COURSE NOTES

Partial Differential Equations

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1/9/2024. Lecture 1

- Introduction: applications, contexts, examples.
- classification & Solutions
- Solution methods.

Introduction

PDEs: Systems that evolve in space & time are often described via PDEs

Applications → {
- thermo.
- mechanics of solid/liquid/gas
- Electromagnetics
- chemical
- dynamical
- material science.
- population

PDE: An equation that relates a multivariable function φ & its partial derivatives in 2 or more Independent variables.

$\underset{\text{dependent variable}}{\varphi}(\underset{\text{Ind. vars.}}{x,t})$

$\varphi(x,y,z,t)$ or $\varphi(\vec{x},t)$ → general: tensor

or $\varphi(\vec{x})$.

φ {
- Scalar: temp., pressure, density, ~potentials.
- Vectors: velocity, $\vec{E}$, $\vec{B}$, force, ...
- tensor: Stress, strain, Reynolds stress.

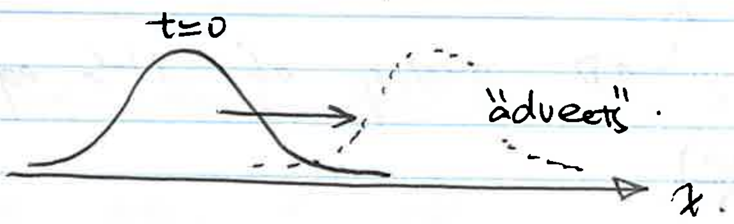
Examples

- Advections.

$$\frac{\partial\varphi}{\partial t} + \underset{\text{const.}}{U}\frac{\partial\varphi}{\partial x} = 0 \qquad (1D)$$

operator applies on φ.

$$\frac{\partial\varphi}{\partial t} + (U\cdot\nabla)\varphi = 0 \qquad (N\text{-dim.})$$



$\mathcal{L}(\varphi)=0.$

*...? $\mathcal{L}(\alpha\varphi_1+\beta\varphi_2) = \alpha\mathcal{L}(\varphi_1)+\beta\mathcal{L}(\varphi_2)$

consts.

flux for φ

Conservation form: $\frac{\partial\varphi}{\partial t} + \frac{\partial(U\varphi)}{\partial x} = 0$

general form for conservation law

$$\frac{\partial\varphi}{\partial t} + \frac{\partial(F(\varphi))}{\partial x} = 0$$

F: flux of φ

*... ? flux.

*... ? $\nabla\cdot$

in N-dim: $\frac{\partial \varphi}{\partial t} + \nabla\cdot\vec{F} = 0.$

$\hookrightarrow \vec{F}(\varphi)$

if F only depends φ:

Remark: F has to be one dimension higher than φ.

conservation law: $\frac{\partial \varphi}{\partial t} + \underbrace{\frac{\partial f}{\partial \varphi}}_{\text{adv. vel.}} \frac{\partial \varphi}{\partial x} = 0$

* Nonlinear adv. eqn.: Burgers eqn.

$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$ inviscid Burgers.

$\downarrow$ $\uparrow$ 1D analog of N-S equation.

$\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right).$

Characteristics Method.

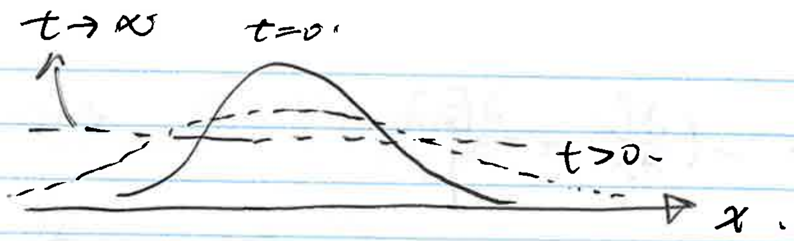
* Diffusion $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}.$

T: temperature.

α: thermal diffusivity.

$= \frac{k}{\rho C_p}.$ $(m^2/s).$

⇝ heat, mass (concentration), momentum, ...



N-dim.: $\frac{\partial T}{\partial t} = \alpha \nabla^2 T$

$$\nabla^2(\cdot) = \frac{\partial^2(\cdot)}{\partial x_i \partial x_i} = \frac{\partial^2(\cdot)}{\partial x^2} + \frac{\partial^2(\cdot)}{\partial y^2} + \frac{\partial^2(\cdot)}{\partial z^2}$$

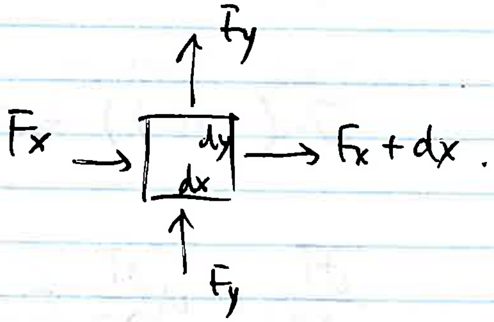
$$T_t = \alpha(T_{xx} + T_{yy} + T_{zz}).$$

⇒ how to use conservation law to derive the heat equation?

Note on PDE derivation

Control volume balance (Eulerian):

Heat eqn. in 2D:



$$(dx+dy)\, \rho\, C_p \frac{\partial T}{\partial t} = (F_x - F_{x+dx})\,dy$$
$$+ (F_y - F_{y+dy})\,dx.$$

$$\rho\, C_p \frac{\partial T}{\partial t} = -\left[\frac{F_{x+dx} - F_x}{dx} + \frac{F_{y+dy} - F_y}{dy}\right]$$

Take lim of dx, dy → 0

$$\rho C_p \frac{\partial T}{\partial t} = -\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right).$$

$$\vec{F} = -k\nabla T. \longrightarrow \begin{cases} F_x = -k\frac{\partial T}{\partial x} \\ F_y = -k\cdot\frac{\partial T}{\partial y}. \end{cases}$$

"Fourier's law"

$$\rho C_p \frac{\partial T}{\partial t} = k\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)T = k\nabla^2 T.$$

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p}\nabla^2 T = \alpha\nabla^2 T.$$

$$\rho C_p \frac{\partial T}{\partial t} + \nabla\cdot\vec{F} = 0$$

↳ $$\rho C_p \frac{\partial T}{\partial t} + \nabla\cdot(k\nabla T) = 0.$$

$\nabla\cdot(\nabla(\cdot)) = \nabla^2(\cdot)$ ⤳ Heat eqn.

$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \alpha\cdot\frac{\partial^2 T}{\partial x^2}$ adv.-diff. eqn.

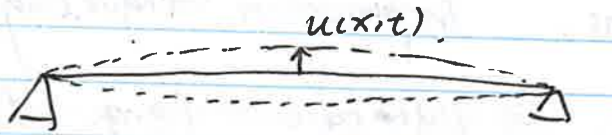
$\frac{\partial T}{\partial t} = \alpha\nabla^2 T. \xrightarrow[\text{state}]{\text{steady}}$ $\underbrace{\nabla^2 T = 0.}_{\text{Laplace equation.}}$

$\underset{\uparrow\ \text{source}}{\nabla^2 T = S.}$ ← Poisson's eqn.

Waves & Vibration

$\frac{\partial^2 u}{\partial t^2} - C_0^2\cdot\frac{\partial^2 u}{\partial x^2} = 0.$ Dalambert's eqn.

$\frac{\partial^2 u}{\partial t^2} - C^2\nabla^2 u = 0$ wave eqn.



String vibration.

surface sound waves.

assume sm

$C_0^2 = \frac{T}{m}$ ← tension

↖ mass per length.

Classification of PDE.

- Order or degree (of partial derivatives)

$\varphi(x, y)$: First-order:

$a(\varphi, x, y)\frac{\partial\varphi}{\partial x} + b(\varphi, x, y)\frac{\partial\varphi}{\partial y} + c(\varphi, x, y) + d$

=

Second-order:

1st ord

terms =

$A(\varphi_x, \varphi_y, \varphi, x, y)\varphi_{xx} + B(\cdots)\varphi_{xy} + C(\cdots)\varphi_{yy} +$

$A^2+B^2+C^2 \neq 0$.

B^2-4AC : discriminant

① $B^2-4AC > 0 \Rightarrow$ hyperbolic (e.g. wave eqn).

wave-like solns, information traveling characteristics.

② $B^2-4AC = 0 \Rightarrow$ parabolic (e.g. diffusion)

③ $B^2-4AC < 0 \Rightarrow$ elliptic (e.g. Laplace, Poisson)

- Linear or nonlinear.

