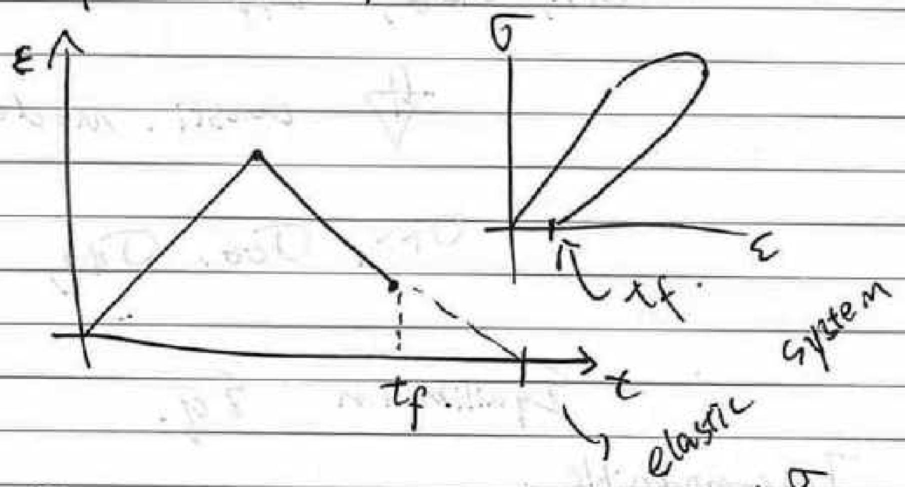


asked to evaluate the stress history.

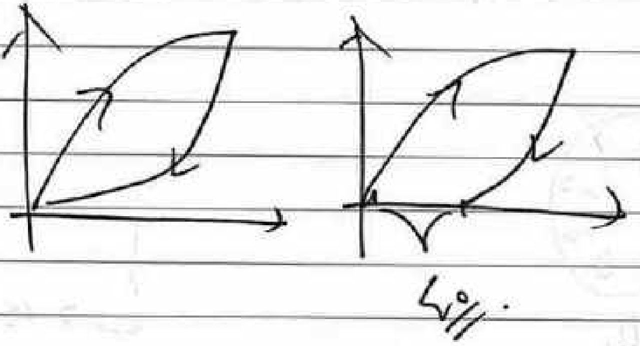
load fast → strain rate → high →

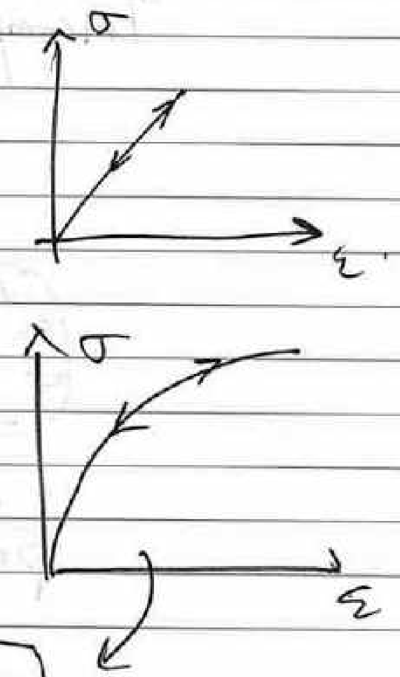
End: should not have stress in Spring 1 & Spring 2.

After 2b. you should be able to see.



Q:





You have:

creep function $\boxed{C(t). \quad \Upsilon(t).}$

find out equation of how $\Downarrow$.

Only one disp. field (r)

$\downarrow$

Strain ⟼ 3 ε_{rr} & ε

u_r

$u_r \longrightarrow \varepsilon_{rr}, \varepsilon_{\theta\theta}, = \varepsilon_{\phi\phi}$

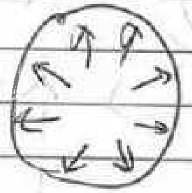
$\Downarrow$ consti. model

$\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{\phi\phi}, \quad \phi,$

Equilibrium Eq.

Incompressible.

$$\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\phi\phi} = 0$$



$\Downarrow$

pressure = const.

$\rightarrow u \xrightarrow{\infty} 0.$

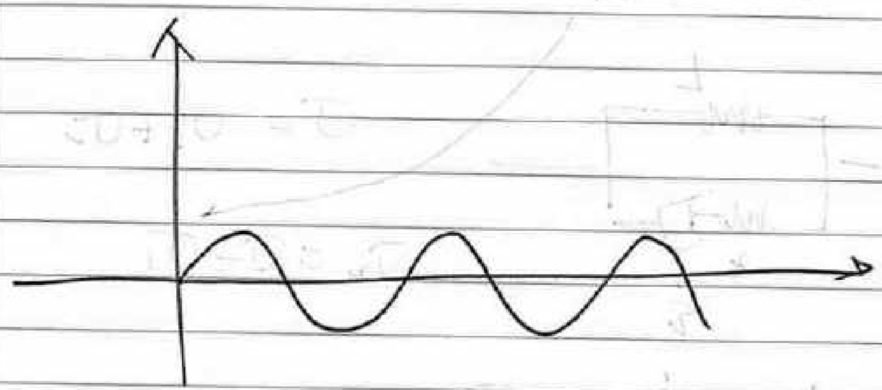
$r = r_0 \quad , \sigma_{rr} = -P.$

integrate incompressibility.

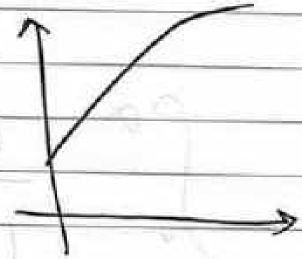
$$\sigma_{ij} = \cancel{+} 2G\varepsilon_{ij} + \overset{\text{hydrostatic.}}{P}\,\delta_{ij}$$

↓ incompressibility

$\varepsilon_{kk} = 0$

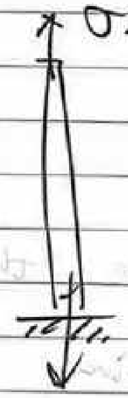


$\sigma_{rn} = P_0 \sin\omega\theta.$



$$\sigma = \frac{(\cos\omega t)e^t}{(1+t^3)}$$

only difference: ↓ strain

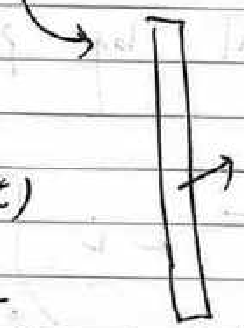


$\sigma = \sigma_0 \cos\omega t.$

$\hookrightarrow \varepsilon = \dfrac{\sigma_0 \cos\omega t}{E}$

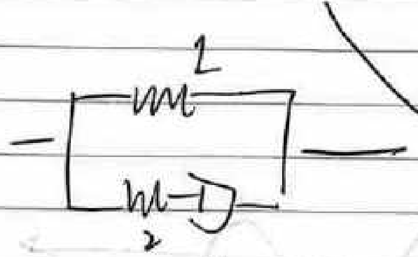
$\Downarrow$

elastic.



$$\varepsilon = \sigma(0^+) C(t) + \int_0^t C(t-\tau)\frac{\partial\sigma}{\partial\tau}d\tau$$

$\varepsilon = \sigma_1/E_1$



$\sigma = \sigma_1 + \sigma_2$

$\sigma_2 = \sigma - \sigma_1$

$\dot{\varepsilon}$

$$\dot{\varepsilon} = \frac{\dot{\sigma}_2}{E_2} + \frac{\sigma_2}{\eta}$$

$$\int_0^\infty \frac{e^{-i\omega\eta}}{1+\eta/t_R}\, d\eta \quad G_0 - G_\infty$$

$\frac{\eta}{t_R} = u$

Exponential Integral $= \underbrace{\int_0^\infty \frac{e^{-i\tilde{\omega}\eta}\, du}{1+u}}$

$d\eta = u\, t_R$

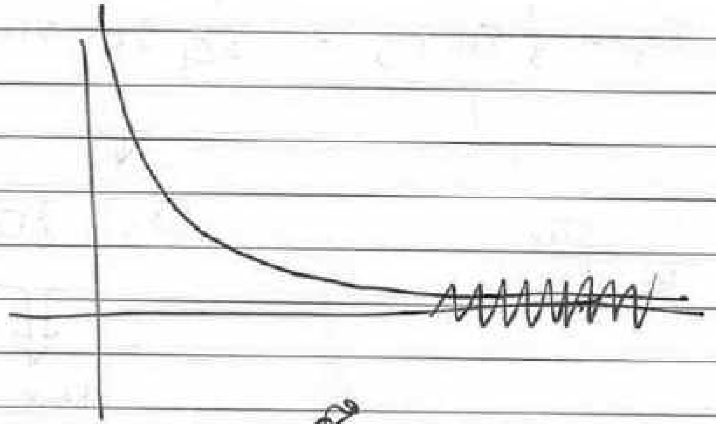
$\tilde{\omega} = \omega t_R$

basic idea ↦ integrate this term.

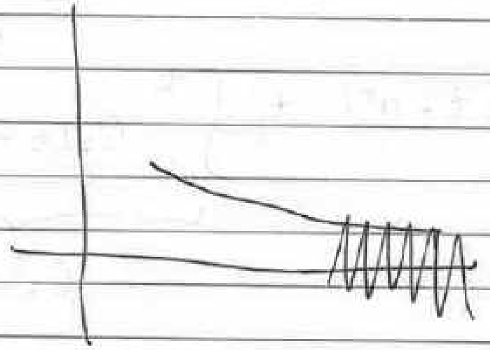
Compute integral by picking ~~Integral~~ infinite value

$$\int_0^\infty \frac{\cos \tilde{\omega} u\, du}{(1+u)} - i \int_0^\infty \frac{\sin \tilde{\omega} u\, du}{(1+u)}$$

Riemann - Lamé theory



$$\int_0^\infty \frac{\cos \omega x\, dx}{1+x}.$$



Nov, 22, Mon, 2021. Wk 14.

Linear visco - isotropic material.

$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij} = \frac{1}{2G} S_{ij}$ Elasticity.

$\varepsilon_{kk} = \frac{1}{K}\cdot\frac{\sigma_{kk}}{3}$ $\Downarrow$ bulk deformation.

$\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$ $\Downarrow$ shear deformation

Visco. $e_{ij} = C_1(t) S_{ij}(0^+) + \underbrace{\int_{0^+}^{t} C(t-\tau)\frac{\partial S_{ij}}{\partial \tau} d\tau}_{C_1^* S_{ij}'}$

~~S_{ij}~~

$\varepsilon_{kk} = C_2(t)\sigma_{kk}(t=0^+) + \underbrace{\int_{0^+}^{t} C_2(t-\tau)\frac{\partial \sigma_{kk}}{\partial \tau} d\tau}_{C_2^* \sigma_{kk}'}$

$\downarrow$ transform variable.