$\rightsquigarrow \mathcal{L}(\alpha\varphi_1 + \beta\varphi_2) \overset{?}{=} \alpha\mathcal{L}(\varphi_1) + \beta\mathcal{L}(\varphi_2)$.

↑ PDE.

- Homogeneous or inhomogeneous ⇝ trivial soln.

  ↳ $\varphi = 0$. forcing $(\vec{x}, t)$

- I.C.s & B.C.s → well-posedness.

  (existence, uniqueness of solns)

  ↳ depends on domain & independent variables.

Solution Methods

Methods	Linear	Nonlinear
· characteristics.	✓	✓
· Separation of vars. (eigenval. expns).	✓	×
· Integral transformations	✓	×
· Similarity solutions.	✓	✓

Super-position }

built solutions based on basis functions.

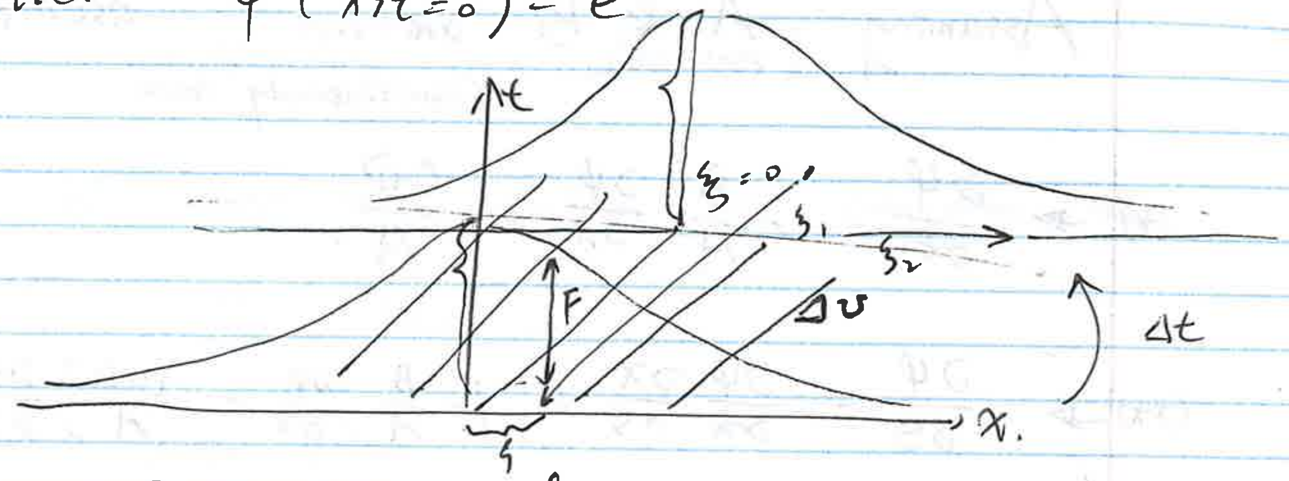
⇓

General theme:

Convert "PDE" to a syste

of ODEs.

I.C. : $\varphi(x,t=0) = e^{-x^2}$



Goal: Find solution for all s

Along charac. : $\frac{dx}{dt} = \frac{\upsilon}{1}$. & $\frac{d\varphi}{dt} = 0$.

$dx = \upsilon dt$, $\Longrightarrow$ $x = \upsilon t + \xi$. & $\varphi = F$.

characteristics are labeled by ξ.

$$\Downarrow$$

$F = F(\xi)$.

Goal: $\rightarrow \varphi(x,t)$?

$\xi = x - \upsilon t$

$\varphi = F(\xi) = F(x - \upsilon t)$.

I.C : $\varphi(x,t=0) = F(x) = e^{-x^2}$

$$\Downarrow$$

$$\varphi(x,t) = F(\xi) = F(x - \upsilon t). = e^{-(x-\upsilon t)^2}$$

Example 2

$u(x,t)$.

$\frac{\partial u}{\partial t} - \frac{x}{2} \cdot \frac{\partial u}{\partial x} = 0$. & $u(x, t=0) = e^{-x^2}$

on characteristics : $\left.\frac{dx}{dt}\right|_{\xi} = \frac{-\frac{x}{2}}{1} = -\frac{x}{2}$

& $\left.\frac{du}{dt}\right|_{\xi} = 0$.

$\Rightarrow \frac{dx}{x} = -\frac{1}{2} dt$, $\rightarrow$ $\ln x = -\frac{t}{2} + C$.

$x = e^{c} \cdot e^{-t/2}$. $= \xi \cdot e^{-t/2}$ $\hookrightarrow \xi = x e^{t/2}$.

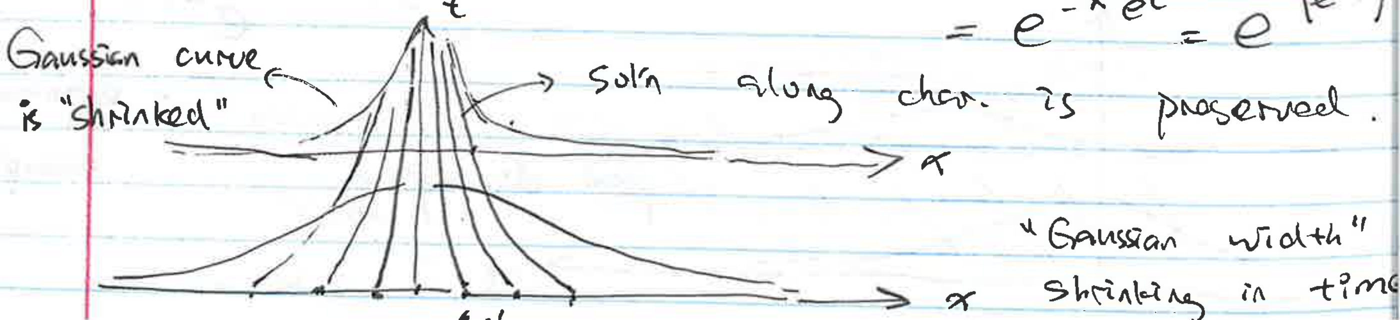
$\left.\frac{du}{dt}\right|_{\xi} = 0 \Rightarrow u = F(\xi)$.

I.C. : $u(x, t=0) = F(\xi) = F(x \cdot e^{t/2})$.

$= F(x) = e^{-x^2}$.

$u(x,t) = F(\xi) = F(x e^{t/2}) = e^{-(x \cdot e^{t/2})^2}$

$= e^{-x^2 e^t} = e^{-\left(\frac{x}{e^{-t/2}}\right)^2}$



Practice problem: $\frac{\partial u}{\partial t} + t\frac{\partial u}{\partial x} = 0.$

$$u(x, t=0) = e^{-x^2}$$

Example 4 $u(x,t)=0.$

$\xrightarrow{}$ sol'n not preserved along char.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = -u, \quad u(x,t=0) = e^{-x^2}.$$

linear & homogeneous.

along char.: $\left.\frac{dx}{dt}\right|_\xi = 1, \quad \left.\frac{du}{dt}\right|_\xi = -u.$

$\left.\frac{dx}{dt}\right|_\xi = 1, \Rightarrow x = t + \xi. \rightsquigarrow \xi = x - t.$

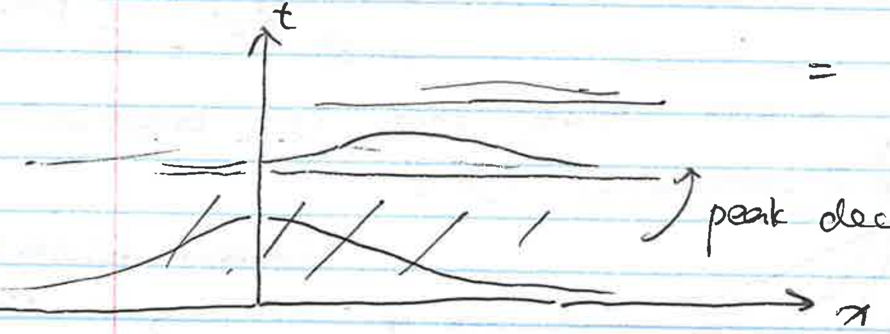
$\left.\frac{du}{dt}\right|_\xi = -u \Rightarrow \frac{du}{u} = dt \Rightarrow. u = u_0(\xi)e^{-t}.$

I.C.: $u(x, t=0) = u_0(\xi)e^{-t} = \underset{\underset{e^{-x^2}}{\parallel}}{u_0(x)}\, \underbrace{e^{-t}}_{=1 \text{ when } t=0}.$

$$u(x,t) = u_0(\xi)e^{-t} = e^{-\xi^2}e^{-t}$$

$$= e^{-(x-t)^2}e^{-t}.$$

↖ exponential damping



Example 5 Burger's Eqn. (inviscid).

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

$$u(x, t=0) = e^{-x^2}$$

$$u(x,t) \longrightarrow u(x(t), t).$$

on the characteristics:

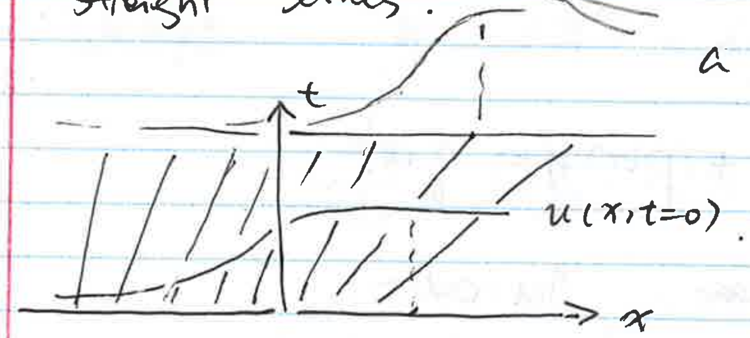
$\left.\frac{dx}{dt}\right|_\xi = u. \quad \& \quad \left.\frac{du}{dt}\right|_\xi = 0.$

↓

along char., sol'n doesn't change, u is a slope of char.

*** IMPORT

char. should be straight lines. ←



Slope of the char. does not change. ($u(\xi)$)

↓

Straight lines.

more rigorously, integrate:

$$x = ut + \xi. \Rightarrow \xi = x - ut.$$

$$u = F(\xi).$$

therefore, $u = F(x - ut)$. ← implicit sol'n.

↖ depends on I.C.

Problem Session I

Review of ODEs.

Classification.

- Order: highest derivative.
- Linearity: are there y^2, yy', etc. terms?
- Homogeneity: is $y=0$ a solution or not.
- Coefficient: are coefficients $y, y', y'', \dots$ function of x or not?

1st-order ODE:

$$y'(x) + p(x)\,y = q(x).$$

→ linear, 1st-order.

integrating factor: "reverse product rule".

Example 1: $y' + 2y = 1$.

Multiply by $\mu(x) = e^{2x} \to \mu'(x) = 2e^{2x}$.

$$e^{2x}y' + 2e^{2x}y = e^{2x}$$

$$(e^{2x}y)' = e^{2x}.$$

$$e^{2x}y = \frac{1}{2}e^{2x} + C$$

$$y = \frac{1}{2} + Ce^{-2x}.$$

In general, $\mu(x) = \exp\left(\int p(x)\,dx\right)$.

Separable equations.

$$N(y)\frac{dy}{dx} = M(x).$$

"Separate" the derivative.

$$\int N(y)\,dy = \int M(x)\,dx.$$

Example: $y' - 6y^2x - x = 0$.

$$\frac{dy}{dx} = (6y^2+1)x.$$

$$\frac{dy}{6y^2+1} = x\,dx.$$

$$\frac{\arctan(\sqrt{6}y)}{\sqrt{6}} = \frac{1}{2}x^2 + C$$

$$\arctan(\sqrt{6}y) = \sqrt{6}\left(\frac{1}{2}x^2 + C\right).$$

$$y = \frac{1}{\sqrt{6}}\tan\left(\sqrt{6}\left(\frac{1}{2}x^2 + C\right)\right)$$

1/11/2024. Lecture 2

Characteristics (6 lectures).

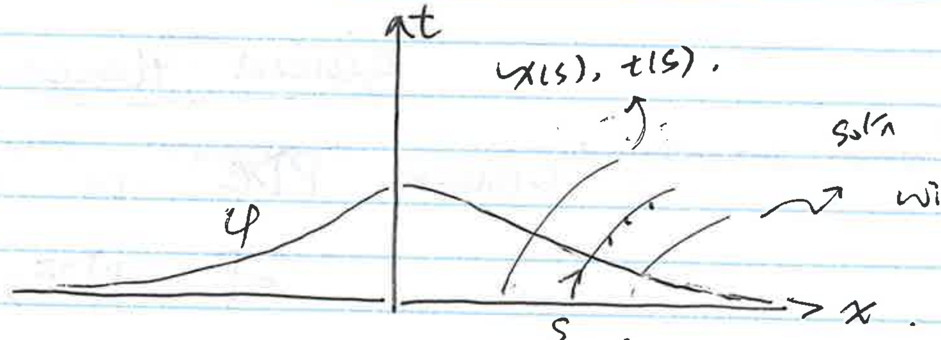
"find coordinate transform to transform the PDE to ODEs".

First-order PDE

$\varphi(x,t)$

$$A(\varphi, x, t)\frac{\partial\varphi}{\partial t} + B(\varphi, x, t)\frac{\partial\varphi}{\partial x} + C(\varphi, x, t) + D(x,t) = 0 \qquad (*)$$

Think geometrically,



family of characteristic curves along ⇝ PDE → ODEs.

⇑

soln along the traj. will be ODE.

s ↙ varies along traj. ↳ label.

$$\varphi(x,t) \xrightarrow{\text{on traj.}} \varphi(x(s), t(s)).$$

$$\frac{\partial\varphi}{\partial s} = \frac{\partial\varphi}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial\varphi}{\partial t}\cdot\frac{\partial t}{\partial s}. \qquad (**)$$

Assuming A & B are not simultaneously zero, assume $A \neq 0$

$$(*) \Rightarrow \frac{\partial\varphi}{\partial t} = -\frac{B}{A}\cdot\frac{\partial\varphi}{\partial x} - \frac{C+D}{A}.$$

$$(**) \Rightarrow \frac{\partial\varphi}{\partial s} = \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial s} + \left(-\frac{B}{A}\cdot\frac{\partial\varphi}{\partial x} - \frac{C+D}{A}\right)\frac{\partial t}{\partial s}.$$

↓ along traj.

$$= \frac{\partial\varphi}{\partial x}\left\{\frac{dx}{ds} - \frac{B}{A}\cdot\frac{dt}{ds}\right\} - \frac{C+D}{A}\cdot\frac{dt}{ds}.$$

choose $s = t$:

$$\boxed{\frac{dx}{ds} = \frac{B}{A}(\varphi, x, t) = \frac{dx}{dt}.}$$

$$\boxed{\frac{d\varphi}{dt} = -\frac{C(\varphi,x,t) + D(x,t)}{A(\varphi,x,t)}}$$

1st-order ODE

⇝ family of characteristic curves. $(x(s), t(s), \varphi$

Example 1

$$\frac{\partial\varphi}{\partial t} + U\frac{\partial\varphi}{\partial x} = 0$$

$$-\infty < x < \infty, \qquad 0 \leq t < \infty$$

Exact equations.

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0.$$

Trying to find some function $\psi(x,y)$ s.t.

$$\psi_x = M \quad \& \quad \psi_y = N.$$

Condition: $\psi_{xy} = \psi_{yx} \rightarrow M_y = N_x$

Compute: $\psi = \int M dx$ or $\psi = \int N dy$

and compare $\psi_y = N$ or $\psi_x = M$.

Example $\underbrace{2xy - 9x^2}_{M} + \underbrace{(2y + x^2 + 1)}_{N}\frac{dy}{dx} = 0$

check if exact: $M_y = 2x = N_x$.

Then: $\psi_x = M \rightarrow \psi = \int M dx$.

$$\psi = \int 2xy - 9x^2 dx.$$

$$\psi = x^2y - 3x^2 + h(y).$$

how to find $h(y) \leftarrow \psi_y$

$$\psi_y = x^2 + h'(y) = 2y + x^2 + 1 = N.$$

$\rightarrow h'(y) = 2y + 1 \rightarrow h(y) = y^2 + y + \text{const.}$

$\rightarrow \psi = x^2y - 3x^2 + y^2 + y + \text{const.} = \text{const.}$

$$y^2 + (x^2+1)y - 3x^2 = C.$$

Second-order ODEs:

$$p(t)y'' + q(t)y' + r(t)y = g(t).$$

$$ay'' + by' + cy = g(t)$$

Usually encounter and solve const. coeff. OD

homog. $g(t) = 0$ vs. inhomog. $g(t) \neq 0$

homogeneous soln will be superposition of 2 s

$$y_h(t) = C_1 y_1(t) + C_2 y_2(t)$$

how to get y_h? Ansatz: $y = \exp(rt)$

$$y' = r\exp(rt)$$

$$y'' = r^2\exp(rt)$$

$$(ar^2 + br + c)\exp(rt) = 0$$

↓

Characteristic equation: $ar^2 + br + c = 0$

For those 2nd-order, linear, const-coeff ODEs, you can start w/ characteristic eqns:

Quadratic equation: $r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

- ∘ $b^2 - 4ac > 0$ → 2 real roots
- ∘ $b^2 - 4ac < 0$ → 2 complex conjugate roots
- ∘ $b^2 - 4ac = 0$ → repeated real roots.