$\tilde{e}_{ij} = s\tilde{C}_1(s)\tilde{S}_{ij}$

$\tilde{\varepsilon}_{kk} = s\tilde{C}_2(s).\tilde{\sigma}_{kk}$

$\frac{1}{2G} \longleftrightarrow s\tilde{C}_1(s)$

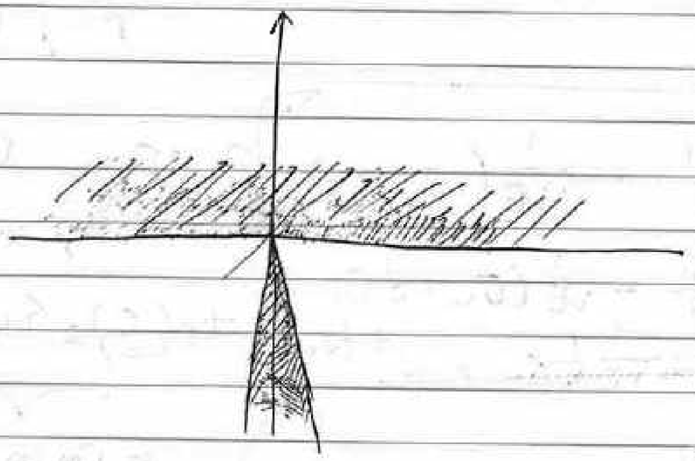
$\frac{1}{3k} \longleftrightarrow s\tilde{C}_2(s)$.

$$\frac{s3\tilde{C}_2(\tilde{C}_1/2)}{\tilde{C}_2 + (\tilde{C}_1/2)}$$

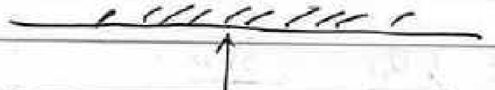
$\uparrow$

$E = \frac{9kG}{3k+G}$.

$$= \frac{s3\tilde{C}_1\tilde{C}_2}{2\tilde{C}_2 + \tilde{C}_1} \equiv s\tilde{E}(s)$$



plane strain problem.

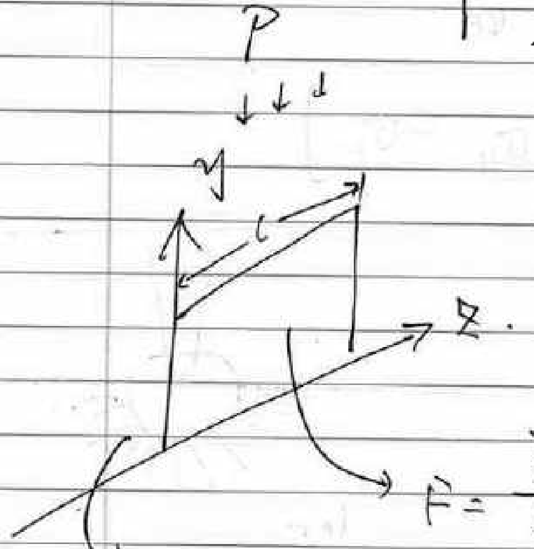


$t_{22} = \frac{F}{A} \to 0$ > fundamental sol. for Elas.

line force F.

(force per unit length).

in direction of blade.



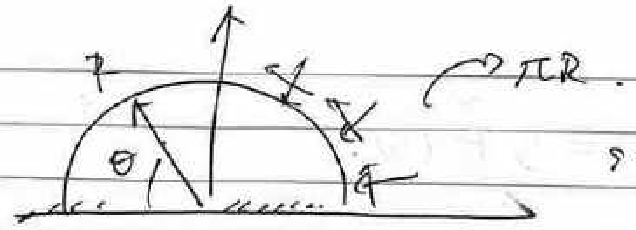
$F = \frac{P}{L}$.

assume infinitely long.

F, G, r, θ (go away)

$\sigma = \frac{F}{r} f(\cancel{\frac{F}{Gr}}, \theta)$

force per unit length.



plane strain

solution has to be:

$$\underline{\sigma} = \frac{F}{r}\underline{f}(\theta) = 0$$

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \frac{1}{2E}(\sigma_{22}+\sigma_{33}) \qquad \underbrace{\underline{\varepsilon} \cdot \varepsilon_{12},\ \varepsilon_{22},\ \varepsilon_{12}}$$

$$\frac{\sigma_{11}}{E} - \frac{1}{2E}(\sigma_{22}+\frac{1}{2}\sigma_{11}$$

$$~~= \frac{\sigma_{22}}{E} - \frac{1}{2E}(\sigma_{21}+\sigma_{22}~~ + \frac{1}{2}\sigma_{22}) \qquad \underbrace{tr(\underline{\varepsilon}) = \varepsilon_{11}+\varepsilon_{22} = 0}$$

⇐ incompressibility

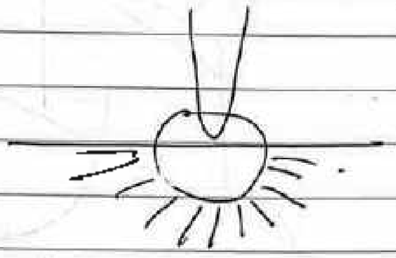
$$\Downarrow$$

$$= \sigma_{11}\left[1-\frac{1}{4}\right] - \frac{1}{2E}\cdot\left[\frac{3\sigma_{22}}{2}\right].$$

$$\frac{3}{4E} = \frac{1}{4G} = \frac{3\sigma_{11}}{4E} - \frac{3\sigma_{22}}{4E}.$$

$$= \frac{1}{4G}\left[\sigma_{11} - \sigma_{22}\right].$$

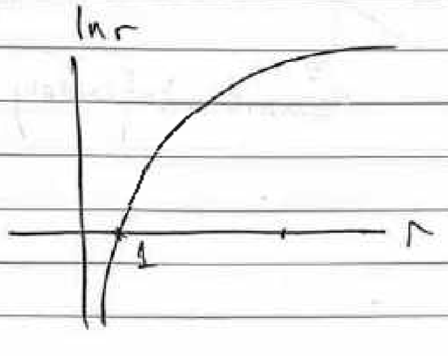
$$\underline{\sigma} = \frac{F}{r}\underline{f}(\theta).$$

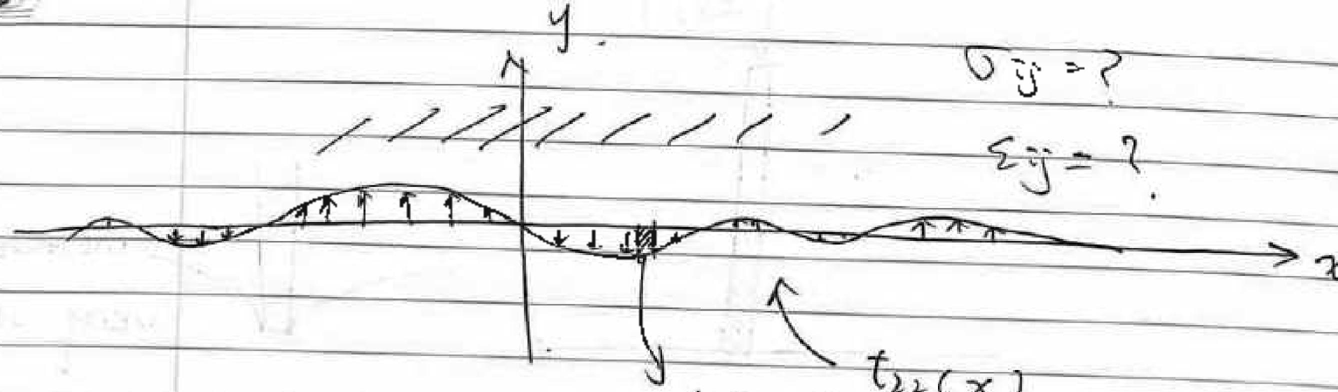


$$\boxed{\underline{\varepsilon} = \frac{F}{Gr}\underline{g}(\theta)}$$

$$\Downarrow$$

displacement: $\underline{u}$.

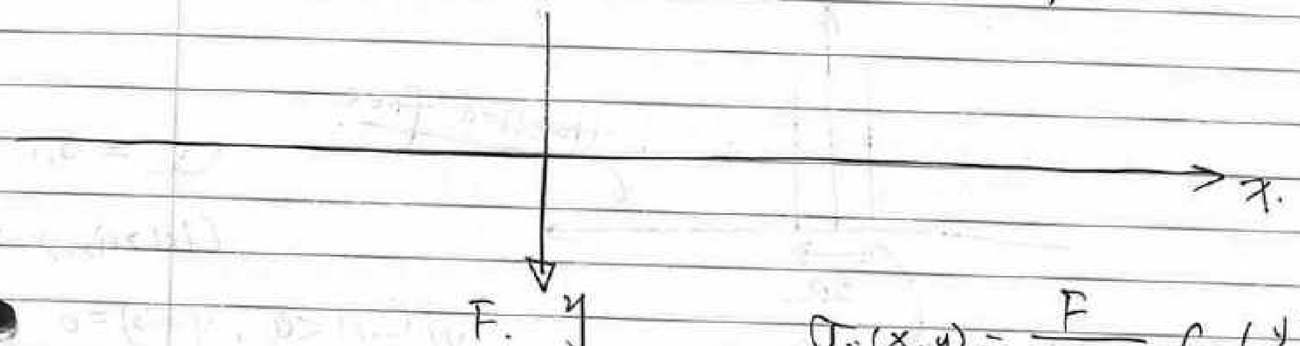




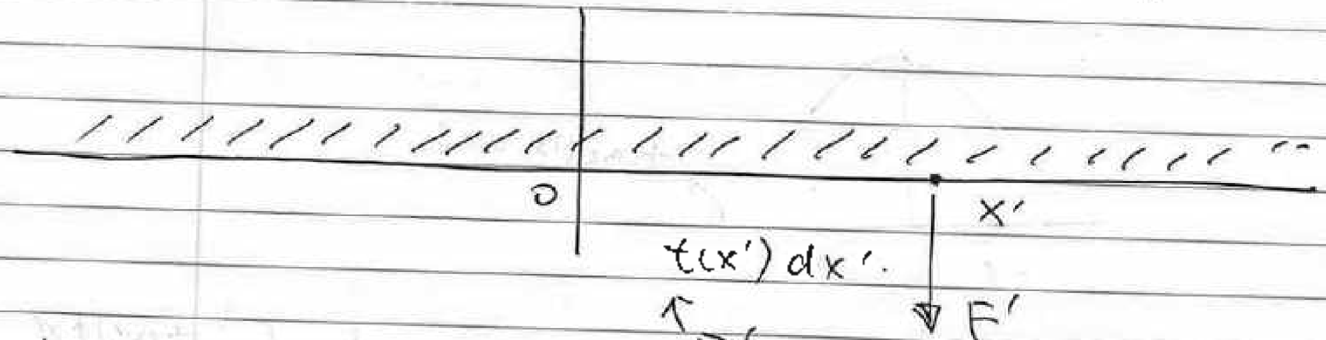
$\sigma_{ij} = ?$

$\varepsilon_{ij} = ?$

$$dF(x') = t_{22}(x')\,dx'.$$



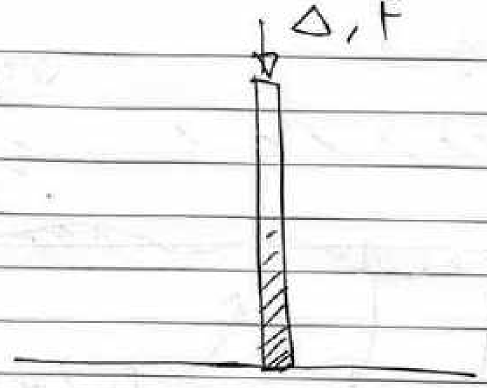
$$\sigma_{ij}(x,y) = \frac{F}{\sqrt{x^2+y^2}} f_{ij}\left(\frac{y}{x}\right).$$

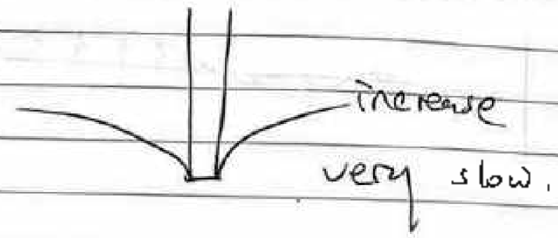


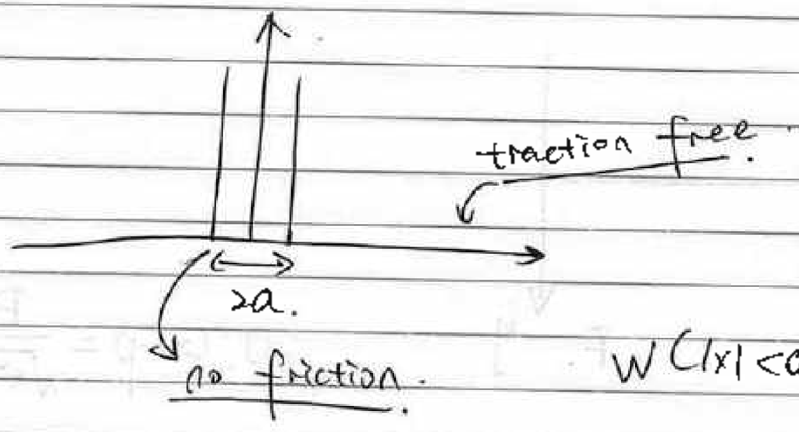
$$d\sigma_{ij}(x,y,x') = \frac{F'}{\sqrt{(x-x')^2+y^2}} f_{ij}\left(\frac{y}{x-x'}\right)$$

$$\sigma_{ij}(x,y) = \int_{-\infty}^{\infty} \frac{t(x')\,f_{ij}\left(\frac{y}{x-x'}\right)dx'}{\sqrt{(x-x')^2+y^2}}.$$

☆☆☆ → general solution. → you need to find



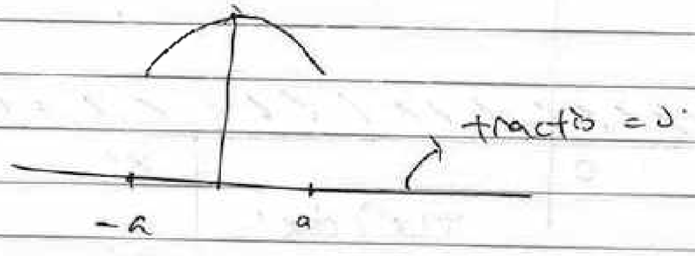


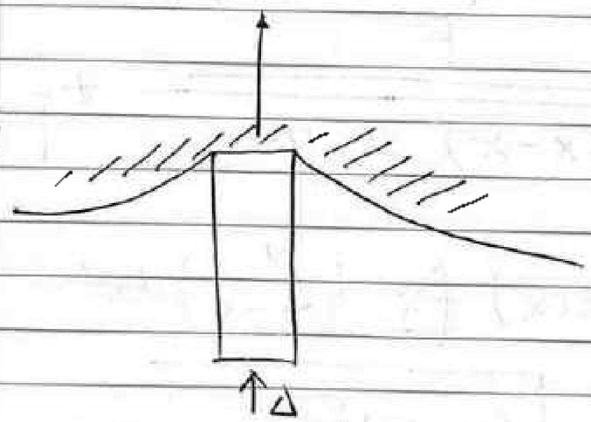


$\sigma_{12} = \sigma_{22}$

$(|x|>a, y=0)=0$

$W(|x|<a, y=0)=0$





$$\frac{1}{2\pi G}\int_{-a}^{a}\frac{t(x')dx'}{x-x'}=0$$

principal value integrate.

$$V_{,x} = -\frac{1}{2\pi G}\int_{-a}^{a}\frac{t(x')\,dx'}{x-x'} = 0.$$

gradient of disp.

Solution : $t(x) = \dfrac{A}{\sqrt{a^2-x^2}} = \dfrac{F}{\pi\sqrt{a^2-x^2}}$.

HW 11. Q3.

First, ~~substitute~~ given the BCs :

$\sigma_{rr}(r=b) = -p$

Compatibility Eq.

$$\frac{\partial^2\varepsilon_{xx}}{\partial y^2} + \frac{\partial^2\varepsilon_{yy}}{\partial x^2} = \frac{\partial^2\varepsilon_{xy}}{\partial x\partial y} \quad \cdots (1)$$

Constitutive Eq.

$$\sigma_{ij} = 2G\varepsilon_{ij} + P\delta_{ij} \quad \cdots (2)$$

Equilibrium Eq.

$$\sigma_{ij,j} = 0 \quad \cdots (3)$$

Substitute the BCs into Eqs. (1), (2), (3), we can hence compute the stress (radial).

▷ Taking BG's advice, we take ~

from Eq. (2). $\varepsilon_{ij} = (\sigma_{ij} - P\delta_{ij})/2G$.

From compatibility Eq. (1) :

$$\sigma_{rr,\theta\theta} - P\delta_{rr,\theta\theta} + \sigma_{\theta\theta,rr} - P\delta_{\theta\theta,rr} = \sigma_{r\theta,r\theta} - P\delta_{r\theta,r\theta}$$

▷ Equilibrium:

$$\sigma_{r\theta,\theta} + \sigma_{\theta\theta,r} + \sigma_{r\theta,r} + \sigma_{\theta\theta,\theta} = 0 \dots (4)$$

▷ Modified compatibility

$$\sigma_{rr,\theta\theta} + \sigma_{\theta\theta,rr} - \sigma_{r\theta,r\theta} = P \dots (5).$$

↳ ~~Now, Substitute BCs~~ ▷ Nah

→ Equilibrium: $\int(\sigma_{\theta r} + \sigma_{rr})d\theta - \int(\sigma_{\theta\theta} + \sigma_{r\theta})d\theta$

Laplace trans.

$$\begin{cases} \hat{\sigma}_{ij,i} = 0 \\ \tilde{\varepsilon}_{ij} = (\tilde{u}_{i,j} + \tilde{u}_{j,i})/2. \\ \tilde{\varepsilon}_{kk} = s\hat{C}_2\tilde{\sigma}_{kk}, \quad \tilde{e}_{ij} = s\hat{C}_1\tilde{S}_{ij} \end{cases}$$

$$\tilde{\sigma}_{rr}(r=b) = -\frac{P}{s}.$$

$$\frac{d\sigma_{rr}}{dr} + \frac{2}{r}(\sigma_{rr} - \sigma) = 0.$$

same in Laplace space:

$$\frac{d\tilde{\sigma}_{rr}}{dr} + \frac{2}{r}(\tilde{\sigma}_{rr} - \tilde{\sigma}) = 0.$$

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}.$$

$$\sigma_{ij} = 2G\varepsilon_{ij} + P\delta_{ij}$$

$$\frac{\partial^2 \varepsilon_{ii}}{\partial x_j^2} + \frac{\partial^2 \varepsilon_{jj}}{\partial x_i^2} = 2\frac{\partial^2 \varepsilon_{ij}}{\partial x_i \partial x_j}.$$

$$\begin{cases} \sigma_{ii} = 2G\varepsilon_{ii} + P\delta \\ \sigma_{jj} = 2G\varepsilon_{jj} + P\delta. \\ \sigma_{ij} = 2G\varepsilon_{ij} + P\delta \end{cases} \quad \delta$$

$$\Leftarrow \begin{cases} \varepsilon_{ii} = \frac{\sigma_{ii} - P\delta_{ii}}{2G} \\ \varepsilon_{jj} = \frac{\sigma_{jj} - P\delta_{jj}}{2G} \\ \varepsilon_{ij} = \frac{\sigma_{ij} - P\delta_{ij}}{2G}. \end{cases}$$

$$\rightarrow \frac{\partial^2 \sigma_{ii}}{\partial x_j^2} + \frac{\partial^2 \sigma_{jj}}{\partial x_i^2} = \frac{\partial^2 \sigma_{ij}}{\partial x_i \partial x_j}.$$

sperical:

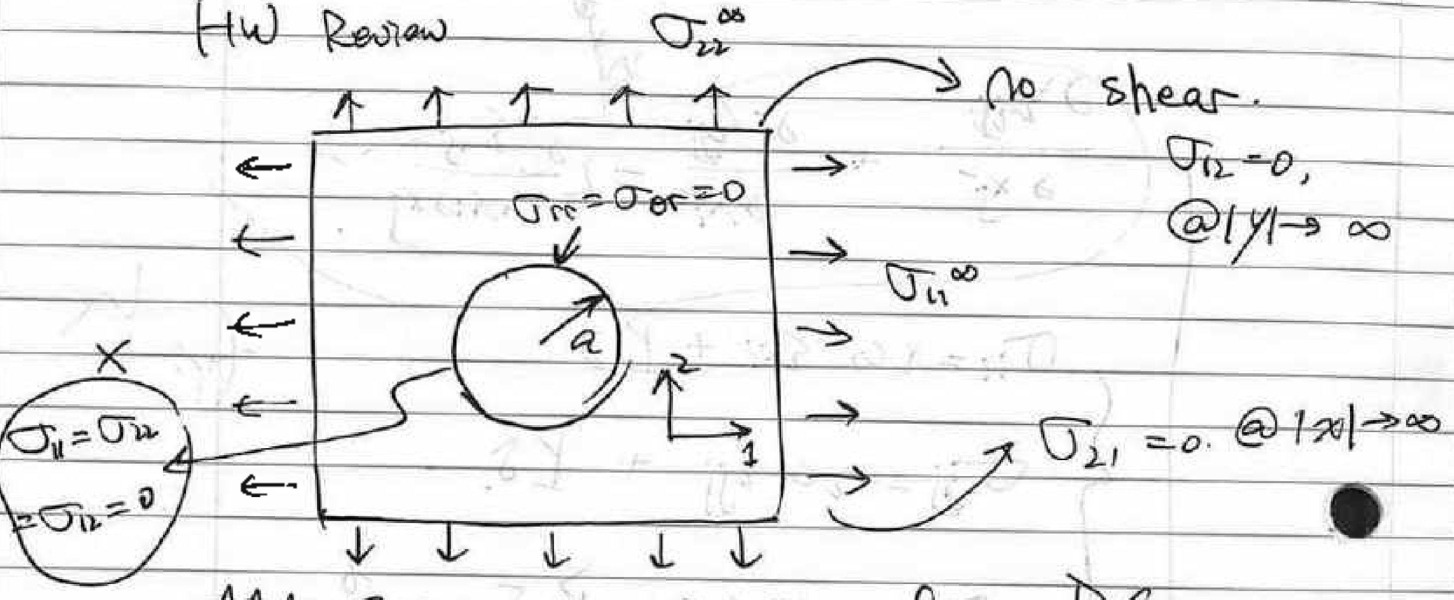
$$\frac{\partial^2 \tilde{\sigma}_{rr}}{\partial \tilde{\theta}^2} + \frac{\partial^2 \tilde{\sigma}_{\theta\theta}}{\partial \tilde{r}^2} = \frac{\partial \tilde{\sigma}_{r\theta}}{\partial \tilde{r}\partial\tilde{\theta}}$$

in Laplace domain: $\dfrac{\partial^2 \tilde{\sigma}_{ii}}{\partial \tilde{x}_j^2} + \dfrac{\partial^2 \tilde{\sigma}_{jj}}{\partial \tilde{x}_i^2} = \dfrac{\partial^2 \tilde{\sigma}_{ij}}{\partial \tilde{x}_i \partial \tilde{x}_j}$