For 2 real roots r_1 & r_2:

$$y_h(t) = C_1 \exp(r_1 t) + C_2 \exp(r_2 t)$$

For 2 complex conjugate roots ($r_{1,2} = \alpha \pm \beta i$).

$$y_h(t) = \exp(\alpha t)\left[C_1 \cos(\beta t) + C_2 \sin(\beta t)\right]$$

$y_1 = \exp[(\alpha + \beta i)t]$, $y_2 = \exp[(\alpha - \beta i)t]$

$= \exp(\alpha t)[\cos\beta t + i\sin\beta t]$ $= \exp(\alpha t)[\cos\beta t - i\sin\beta t]$

$y_3 = C_1 y_1 + C_2 y_2 \rightarrow C_1 = C_2 = 1/2$

↳ $y_3 = \exp(\alpha t)\cos\beta t$

$y_4 = C_1 y_1 + C_2 y_2 \rightarrow C_1 = 1/2i$, $C_2 = 1/2i$.

$y_4 = \exp(\alpha t)\sin\beta t$.

For repeated real root (r).

$$y_h(t) = C_1 \exp(rt) + C_2 t \exp(rt).$$

↓ $y_1 = \exp(rt)$.

guess $y_2 = v(t)\, y_1$.

Calculate y_2' & y_2'' & substitute.

Everything cancels except for $v'' = 0$.

→ ODE for v → $v = \underset{\uparrow}{c}t + \underset{\uparrow}{k} = t$.

const.

Always need $y_h(t)$.

If inhomogeneous → $y(t) = y_h(t) + y_p(t)$.

2 methods: {
Methods of undetermined coefficients.
Variation of parameters.

∘ Apply BCs/ICs after getting full sol'n.

→ Undetermined coefficients.

Example: $y'' - y = 3t^2 + t + 1$

try: $y_p(t) = at^2 + bt + c$

$$y_p' = 2at + b \ , \quad y_p'' = 2a$$

Substitute: $2a - (at^2 + bt + c) = 3t^2 + 2t + 1$

$$at^2 + bt + c - 2a = -3t^2 - 2t - 1$$

$\rightarrow a = -3, \quad b = -2, \quad c = -7$

$$y_p = -3t^2 - 2t - 7$$

general sol'n: $y = y_p(t) + y_h(t)$.

to formulate (guess) y_p form:

n-deg. polynomial	↔	n-deg poly.
e^{at}	↔	e^{ct}
$\cos\beta t$	↔	$\cos\beta t + \sin\beta t$.
$\sin\beta t$	↔	...

→ Variation of Parameters.

More general method to find particular sol'n

$$y'' + q(t)y' + r(t)y = g(t)$$

* Requires homogeneous solution.

$$y_h(t) = C_1 y_1(t) + C_2 y_2(t).$$

Trying to find u_1 & u_2 s.t.

$$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$$

Is a solution to the inhomogeneous sys.

then $y_p'(t) = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$

Now, assume $u_1 y_1' + u_2' y_2 = 0$

$$y_p''(t) = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

Plug in & simplify:

$$u_1' y_1' + u_2' y_2' = g(t)$$

Solve for u_1', u_2' and integrate

$$\begin{cases} u_1(t) = -\displaystyle\int \frac{y_2 g(t)}{W(y_1, y_2)}\, dt. \\ u_2(t) = \displaystyle\int \frac{y_1 g(t)}{W(y_1, y_2)}\, dt. \end{cases}$$

← For VoM, a
these to fo
$y_p(t)$.

Wronskian $W(y_1, y_2) = y_1 y_2' - y_2 y_1' \neq 0$

then $y_p(t) = u_1 y_1 + u_2 y_2$.

Example $y'' - 4y' + 3y = e^{-t}$.

$\rightarrow y_1 = e^{3t}, \quad y_2 = e^{t}$

$W = y_1 y_2' - y_2 y_1' = -2e^{4t}$

$u_1 = \int \dots dt, \quad u_2 = \int \dots dt.$

$\rightarrow y_p = \frac{1}{8} e^{-t}$

Wronskian.

$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

↳ check whether y_1 & y_2 are a fundamental set of solutions.

↳ fundamental set if $W \neq 0$.

* Basically a check on linear independence.

If $y_2 = \text{const} \cdot y_1 \rightarrow$ are not truly 2 solutions

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Lecture 3 1/16/2024.

* Characteristics as coordinate transformations.

* Nonlinear PDE (Burgers' equation)

* Expansion & Compression waves & shocks.

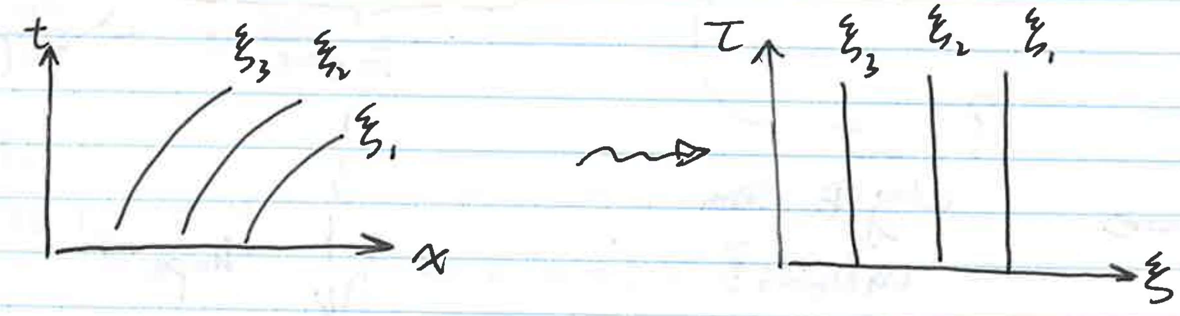
1st order PDE: $\varphi(x,t)$.

(*)

$$A(\varphi, x, t)\frac{\partial \varphi}{\partial t} + B(\varphi, x, t)\frac{\partial \varphi}{\partial x} + \tilde{C}(\varphi, x, t) =$$

$$\tilde{C}(\varphi, x, t) = C(\varphi, x, t) + D(x,t).$$

$$(x,t) \longrightarrow (\xi(x,t), \tau(x,t))$$



$$\frac{\partial \varphi}{\partial t} = \frac{\partial \tau}{\partial t}\frac{\partial \varphi}{\partial \tau} + \frac{\partial \xi}{\partial t}\frac{\partial \varphi}{\partial \xi}$$

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \tau}{\partial x}\cdot\frac{\partial \varphi}{\partial \tau} + \frac{\partial \xi}{\partial x}\cdot\frac{\partial \varphi}{\partial \xi}.$$

$$(*): \left\{A\frac{\partial \tau}{\partial t} + B\frac{\partial \tau}{\partial x}\right\}\frac{\partial \varphi}{\partial \tau} + \left\{A\cdot\frac{\partial \xi}{\partial t} + B\frac{\partial \xi}{\partial x}\right\}$$

(**) $= -\tilde{C}$

We need the PDE → ODE along characteristics

trajectory of solutions.

along characteristics : $\xi(x,t) = \text{const.}$

let $\tau = t \implies (\xi(x,t), \tau)$

const. along char.

varies along char. (plays role of time).

$$(**): \quad A\frac{\partial \varphi}{\partial \tau} + \underbrace{\left\{A\frac{\partial \xi}{\partial t} + B\frac{\partial \xi}{\partial x}\right\}}_{=0} \cdot \frac{\partial \varphi}{\partial \xi} = -\tilde{C}$$

(□).

? . . . why B term vanishes?

"hope".