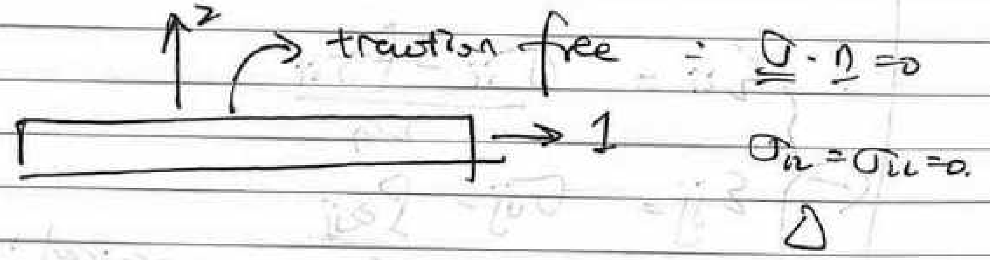
Nov. 29, Mon. Wk 15.

☆ Exam: Dec. 11. 9am - 9pm.

HW Review



☆☆☆ Setup the traction free BCs



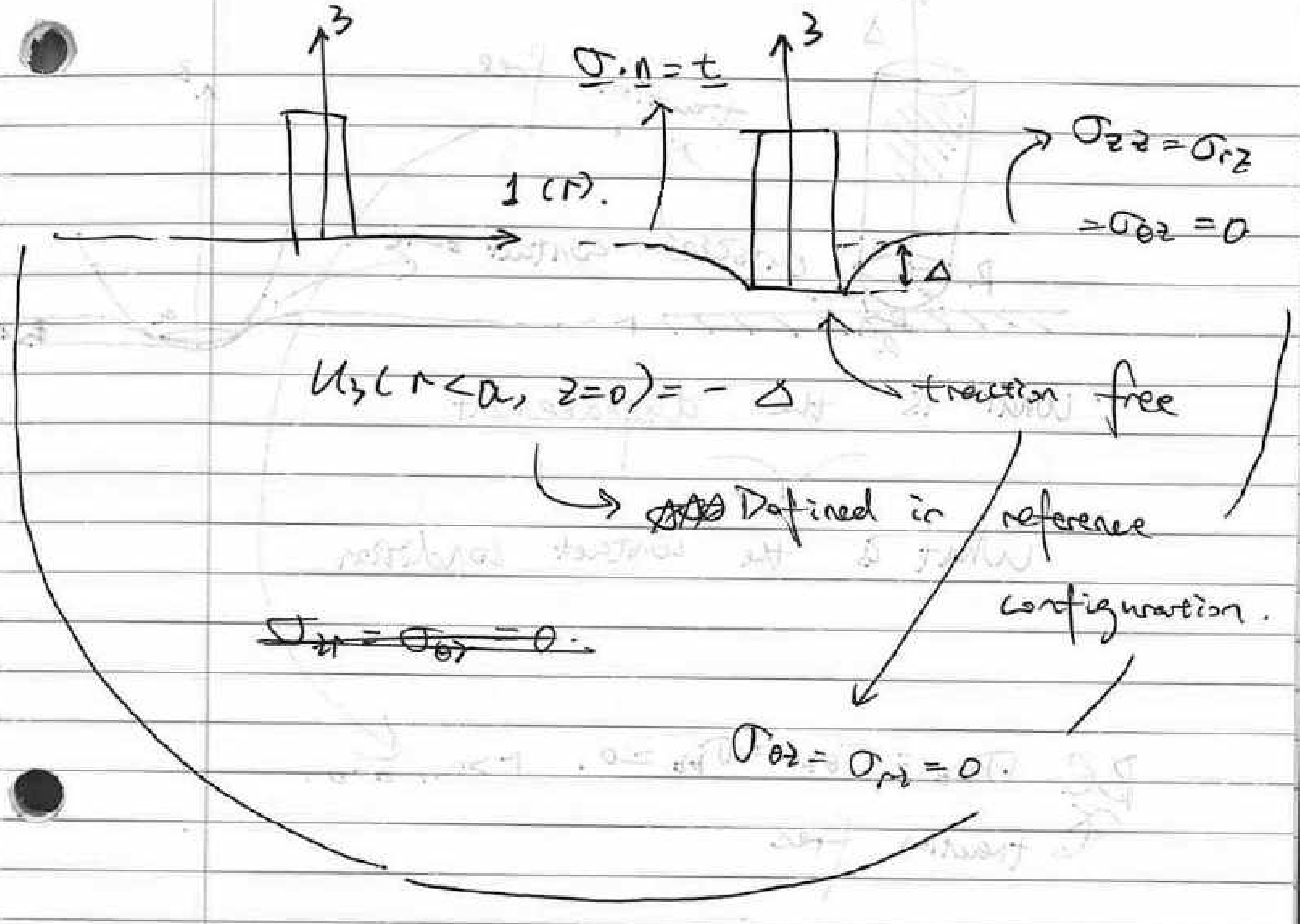
traction free : $\underline{\sigma}\cdot\underline{n}=0$

$\sigma_{12}=\sigma_{22}=0.$

remember how the setup the traction free BCs

$\underline{n}=\cos\theta\,\underline{e}_1+\sin\theta\,\underline{e}_2.$

☆☆☆ $\underline{n}$ should always normal to the surface BECAUSE it identifies the surface



$\underline{\sigma}\cdot\underline{n}=\underline{t}$

$\sigma_{zz}=\sigma_{rz}$
$=\sigma_{\theta z}=0$

$u_3(r<a,\ z=0)=-\Delta$

☆☆☆ Defined in reference configuration.

~~$\sigma_{zr}=\sigma_{\theta r}=0.$~~

$\sigma_{\theta z}=\sigma_{rz}=0.$

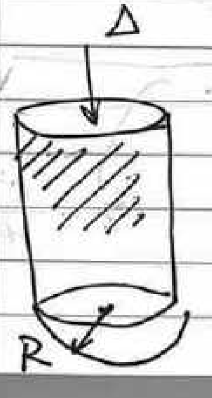
$\rho=\sqrt{r^2+z^2}$

$\underline{\sigma}(\rho\to\infty)\to 0.$

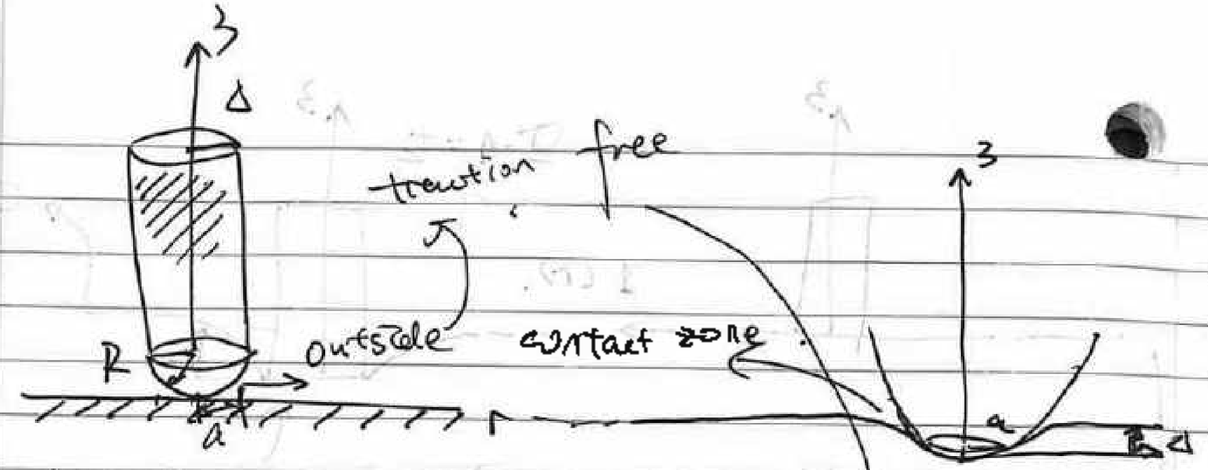
frictionless BCs: Cannot take any load.