Along char., we want to be zero :

$$d\xi = 0 = \frac{\partial \xi}{\partial x}dx + \frac{\partial \xi}{\partial t}dt \Rightarrow \frac{\partial \xi}{\partial t} = -\left.\frac{dx}{dt}\right|_{\xi}\frac{\partial \xi}{\partial x}.$$

Along char:

$$\square: \quad A\left(-\left.\frac{dx}{dt}\right|_{\xi}\frac{\partial \xi}{\partial x}\right) + B\left(\frac{\partial \xi}{\partial x}\right) = \frac{\partial \xi}{\partial x}\left(-A\left.\frac{dx}{dt}\right|_{\xi} + B\right)$$

$$\rightsquigarrow \left.\frac{dx}{dt}\right|_{\xi} = \frac{B}{A}.$$

$$\left.\frac{\partial \varphi}{\partial \tau}\right|_{\xi} = -\frac{\tilde{C}}{A}.$$

} System of ODEs.

↳ what we have derived last week for 1st-order ODE

Burgers' Equation

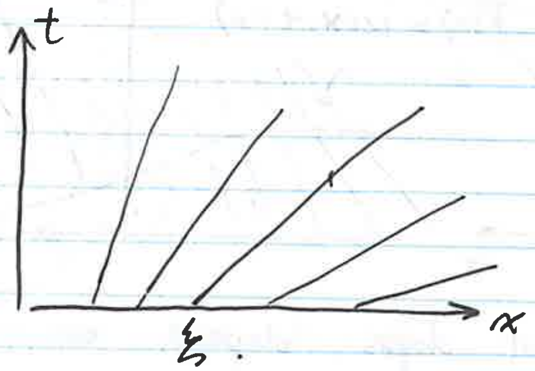
1st order nonlinear PDE.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

$$\left.\begin{aligned} \left.\frac{dx}{dt}\right|_{\xi} &= u. \\ \left.\frac{du}{dt}\right|_{\xi} &= 0 \end{aligned}\right\} \rightarrow u(x,t) = G(\xi).$$

$$\left.\frac{dx}{dt}\right|_{\xi} = \text{const.}$$

char. are straight lines.



$x = G(\xi)t + \xi.$

char. curves. → family of straight lines

→ Impose I.C. : $u(x, t=0) = G(x - G(\xi)t) = G(x)$

$= F(x)$

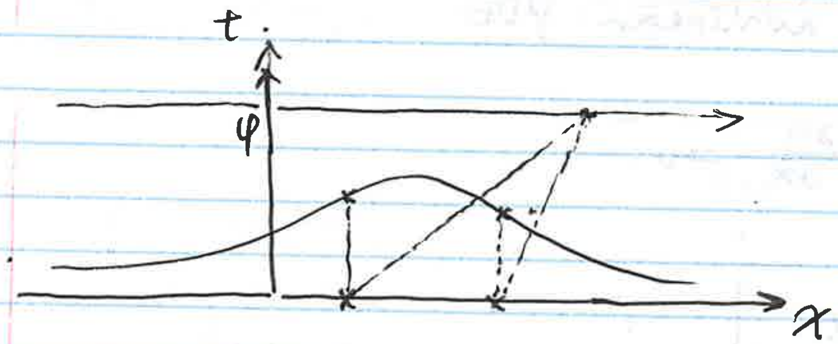
eq., $u(x, t=0) = F(x)$.

Solution: $u(x,t) = F(\xi)$.

$u(x,t) = F(x - ut)$.

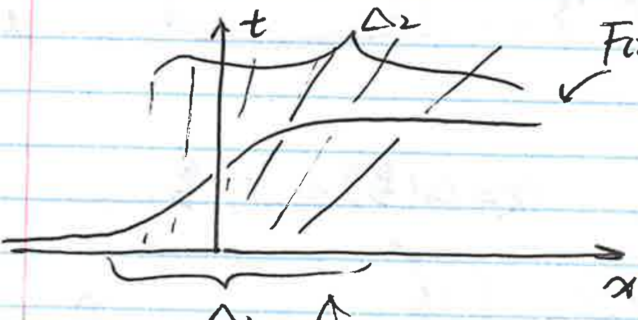
Implicit equation.

$F(x) = e^{-x^2} \rightsquigarrow u = e^{(x-ut)^2}$.

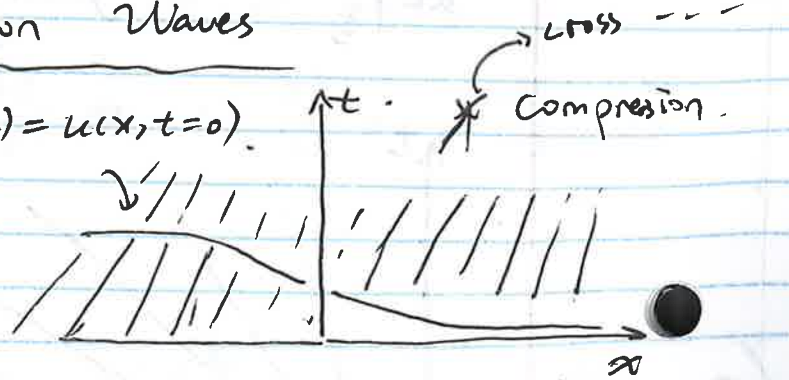


e.g., Bisection methode, Newton-Raphson, Secant meth.)

Expansion & Compression Waves



$F(x) = u(x, t=0)$.



change of slopes depend on $u(0)$.

$\Delta_2 > \Delta_1 \rightarrow$ expansion waves

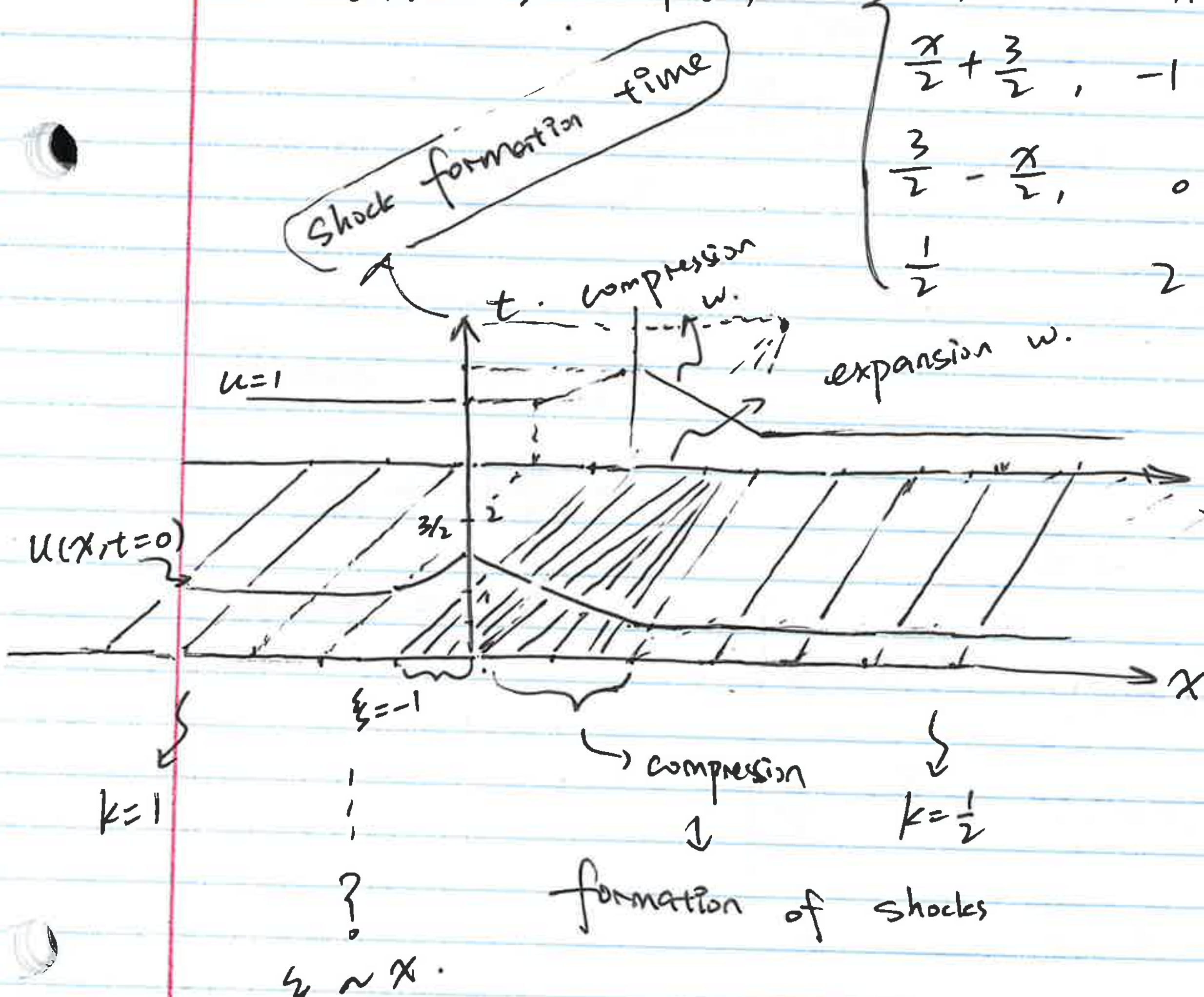
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

→ if "characteristic crossing" : formation of shocks.