⇓

☆☆☆ there are no shear



→ Next page



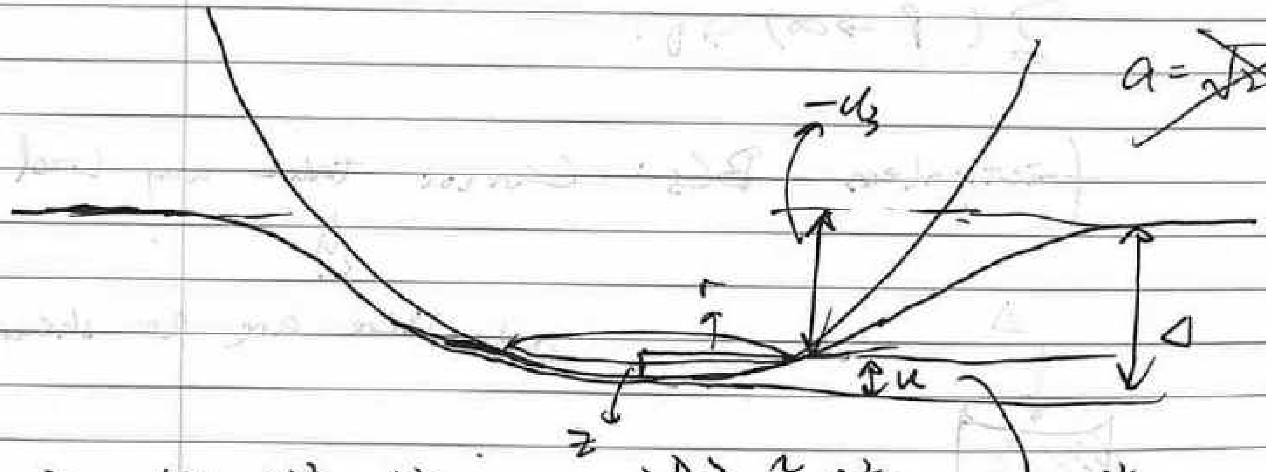
What is the displacement

What is the contact condition

BC $\sigma_{rz} = \sigma_{\theta z} = \sigma_{zz} = 0$, $r > a$, $z = 0$.

traction free

inside the contact region: $\sigma_{rz} = \sigma_{\theta z} = 0$,
$r < 0$, $z = 0$



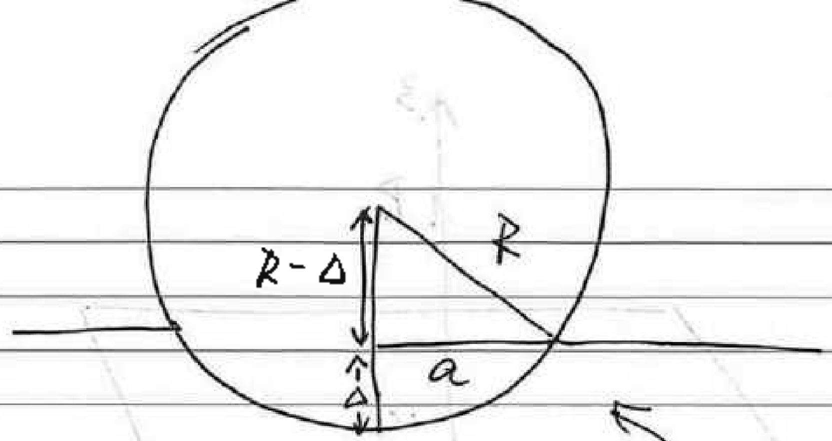
$a = \cancel{\sqrt{2R\Delta}}$

$r^2 + (z-R)^2 = R^2$ ↳

$2Rz \simeq r^2$.

$z \simeq \frac{r^2}{2R}$

$z - \frac{r^2}{2R} =$
$-u_3 \, (r < a, z = 0)$



$(R-\Delta)^2 + a^2 = R^2$.

$R^2 - 2R\Delta + \Delta^2 + a^2 = 0$.

$a^2 \approx 2R\Delta$

$\Delta = \frac{a^2}{2R}$.

wrong

⇓

like a fluid.

Actual Hertz soln. $\Delta = \frac{a^2}{R}$.

$\Delta = \frac{P}{G \pi a} \cdot \alpha$ ← numerical const.

← measure the load F

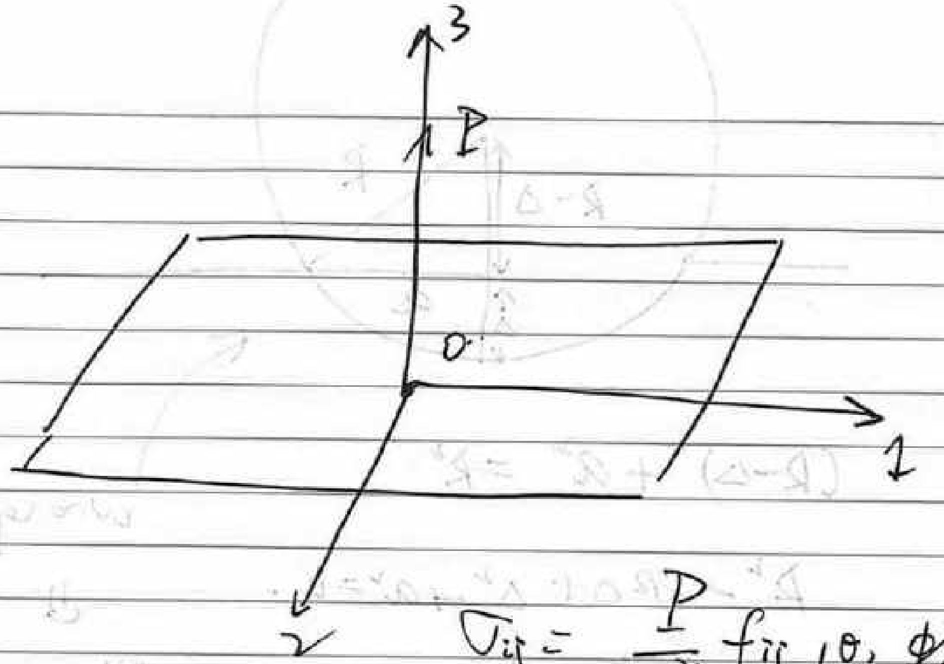
$$\boxed{\Delta = \left[\frac{9}{16R\,(4G)^2} \right] F^{2/3}}$$

⇓

incompressible solid.

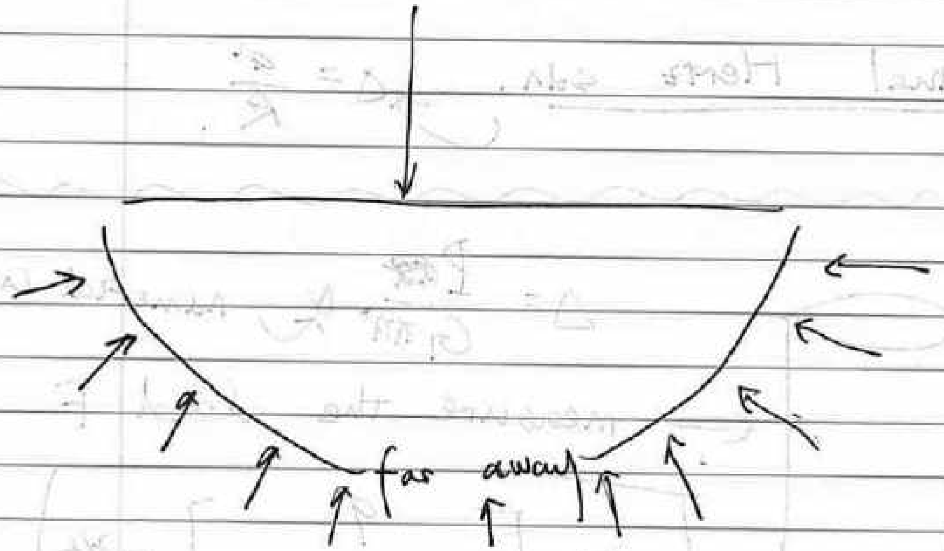
$4G \longrightarrow E^* = \frac{E}{1-\nu^2}$.

↳ shear modulus



$$\sigma_{ij} = \frac{P}{r^2} f_{ij}(\theta, \cancel{\phi})$$

independent of ϕ.

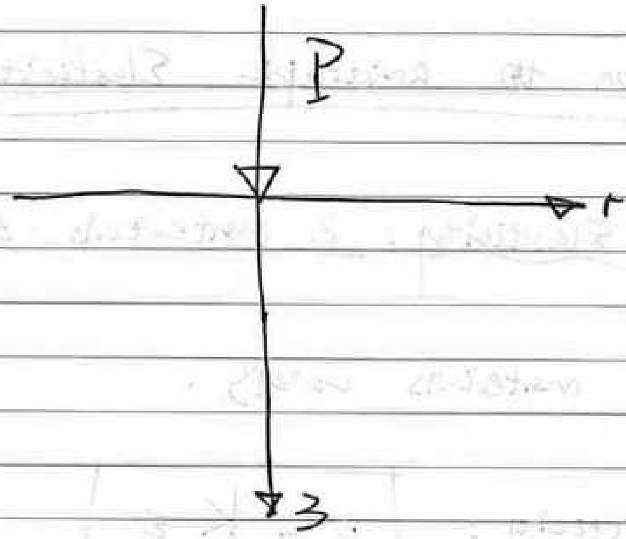


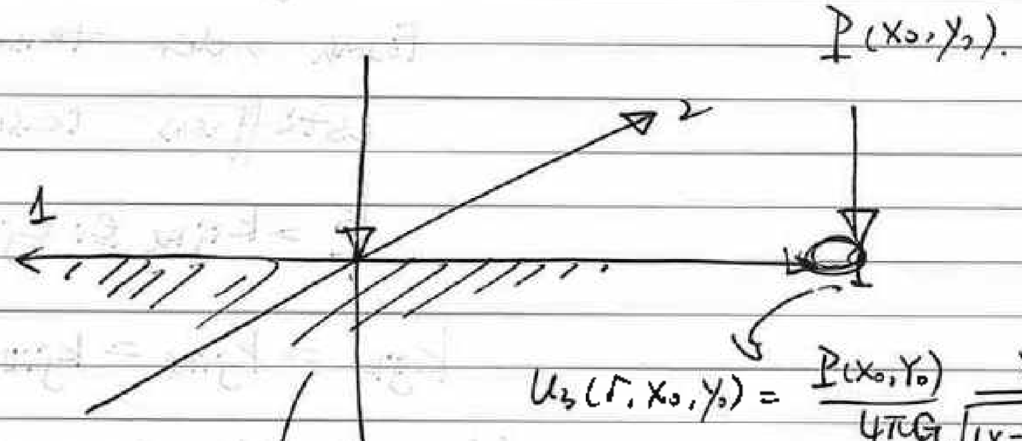
$4\pi r^2 \sigma_{rr} \approx P$

$r \gg 1$

$$u_i \approx \frac{P}{Gr} \hat{u}_i(\theta)$$

Standard Bushiness soln. $\underset{(r, z=0)}{u_3} = \frac{P}{4\pi G} \cdot \frac{2(1-\nu)}{\underset{\sqrt{x^2+y^2}}{\underbrace{r}}}$





$$u_3(r, x_0, y_0) = \frac{P(x_0,y_0)}{4\pi G} \frac{2(1-\nu)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$$

line load

$$u_3(r, z=0) = \frac{P}{4\pi G} \frac{2(1-\nu)}{\sqrt{x^2+y^2}}$$

distributed load

Superposition: $P(x_0, y_0) \Rightarrow p(x_0, y_0)\, \underbrace{dx_0\, dy_0}_{\text{very small area}}$.

$$u_3 = \frac{1-\nu}{2\pi G} \iint_A \frac{p(x_0, y_0)\, dx_0\, dy_0}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}$$

wdd.

Nov. 24, Wed., 2021. Wk 13.

Introduction to anisotropic Elasticity

Isotropic Elasticity: 2 materials consts

anisotropic:

▷ worst: 21 materials consts.

linear elasticity: $\boxed{\underline{\sigma} = \underline{\underline{k}}\,\underline{\varepsilon}}$ ①

Fourth order tensor

(stiffness tensor).

$$\underline{\underline{k}} = k_{ijkl}\,\underline{e}_i\,\underline{e}_j\,\underline{e}_k\,\underline{e}_l$$

$$k_{ijkl} = k_{jikl} = k_{jilk}$$

(due to the symmetry of stress & strain tensors).

36 characteristics.

Existence of strain energy density W.

$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}$ $\underline{\underline{k}}$ has 21 independent consts.

$$\sigma_{ij} \longrightarrow \underline{\sigma} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix}.$$

$\underline{\sigma} = \sigma_{ij}\,\underline{e}_i\,\underline{e}_j$,

$$\varepsilon_{ij} \longrightarrow \underline{\varepsilon} = \begin{pmatrix} \varepsilon_{11} = \varepsilon_1 \\ \varepsilon_{22} = \varepsilon_2 \\ \varepsilon_{33} = \varepsilon_3 \\ \varepsilon_{23} = \varepsilon_4 \\ \varepsilon_{13} = \varepsilon_5 \\ \varepsilon_{12} = \varepsilon_6 \end{pmatrix}$$

Eq. ① <=> $\underline{\sigma} = \underline{\underline{k}}\,\underline{\varepsilon}$

$$\underline{\underline{k}} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ \swarrow & & k_{33} & k_{34} & k_{35} & k_{36} \\ & \swarrow & & k_{44} & \cdot & \\ & & \swarrow & & k_{55} & \\ & & & & & k_{66} \end{bmatrix}$$

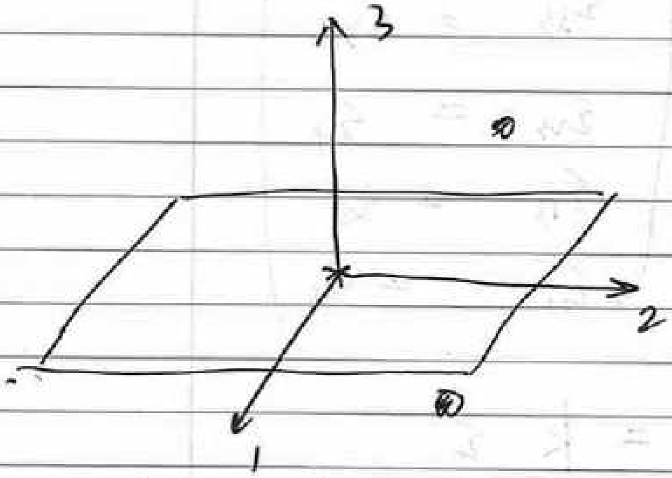
$$\sigma_{11} = K_{1111}\varepsilon_{11} + K_{1112}\varepsilon_{12} + K_{1113}\varepsilon_{13} + K_{1121}\varepsilon_{21}$$
$$+ K_{1122}\varepsilon_{22} + K_{1123}\varepsilon_{23} + K_{1131}\varepsilon_{31} + K_{1132}\varepsilon_{32}$$
$$+ K_{1133}\varepsilon_{33}.$$

$$\sigma_1 = K_{11}\varepsilon_1 + K_{16}\varepsilon_6 + K_{15}\varepsilon_5 + K_{14}\varepsilon_4 + K_{12}\varepsilon_2$$
$$+ K_{13}\varepsilon_3 \implies \underline{\sigma} = K\underline{\varepsilon}.$$

<u>Plane of symmetry</u> : (Material).

anisotropic

2 bases: $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\} = B_1$
$\{\underline{e}_1', \underline{e}_2', \underline{e}_3'\} = B_2$.



(Reflection).

$\underline{e}_1' \rightarrow \underline{e}_1$

$\underline{e}_2' = \underline{e}_2$

$\underline{e}_3' = -\underline{e}_3$

$\to K_{ijkl} \mapsto K_{rstu}'$

$\underline{\underline{K}}$ is the same for both basis. $(\underline{B}, \underline{B}')$.

transform into $\underline{B}'$ basis

$K_{ijkl}' = K_{ijkl}.$

$$K_{rstu}' = K_{ijkl}\underbrace{(\underline{e}_i \cdot \underline{e}_r')}_{P_{ri}}\underbrace{(\underline{e}_j \cdot \underline{e}_s')}_{P_{sj}}\underbrace{(\underline{e}_k \cdot \underline{e}_t')}_{P_{tk}}\underbrace{(\underline{e}_l \cdot \underline{e}_u')}_{P_{lu}}.$$

$\downarrow$ general transformation formula

P matrix is simple

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

$$K_{1111}' = K_{1111} = K_{ijkl}\underbrace{P_{1i}}_{\delta_{1i}}\underbrace{P_{1j}}_{\delta_{1j}}\underbrace{P_{1k}}_{\delta_{1k}}\underbrace{P_{1l}}_{\delta_{1l}}$$

(Hypothesis)

$$= K_{1111}$$

$$K_{2222}' = K_{2222} = K_{ijkl}P_{1i}P_{2j}P_{2k}P_{2l} = K_{2222}.$$

$$K_{1122}' = K_{ijkl}P_{1i}P_{1j}P_{2k}P_{2l}. = K_{1122}.$$
$$\delta_{1i}\ \delta_{1j}\ \delta_{2k}\ \delta_{2l}.$$

in the same way you can show for all.

$$K'_{1123} = K_{ijkl}\, P_{1i}\, P_{1j}\, P_{2k}\, P_{3l} = -K_{1123}$$

~~$\delta_{1i}\ \delta_{1j}$~~

$\delta_{1i}\ \delta_{1j}\ \delta_{2k}\ (-\delta_{3l})$

we know in prior : $K'_{1123} = K_{1123}$

Hence : $K_{1123} = -K_{1123}$.

$\therefore\ K_{1123} = 0$.

$K_{14} = 0$.

$K_{24} = K_{25} = K_{34} = K_{35} = K_{46} = K_{56} = 0$

reduce the num. of consts. to 8.

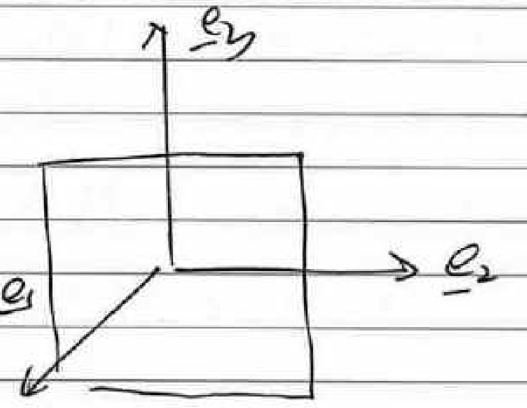
$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} & 0 & 0 & K_{16} \\ & K_{22} & K_{23} & 0 & 0 & K_{26} \\ & & K_{33} & 0 & 0 & K_{36} \\ & & & K_{44} & K_{45} & \cancel{K_{46}}\,0 \\ & & & & K_{55} & \cancel{K_{56}}\,0 \\ & & & & & K_{66} \end{bmatrix}$$

13 Mateial constants.

I already have material plane of symmetry

(3)

$$\boxed{\begin{aligned} \sigma_1 &= K_{11}\varepsilon_1 + K_{12}\varepsilon_2 + K_{13}\varepsilon_3 + K_{16}\varepsilon_6. \\ \sigma_2 &= K_{12}\varepsilon_1 + K_{22}\varepsilon_2 + K_{23}\varepsilon_3 + K_{26}\varepsilon_6 \\ \sigma_3 &= K_{13}\varepsilon_3 + K_{23}\varepsilon_2 + K_{33}\varepsilon_3 + K_{36}\varepsilon_6. \\ \sigma_4 &= K_{44}\varepsilon_4 + K_{45}\varepsilon_5. \\ \sigma_5 &= K_{54}\varepsilon_4 + K_{55}\varepsilon_5. \\ \sigma_6 &= K_{61}\varepsilon_1 + K_{62}\varepsilon_2 + K_{63}\varepsilon_3 + K_{66}\varepsilon_6. \end{aligned}}$$



define anew basis :

$e_1' = -e_1$

$e_2' = e_2$.

$e_3' = e_3$

put an additional plane of symmetry.

$\sigma_{ij} \rightarrow \sigma'_{ij}$

$\varepsilon_{ij} \rightarrow \varepsilon'_{ij}$

$$[\sigma'_{ij}] = \begin{bmatrix} \sigma_{11} & -\sigma_{12} & -\sigma_{13} \\ -\sigma_{12} & \sigma_{22} & \sigma_{23} \\ -\sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

$\sigma_4' = \sigma_4 = K_{44}\varepsilon'_{23}$

$- \oplus K_{45}\varepsilon_{13}$

$\sigma_4 = K_{44}\varepsilon_{23} - K_{45}\varepsilon_5$

$$\boxed{\begin{matrix} K_{45} = 0 & K_{36} = 0 \\ K_{16} = 0 & K_{26} = 0 \end{matrix}}$$

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} & 0 & 0 & 0 \\ & K_{22} & K_{23} & 0 & 0 & 0 \\ & & K_{33} & 0 & 0 & 0 \\ & & & K_{44} & 0 & 0 \\ & & & & K_{55} & 0 \\ & & & & & K_{66} \end{bmatrix}$$

HW 11

In torsion rheology test, circular cylinder

R. h $\gamma(t) = \gamma_0 e^{i\omega t}$.

Use cylinder coordinate,

only stress exist: $\sigma_{\theta z}$.

Initial condition: $\varepsilon_{ij} = \sigma_{ij} = 0$, $t=0$.

boundary condition:

$$\begin{cases} u_i(r, \theta, z=0, t>0) = 0, \\ u_r(r, \theta, z=h, t>0) = 0, \ u_z(r, \theta, z=h, t>0) = 0, \\ u_\theta(r \le R, \theta, z=h, t>0) = rh\gamma = rh\gamma_0 e^{i\omega t} \\ \sigma_{rr}(r=R, \theta, 0<z<h, t>0) \ \ldots \\ = \sigma_{zr}(r=R, \theta, 0<z<h, t>0) = 0. \end{cases}$$

(traction free on side walls).

governing Eqs. for torsion:

$$u_r = u_z = 0. \quad u_\theta = \gamma r z.$$

the only non-vanishing strain: $\underline{\varepsilon_{z\theta} = \frac{r\gamma}{2}}$

In cylindrical coor., all equilibrium satisfied!