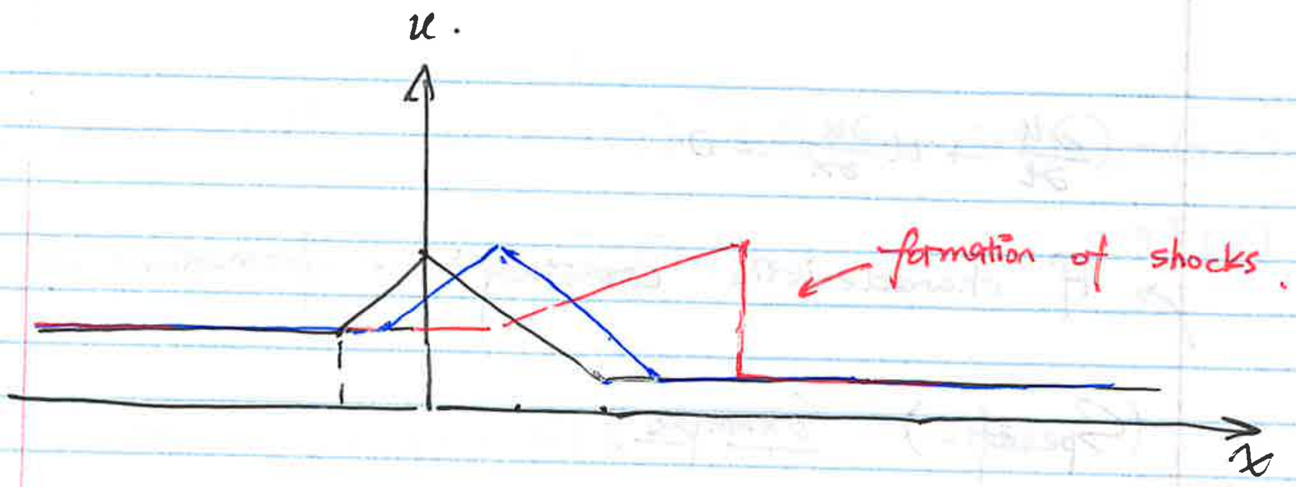
(Specific) Example.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

$$u(x, t=0) = F(x) = \begin{cases} 1, & -\infty < x < -1 \\ \frac{x}{2} + \frac{3}{2}, & -1 \le x < 0 \\ \frac{3}{2} - \frac{x}{2}, & 0 \le x < 2 \\ \frac{1}{2} & 2 \le x < \infty \end{cases}$$



Remark: the definition of "slope" is reversed w.r.t. char. lines compared w/ normal context



Lecture 4. 1/18/2024.

Recap for HW: Equation of char.: $\left.\frac{dx}{dt}\right|_{\xi} = \frac{B}{A}$.

Pb. 2 ~ char. solution: $\left.\frac{du}{dt}\right|_{\xi} = -\frac{\hat{C}}{A}$.

unique analytical: $u(x,t)$

Pb. 3 ✓ $\vec{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$ ⇝ 1D: $\frac{d\varphi/dx}{|d\varphi/dx|}$

φ_0 :

$(t=0)$.

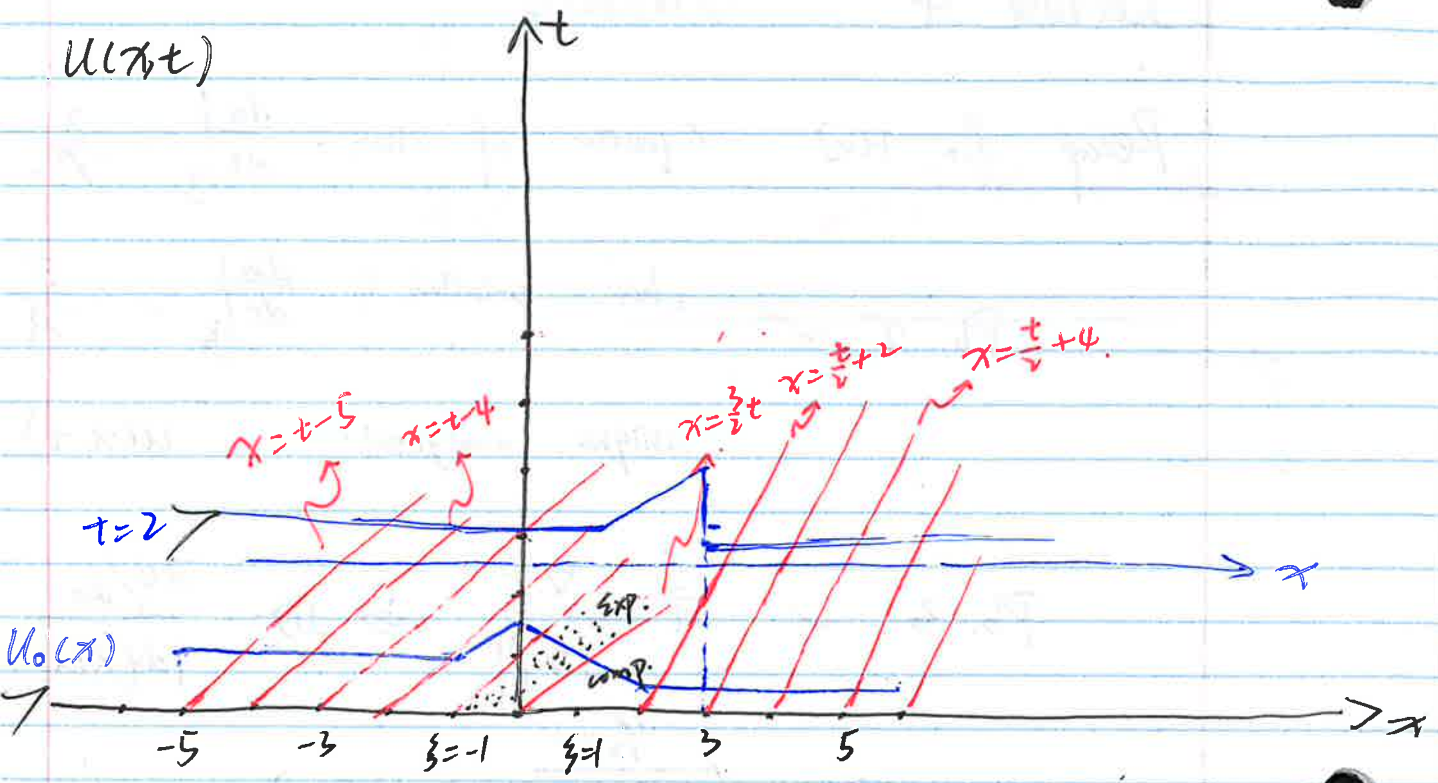
$\Downarrow$

$n=1$

Burgers' eqn.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

$$u(x,0) = u_0(x). = \begin{cases} 1 & x < -1 \\ \frac{3}{2} + \frac{x}{2}. & -1 \le x < 0. \\ \frac{3}{2} - \frac{x}{2}. & 0 \le x < 2 \\ \frac{1}{2} & 2 < x. \end{cases}$$



Equation for char.: $\left.\frac{dx}{dt}\right|_{\xi} = u$. $\quad x = u(\xi)t + \xi$.

char. soln curve: $u = F(\xi)$.

I.C.s: $F(\xi(x,t=0)) = F(x) = u_0(x)$

$u(x,t=0)$ ⟶ u_0

$F(\xi) = u_0(\xi)$

$\xi < -1$ & $u_0(\xi) = 1 \Rightarrow x = t + \xi$.

$2 < \xi$: $u_0(\xi) = 1/2 \Rightarrow x = \frac{t}{2} + \xi$

$-1 \le \xi < 0$: $u_0(\xi) = \frac{3}{2} + \frac{\xi}{2}$.

$$x = \xi + u_0(\xi)t = \xi + \left(\frac{3}{2} + \frac{\xi}{2}\right)t$$

$$= \xi\left(1 + t/2\right) + \frac{3}{2}t.$$

$$\Rightarrow \xi(x,t) = \frac{x - \frac{3}{2}t}{1 + \frac{t}{2}}$$

$$u(x,t) = u_0(\xi) = \frac{3}{2} + \frac{\xi}{2} = \frac{3}{2} + \frac{x/2}{1+t/2} - \frac{3t/4}{1+t/2}.$$

expansion Zone: $t - 1 \le x < \frac{3}{2}$

Recall: $u(x,t) = \frac{3}{2} + \frac{x/2}{1+t/2} - \frac{3t/2}{1+t/2}$

$0 \le \xi < 2$

Compression zone

$\frac{3}{2}t \le x < \frac{t}{2} + 2$

⟶ $u = u_0(\xi) = \frac{3}{2} - \frac{\xi}{2}$

$$x = u_0(\xi)t + \xi = \left(\frac{3}{2} - \frac{\xi}{2}\right)t + \xi$$

$$x = \xi\left(1 - t/2\right) + \frac{3}{2}t.$$

$$\Rightarrow \xi(x,t) = \frac{x - \frac{3}{2}t}{1 - t/2}$$

@ $t=2$: $x = \xi\left(1 - \frac{2}{2}\right) + \frac{3}{2}t$

$$= \frac{3}{2}t.$$

$$u(x,t) = u_0(\xi) = \left(\frac{3}{2} - \frac{\xi}{2}\right)t + \xi.$$

$$= \left(\frac{3}{2} - \frac{x - 3/2\,t}{2-t}\right)t + \frac{x - \frac{3}{2}t}{1 - t/2}$$

$$u(x,t) = \frac{3}{2} - \frac{x - \frac{3}{2}t}{2-t} = \frac{3}{2} - \frac{x}{2-t} + \frac{\frac{3}{2}t}{2-t}.$$

@ $x=3$, $t=2$ u is multi-valued.

Analytical Solution:

$$u(x,t) = \begin{cases} 1 & x < -1+t. \\ \frac{3}{2} - \frac{3t/4}{1+t/2} + \frac{x/2}{1+t/2} & -1+t \le x < \frac{3}{2}t \\ \frac{3}{2} + \frac{3t/2}{2-t} - \frac{x}{2-t} & \frac{3}{2}t \le x < \frac{t}{2}+2 \\ 1/2 & \frac{t}{2}+2 \le x \end{cases}$$

Shock formation Time. * WHY???

Burgers' eqn. $u(x,t) = \overset{u_0}{F}(\underbrace{x - u(x,t)t}_{\xi})$

$\frac{\partial u}{\partial x}$ blow up. @ shock. $\rightarrow \frac{\partial u}{\partial x} = \frac{\partial F(\xi)}{\partial x}$

$$= \frac{dF}{d\xi}\frac{\partial \xi}{\partial x}$$

$$= \frac{dF}{d\xi}\left(1 - \frac{\partial u(x,t)}{\partial x}t\right)$$

$$\frac{\partial u}{\partial x} = \overset{u_0(\xi)}{\frac{dF}{d\xi}} - \frac{dF}{d\xi}t \cdot \frac{\partial u}{\partial x}.$$

$$= \frac{\partial u}{\partial x}\left(1+\frac{dF}{d\xi}t\right). = \frac{dF}{d\xi}.$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\frac{dF}{d\xi}}{1+\frac{dF}{d\xi}t}.$$

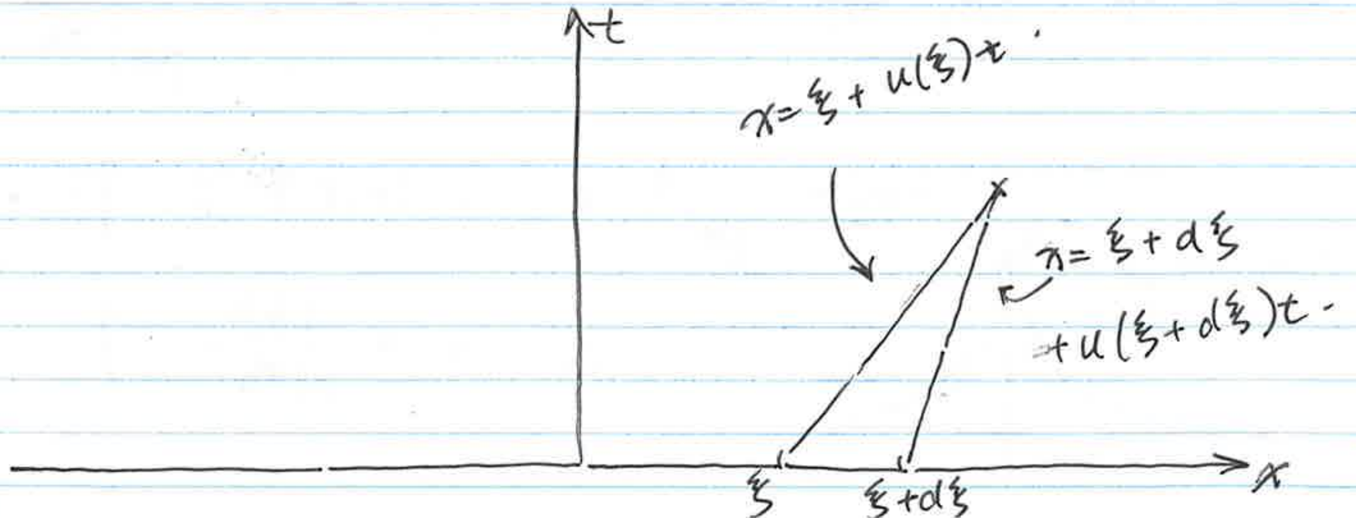
F is a smooth function.

$\frac{\partial u}{\partial x} \rightarrow \infty.$ IFF $1+\frac{dF}{d\xi}t=0.$

$$\Rightarrow t = \frac{-1}{\frac{dF}{d\xi}(\xi)}.$$

$$t_{shock} = \min_{\xi}\left[\frac{-1}{\frac{dF}{d\xi}(\xi)}\right]$$

Geometric Interpretation.



$0 = \cancel{\xi} + d\xi + u_0(\xi + d\xi)t - \cancel{\xi} - u_0(\xi)t.$

$d\xi = \left[u_0(\xi) - u_0(\xi + d\xi)\right] t_{shock}.$

assume: u_0 is smooth, → Taylor's Expansion

$u_0(\xi) + \frac{\partial u_0}{\partial \xi} d\xi + \mathcal{O}(d\xi^2).$

$d\xi = -\frac{\partial u_0}{\partial \xi} \times d\xi \; t.$

more generally: $x = u_0(\xi)t + \xi.$

$$\boxed{\left.\frac{dx}{d\xi}\right|_t = 0}$$

$\frac{\partial u_0(\xi)}{\partial \xi} t + 1 = 0 \Rightarrow t_{shock} = \frac{-1}{\frac{\partial u_0}{\partial \xi}}.$

Problem Session 2

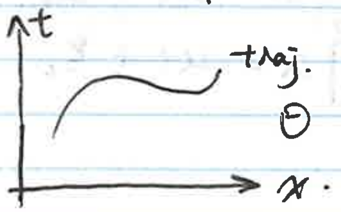
1/19/2024.

$$\frac{\partial \phi}{\partial t} + u(\phi, x, t)\frac{\partial \phi}{\partial x} = S(\phi, x, t).$$

domain: $x \in (-\infty, +\infty)$, $t \in [0, \infty)$.

I.C., $\phi(x, t=0) = f(x)$.

Method of Characteristics. (MoC).



→ Characteristics.

⇓

convert to sys. of ODEs.

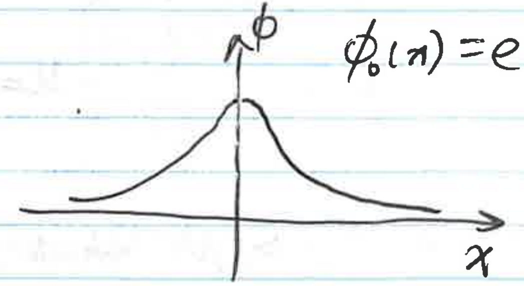
$\frac{d}{d\theta}(\ ) = \frac{dt}{d\theta}\cdot\frac{\partial}{\partial t} + \frac{dx}{d\theta}\cdot\frac{\partial}{\partial x}.$ $\quad x = x(\theta)$, $t = t(\theta)$.