* constitutive model: here linear viscoelasticity comes in

$$\sigma_{z\theta}(r,t) = 2G(t)\varepsilon_{z\theta}(r, t=0^+) + 2\int_{0^+}^{t} G(t-\tau)\frac{\partial \varepsilon_{z\theta}(r,\tau)}{\partial \tau}d\tau$$

$$\rightarrow \sigma_{z\theta}(r,t) = G(t)\, r\gamma(t=0^+) + r\int_{0^+}^{t} G(t-\tau)\frac{d\gamma(\tau)}{d\tau}d\tau$$

$$= G(t) r \gamma_0 + r \int_{0^+}^{t} G(t-\tau) \frac{d \gamma_0 e^{i\omega\tau}}{d\tau} d\tau .$$

$$= \left[G(t) + i\omega \int_{0^+}^{t} G(t-\tau) e^{i\omega\tau} d\tau \right] r \equiv \varphi(\omega, t) \gamma_0 r$$

the torque $M(t)$:

$$M(t) = 2\pi \int_0^R \sigma_{\theta 3} r^2 dr = \pi \gamma_0 \varphi(\omega, t) \int_0^R r^3 dr$$

$$= \frac{\pi \varphi(\omega, t) R^4 \gamma_0}{2} .$$

1b. $\varphi(\omega, t) = G(t) + \underbrace{i\omega \int_{0^+}^{t} G(t-\tau) e^{i\omega\tau} d\tau}_{\text{as } t \to \infty,} .$

integral term:

$$\int_{0^+}^{t} G(t-\tau) e^{i\omega\tau} d\tau = e^{i\omega t} \int_0^t G(t-\tau) e^{-i\omega(t-\tau)} d\tau$$

$$= e^{i\omega t} \int_0^t G(\eta) e^{-i\omega\eta} d\eta = e^{i\omega t} \left[\int_0^t [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta + \int_0^t G_\infty e^{-i\omega\eta} d\eta \right]$$

$$= e^{i\omega t} \left[\int_0^t [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta + \frac{G_\infty}{-i\omega} e^{-i\omega\eta} \Big|_0^t \right]$$

$$= e^{i\omega t} \int_0^t [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta + \frac{G_\infty}{i\omega} (e^{i\omega t} - 1)$$

then we have φ:

$$\varphi(\omega, t) = G(t) + i\omega \left[e^{i\omega t} \int_0^t [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta + \frac{G_\infty}{i\omega} (e^{i\omega t} - 1) \right]$$

$$= (G(t) - G_\infty) + G_\infty e^{i\omega t} + i\omega e^{i\omega t} \int_0^t [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta$$

We already know:

$$\int_0^t [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta$$

$$= \int_0^\infty [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta - \int_t^\infty [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta$$

$$\varphi(\omega, t) = \left[(G(t) - G_\infty) - i\omega e^{i\omega t} \int_t^\infty [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta \right] + \left\{ G_\infty + i\omega \int_0^\infty [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta \right\} e^{i\omega t}$$

since $G(\eta \to \infty) - G_\infty = 0$.

$$\varphi(\omega, t \to \infty) = \left\{ G_\infty + i\omega \int_0^\infty [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta \right\} e^{i\omega t}$$

$$= \mu(\omega) e^{i\omega t} .$$

$$M_{ss}(\omega) = \frac{\pi R^4 \gamma_0}{2} \mu(\omega) e^{i\omega t} .$$

Assuming $G(t) = G_\infty + \frac{G_0 - G_\infty}{(1+\frac{t}{t_R})^n}$., find storage & loss modulus.

- Storage modulus :

$$\mu'(\omega) = Re[\mu(\omega)] = Re\left[G_\infty + i\omega \int_0^\infty \frac{G_0 - G_\infty}{(1+\eta/t_R)^n} e^{-i\omega\eta} d\eta\right]$$

$$= G_\infty + (G_0 - G_\infty)\, \omega Re\left[i \int_0^\infty (1+\eta/t_R)^{-n} e^{-i\omega\eta} d\eta\right]$$

- loss modulus :

$$\mu''(\omega) = Im[\mu(\omega)]$$

$$= (G_0 - G_\infty)\, \omega Im\left[i \int_0^\infty (1+\eta/t_R)^{-n} e^{-i\omega\eta} d\eta\right]$$

to evaluate the integrals, let $\eta/t_R = p$,

so that $\int_0^\infty (1+\eta/t_R)^{-n} e^{-i\omega\eta} d\eta$

$$= t_R \int_0^\infty (1+p)^{-n} e^{-i\Omega p} dp.$$

For our case, $n=1$.

$$\int_0^\infty (1+p)^{-1} e^{-i\Omega p} dp = \int_0^\infty (1+p)^{-1} \cos(\Omega p)\, dp$$

$$- i\int_0^\infty (1+p)^{-1} \sin(\Omega p)\, dp.$$

$$= \int_0^\infty (\Omega+p)^{-1} \cos q\, dq - i\int_0^\infty (\Omega+q)^{-1} \sin q\, dp.$$

$$= \{-Ci(\Omega)\cdot\cos\Omega - si(\Omega)\sin\Omega\} - i\{Ci(\Omega)\sin\Omega$$

$$- si(\Omega)\cos\Omega\}.$$

Ci & Si : sine and cosine integrals.

normalize the storage & loss modulus :

$$\bar{\mu}'(\omega) = 1 + \left(\frac{G_0}{G_\infty} - 1\right)\Omega\, Re\left[i\int_0^\infty (1+p)^{-n} e^{-i\Omega p} dp\right]$$

$$= 1 + \left(\frac{G_0}{G_\infty} - 1\right)\Omega\{Ci(\Omega)\sin\Omega - Si(\Omega)\cos\Omega\}.$$

$$\bar{\mu}''(\omega) = \left(\frac{G_0}{G_\infty} - 1\right)\Omega\, Im\left[i\int_0^\infty (1+p)^{-n} e^{-i\Omega p} dp\right]$$

$$= -\left(\frac{G_0}{G_\infty} - 1\right)\Omega\{Ci(\Omega)\cos\Omega + Si(\Omega)\sin\Omega\}.$$

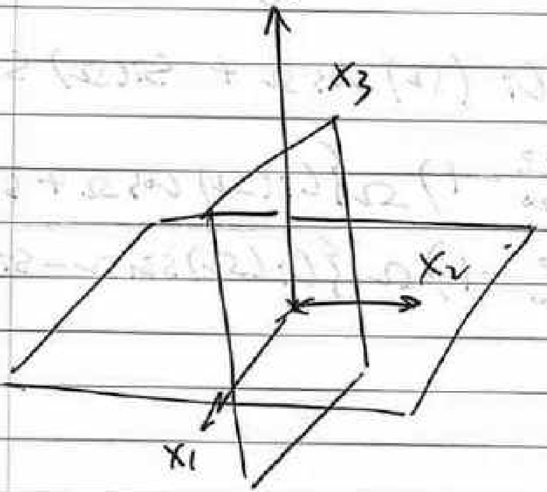
$$\therefore \tan\delta = \frac{-\left(\frac{G_0}{G_\infty} - 1\right)\Omega\{Ci(\Omega)\cos\Omega + Si(\Omega)\sin\Omega\}}{1 + \left(\frac{G_0}{G_\infty} - 1\right)\Omega\{Ci(\Omega)\sin\Omega - Si(\Omega)\cos\Omega\}}$$

Dec. 6., Mon. 2021. Wk 16.

orthotropic material. 9 constants.

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} & 0 & 0 & 0 \\ & K_{22} & K_{23} & & & \\ & & K_{33} & & & \\ & & & K_{44} & & \\ & & & & K_{55} & \\ & & & & & K_{66} \end{bmatrix}$$

Transversely Isotropic

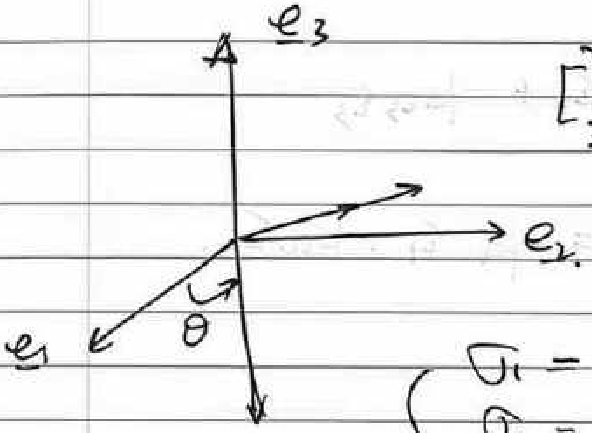


e_3 is an axis of symmetry,

every plane containing this axis is a plane of reflection. symmetry

invariant to Rotation above this axis.

$$[\underline{\underline{R}}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{cases} \underline{e}_1' = \cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_2 \\ \underline{e}_2' = -\sin\theta\, \underline{e}_1 + \cos\theta\, \underline{e}_2 \\ \underline{e}_3' = \underline{e}_3 \end{cases}$$



$$[\underline{\underline{P}}] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

standard isotropic material

$$\begin{cases} \sigma_1 = K_{11}\varepsilon_1 + K_{12}\varepsilon_2 + K_{13}\varepsilon_3 \\ \sigma_2 = K_{21}\varepsilon_1 + K_{22}\varepsilon_2 + K_{23}\varepsilon_3 \\ \sigma_3 = K_{31}\varepsilon_1 + K_{32}\varepsilon_2 + K_{33}\varepsilon_3 \\ \sigma_4 = K_{44}\varepsilon_4. \\ \sigma_5 = K_{55}\varepsilon_5 \\ \sigma_6 = K_{66}\varepsilon_6 \end{cases}$$

$$\underline{\underline{\sigma}}' = \underline{\underline{P}}\,\underline{\underline{\sigma}}\,\underline{\underline{P}}^T.$$

$$\underline{\underline{\varepsilon}}' = \underline{\underline{P}}\,\underline{\underline{\varepsilon}}\,\underline{\underline{P}}^T.$$

$$\begin{cases} \sigma_1' = \sigma_2 \quad \sigma_2' = \sigma_1 \quad \sigma_3' = \sigma_3 \\ \sigma_4' = \cancel{0} - \sigma_5. \quad \sigma_5' = \sigma_4. \\ \sigma_6' = -\sigma_6. \end{cases}$$

→ ~~Stat~~ same thing for strain.

$$\sigma_1' = K_{11}\varepsilon_1' + K_{12}\varepsilon_2' + K_{13}\varepsilon_3'$$

$$\to \sigma_2 \overset{\Downarrow}{=} K_{11}\varepsilon_2 + K_{12}\varepsilon_1 + K_{13}\varepsilon_3'$$

$$\sigma_2' = K_{21}\varepsilon_1 + K_{22}\varepsilon_2 + K_{23}\varepsilon_3'$$

$$\sigma_2' = K_{21}\varepsilon_1' + K_{22}\varepsilon_2' + K_{23}\varepsilon_3'.$$

$$\sigma_1 = K_{21}\varepsilon_2 + K_{22}\varepsilon_1. + K_{23}\varepsilon_3$$

$\sigma_1 = k_{12}\varepsilon_1 + k_{22}\varepsilon_2 + k_{23}\varepsilon_3$

$k_{12}\varepsilon_2 + k_{22}\varepsilon_1 = k_{12}\varepsilon_1 + k_{22}\varepsilon_2.$

~~$k_{11}\varepsilon_2 = k_{22}\varepsilon_2$~~ .

$k_{11}\varepsilon_2 + k_{13}\varepsilon_3 = k_{22}\varepsilon_2 + k_{23}\varepsilon_3.$

$(k_{11} - k_{12})\varepsilon_2 + (k_{13} - k_{23})\varepsilon_3 = 0.$

$k_{11} = k_{22}$

$k_{13} = k_{23}.$

$\sigma_4 = k_{44}\varepsilon_4.$

$\sigma_4' = k_{44}\varepsilon_4'$

$\sigma_5' = k_{55}\varepsilon_5'.$

$\sigma_4' = -\sigma_5 = -k_{44}\varepsilon_5'.$

$\vee$

$\sigma_5 = k_{44}\varepsilon_5'$

$\boxed{k_{44} = k_{55}.}$

$k_{44} = k_{55}. \Rightarrow \left.\begin{array}{l} k_{11} = k_{22}. \\ k_{13} = k_{23}. \end{array}\right\}.$

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & 0 & 0 & 0 \\ & k_{11} & k_{13} & & & \\ & & k_{23} & & & \\ & & & k_{44} & & \\ & & & & k_{55} & \\ & & & & & k_{66} \end{bmatrix}$$

$\theta = 45° \rightarrow \pi/4.$

$$[\underset{=}{P}] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

calculation.

this has to be true.

$\sigma_{12}', \varepsilon_{12}'.$

$\underbrace{k_{66} = \frac{1}{2}(k_{11} - k_{12}).}$

$\frac{1}{2}[k_{11} - k_{12}]$

$\underset{=}{K}^{-1} \cdot \sigma = \underset{=}{\varepsilon}.$

S matrix compliance index

Poisson's ratio for anisotropic elastic material.

Can leave no bounds. Ting, ICT.

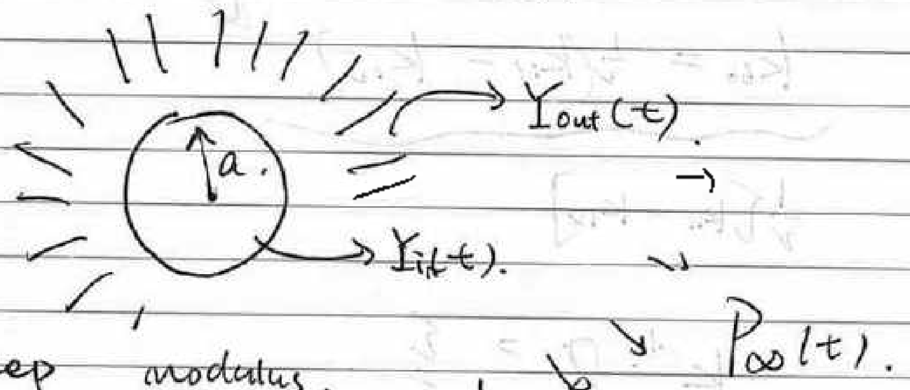
Linear viscoelasticity

- correspondence principle

↳ stress dependent of modulus

$$e_{ij} = S_{ij}(0^+) C_1(t) + \int_{0^+}^{t} C_1(t-\tau). \frac{\partial S_{ij}}{\partial \tau} d\tau .$$

$$\varepsilon_{kk} = \sigma_{kk}(0^+). C_2(t) + \int_{0^+}^{t} C_2(t-\tau). \frac{\partial \sigma_{kk}}{\partial \tau} d\tau .$$



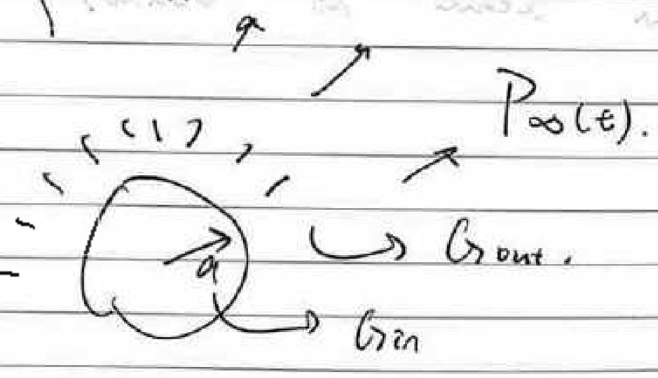
creep modulus.

↳ remote hydrostatic tension.

$P_\infty(t<0) = 0$

Elastic problem:

Plane stress



$$\boxed{\begin{cases} \sigma_{rr} = \frac{A}{r^2} + P_\infty \\ \sigma_{\theta\theta} = -\frac{A}{r^2} + P_\infty . \\ \sigma_{r\theta} = 0 \end{cases} \qquad r>A}$$

$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{in}$.

$\Downarrow$. $r<A$.

Continuity of traction.

$\frac{A}{r^2} + P_\infty = \sigma_{in}$. A. P_∞ → unknown.

$3 G_{out} \varepsilon_{\theta\theta} = \sigma_{\theta\theta} - \frac{1}{2} \sigma_{rr}$. $r>a$.

$$\underset{(r>a)}{\varepsilon_{\theta\theta}} = \frac{1}{3G_{out}}\left[\frac{-3A}{2r^2} + \frac{P_\infty}{2}\right].$$

$\Downarrow$

$\varepsilon_{\theta\theta} = \frac{u}{r}$

$r<a$, $\varepsilon_{in} = \varepsilon_{\theta\theta} = \varepsilon_{rr} = \frac{u}{r}$.

$u = \varepsilon_{in} r$

$$3 G_{in} \varepsilon_{in} = \sigma_{\theta\theta} - \frac{1}{2}\sigma_{rr} .$$
$$= \sigma_{in}/2 .$$

$$\therefore \varepsilon_{in} = \frac{\sigma_{in}}{6G_{in}} = \frac{u}{r}.$$

continuing of Hooke's strain.

$$\frac{1}{2G_{out}}\left[\frac{-3A}{2a^2} + \frac{P_\infty}{2}\right] = \frac{\sigma_{in}}{6G_{in}}.$$

$$\boxed{\begin{gathered}\frac{A}{a^2} = \frac{(\rho-1)P_\infty}{(1+3\rho)}, \quad \rho = \frac{G_{in}}{G_{out}}. \\ \sigma_{in} = \frac{4\rho}{1+3\rho} P_\infty\end{gathered}}$$

$$\frac{G_{in}}{G_\infty} \to \infty \Rightarrow \underline{\frac{A}{a^2} = \frac{1}{3}P_\infty}$$

$$\sigma_{\theta\theta}(r = a^+). = -\frac{2}{3}P_\infty.$$

$$G_{in} = \frac{1}{2} s \tilde{Y}_{in}(s).$$ ☆☆☆

$$G_{out} = \frac{1}{2} s \hat{Y}_{out}(s).$$

$$\tilde{\sigma}_{in} = \frac{4\frac{\tilde{Y}_{in}(s)}{\hat{Y}_{out}(s)}}{1 + 3\frac{\tilde{Y}_{in}(s)}{\hat{Y}_{out}(s)}}\tilde{P}_\infty(s)$$

$$Y_{in}(t). = Y_{\infty in} + (Y_{0in} - Y_{\infty in}) e^{-t/t_{in}}$$

$$\mathcal{L}(Y_{in}(t).) = \hat{Y}_i(s) = \int_0^\infty e^{-st} Y_i(t)\,dt.$$

$$\tilde{Y}_{in}(s) = \frac{Y_{in\infty}}{s} + \frac{(Y_{in0} - Y_{in\infty})}{s + t/t_{in}}$$

$$\hat{Y}_{out}(s) = \frac{Y_{out\infty}}{s} + \frac{Y_{out0} - Y_{out\infty}}{s + t/t_{out}}$$

$$\sigma_{in}(t) = \frac{1}{2\pi i}\int_{\delta-\infty}^{\delta+\infty} e^{st}\,\tilde{\sigma}_\infty(t)\,ds.$$

☆ Solve ODE with MATLAB.

HW 10. Review:

a. problem formulation:

$$\nabla^2 \phi = 0. \quad \text{in} \quad |x| < \frac{a}{2} \ \& \ |y| < \frac{b}{2}.$$

BCs:

$$\phi(x = \pm a/2, \ |y| < \tfrac{b}{2}) = \tfrac{1}{2}(\tfrac{a^2}{4} + y^2).$$

$$\phi(|x| < \tfrac{a}{2}, \ y = \pm \tfrac{b}{2}) = \tfrac{1}{2}(\tfrac{b^2}{4} + x^2)$$

b. $f = \nabla_{xx} \phi + 1.$

on the boundary $y = \pm \frac{b}{2}$,

$$\Rightarrow f(|x| < a/2, \ y = \pm \tfrac{b}{2}) = 2.$$

on the boundary $x = \pm \frac{a}{2}$.

we know $\partial_{xx} \phi = -\partial_{yy} \phi.$

$$\Rightarrow f(x = \pm \tfrac{a}{2}, \ |y| < \tfrac{b}{2}) = 0.$$

c. find f:

$$f(x,y) = X(x) Y(y).$$

substitute into $\nabla^2 f = 0$:

$$\ddot{X}(x) Y(y) + X(x). \ddot{Y}(y) = 0.$$

$$\Rightarrow \frac{\ddot{X}(x)}{X(x)} = -\frac{\ddot{Y}(y)}{Y(y)} = C = -k^2.$$

we look for solution: satisfy $x = \pm \frac{a}{2}$.

$$\begin{cases} \ddot{X}(x) + k^2 X(x) = 0. \\ \ddot{Y}(y) - k^2 Y(y) = 0 \end{cases}$$

$$\begin{cases} X(x) = B \sin kx + A \cos kx. \xrightarrow{x = \pm \frac{a}{2}, \ X = 0} \\ Y(y) = C \cosh k_n y + D \sinh k_n y. \end{cases}$$

$$f(x,y) = \sum_1 a_n \cos(k_n x) \cosh(k_n y).$$

BCs: $f(x < \frac{a}{2}, \ y = \pm \frac{b}{2}) = 2.$

$$\sum_1 a_n \cos(k_n x) \cosh(k_n b/2) = 2.$$

method of Fourier series:

$$a_n = \frac{2}{a \cosh(k_n b/2)} \int_{-a/2}^{a/2} 2 \cos(k_n x) dx$$

$$= \frac{8(-1)^n}{\pi(2n+1)\cosh(k_n b/2)}.$$

Thus:

$$f(x,y) = \frac{8}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot \frac{\cosh(k_n y)}{\cosh(k_n b/2)} \cos(k_n x),$$

$$k_n = \frac{2n+1}{a}\pi$$

d: find max stress:

Shear stresses:

$$\begin{cases} \sigma_{13} = G\gamma(-y + \phi_{,2}) \\ \sigma_{23} = G\gamma(x - \phi_{,1}) \\ \frac{\sigma_{13}}{G\gamma} + y = \phi_{,2} \\ -\frac{\sigma_{23}}{G\gamma} + x = \phi_{,1} \end{cases}$$

on the boundary $x = \pm\frac{a}{2}$, $\phi_{,2}\big|_{x=\pm\frac{a}{2}} = y$.

$\sigma_{13} = 0$ on $x = \pm\frac{a}{2}$.

$\sigma_{23} = 0$ on $y = \pm\frac{b}{2}$.

$\phi_{,1}\big|_{y=\pm\frac{b}{2}} = x.$