$\theta \equiv t.$ $\quad \left.\frac{d}{dt}\right|_{traj.} = 1\cdot\frac{\partial}{\partial t} + \left.\frac{dx}{dt}\right|_{traj}\cdot\frac{\partial}{\partial x}.$

$\left.\frac{dx}{dt}\right|_{traj.} = u(\phi, x, t).$ → ODE for eqn. of char. lines. *

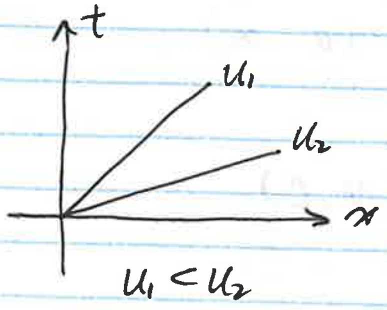
$\left.\frac{d\phi}{dt}\right|_{traj.} = S(\phi, x, t).$ → ODE for char. sol'ns. curves.

- if $S(\phi, x, t) = 0$. → $\left.\phi\right|_{traj.} = const.$
- if $\left.\frac{d\phi}{dt}\right|_{\xi} = \phi$ ...



$\phi_0(x) = e^{-x^2}$

$$\left.\frac{dx}{dt}\right|_{\xi} = u(\phi, x, t).$$ ⤳ propagation speed of information.



$S=0$, $\left.\frac{dx}{dt}\right|_{\xi} = u = \text{const.}$ … ①

$$\begin{cases} x = ut + \xi_1 \\ x = ut + \xi_2. \end{cases}$$

tilted line

parallel & straight

← $\frac{dx}{dt} = u = 0.$ vertical. … ②

not parallel & straight ←-- $\frac{dx}{dt} = u(\phi).$ $S=0$ … ③

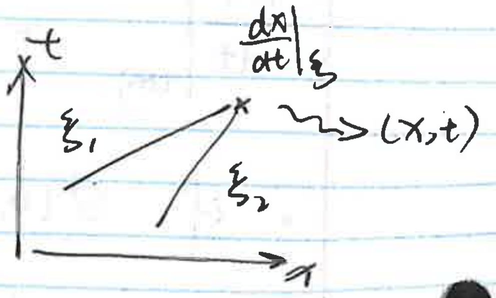
<u>Example</u>: $2xt\frac{\partial \phi}{\partial x} + (1+t^2)\left[\frac{\partial \phi}{\partial t} + \phi\right] = 0.$

I.C., $\phi(x, t=0) = \tanh(x)$

1st-order in x & t, homog., → $\overset{***}{\phi = 0}$ satisfies.

linear. PDE.

$$\frac{\partial \phi}{\partial t} + \frac{2xt}{(1+t^2)}\cdot\frac{\partial \phi}{\partial x} = -\phi.$$



$u = f(x, t).$

↳ uniquely defined.

--> No shock for a linear PDE.

$$\left.\frac{dx}{dt}\right|_{\xi} = u(\quad, x, t) = \frac{2xt}{1+t^2}.$$

$x = \xi(1+t^2).$ → $\xi = \frac{x}{1+t^2}$

$$\left.\frac{d\phi}{dt}\right|_{\xi} = -\phi \rightarrow \phi = \underset{\phi_0(\xi)}{\underset{\downarrow}{\phi(x,0)}} \exp(-t).$$

$= \tanh(\xi).\ \exp(-t).$

↓

$$\phi(x,t) = \tanh\left(\frac{x}{1+t^2}\right)\exp(-t).$$

2\. $\frac{\partial u}{\partial t} + u^2\frac{\partial u}{\partial x} = \beta u.$ (when $\beta > 0$).

(a).

$\frac{dx}{dt} = u^2,$ ⇒ $= [F(\xi)]^2 \exp(2\beta t).$

$\frac{du}{dt} = \beta u.$ ⇒ $u = F(\xi)\exp(\beta t).$

$$\int_{\xi}^{x} dx = \int_{0}^{t} [F(\xi)]^2 \exp(2\beta\tau)\, d\tau.$$

$$\Rightarrow x = \frac{[F(\xi)]^2}{2\beta}\cdot \exp(2\beta t) - \frac{[F(\xi)]^2}{2\beta} + \xi.$$

shock time : $\left.\frac{dx}{d\xi}\right|_{t} = 0$ check for crossing times/shock f

I.C. $u = \exp(-2x^2)$.

$$u(\xi, 0) = F(\xi) = \exp(-2\xi^2).$$

$$\frac{dx}{d\xi} = \frac{(4\xi)\exp(-2\xi^2)}{2\beta}\left[1 - \exp(2\beta t)\right] + 1 = 0.$$

$$\Rightarrow \quad 1 - \exp(2\beta t_c) = \frac{-2\beta\exp(2\xi^2)}{4\xi}.$$

$$t_c = \frac{1}{2\beta}\ln\left[1 + \frac{\beta\exp(2\xi^2)}{2\xi}\right].$$

↗ "Smallest t_c is the shock formation time".

To find t_{shock}, minimize t_c as a function of ξ.

$$\frac{dt_c}{d\xi} = 0$$

$$\hookrightarrow \quad t_{shock} = \frac{1}{2\beta}\ln(1 + \beta\sqrt{e}).$$

for $\beta = 0$. case, $u = F(\xi)$.

$$\frac{dx}{dt} = [F(\xi)]^2$$

$$\Rightarrow x = [F(\xi)]^2 t + \xi.$$

$$x = \left[\exp(-2\xi^2)\right]^3 t + \xi.$$

if $\beta \to 0$, $t_{shock} = \frac{\beta\sqrt{e}}{2\beta} = \frac{\sqrt{e}}{2}$ for $\beta = 0$.

HW: 3b.

$$\begin{cases} x' = x + Ut \\ t' = t \end{cases} \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + \frac{\partial}{\partial x'}\cdot\overset{\curvearrowright U}{\frac{dx'}{dt}}.$$

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x} + \frac{\partial}{\partial t'}\cdot\cancelto{0}{\frac{\partial t'}{\partial x'}}$$

$$\hookrightarrow \frac{\partial\phi}{\partial t'} + \frac{\partial\phi}{\partial x'}U + U\frac{\partial\phi}{\partial x'}$$ ← U not eliminated.

if chose $\begin{cases} x' = x - Ut \\ t' = t. \end{cases}$

$$\frac{\partial\phi}{\partial t'} \underbrace{- \frac{\partial\phi}{\partial x'}U + U\frac{\partial\phi}{\partial x'}}_{\text{cancels.}}$$

$$\frac{\partial\phi}{\partial t'} = RHS.$$

$$= \gamma\frac{d}{dx'}\left[\varepsilon\frac{d\phi}{dx} - \phi(1-\phi)\right]$$

$\phi_{eq}(x')$: $f(x')$ ↖ replace $x' = x + Ut$

Using the coordinate transformation

$$\frac{\partial\phi}{\partial t} + U\frac{\partial\phi}{\partial x} = \gamma\frac{\partial}{\partial x}\left[\varepsilon\frac{\partial\phi}{\partial x} - \phi(1-\phi)\right]$$

↳ ODE. w.r.t. x'

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial t'} - \frac{\partial\phi}{\partial x'}\cdot U$$

choose $x' = x - Ut$.

$$\frac{\partial\phi}{\partial x'} = \frac{\partial\phi}{\partial x}$$

$$\frac{\partial\phi}{\partial t'} - \frac{\partial\phi}{\partial x'}\cdot U + U\frac{\partial\phi}{\partial x'} = \gamma\frac{\partial}{\partial x'}\left[\varepsilon\frac{\partial\phi}{\partial x'} - \phi(1-\phi)\right]$$

Setting LHS $=0$ → Solve for RHS $=0$.

$$\gamma\frac{\partial}{\partial x'}\left[\varepsilon\frac{\partial\phi}{\partial x'} - \phi(1-\phi)\right] = 0$$

$$\varepsilon\frac{\partial^2\phi}{\partial x'^2} = \frac{\partial}{\partial x'}\left[\phi(1-\phi)\right].$$

Integrating on both sides.

$$\varepsilon\frac{d\phi}{dx'} - \phi(1-\phi) = \cancelto{0}{C_1}$$

→ away from interfene LHS=0

Need to satisfy the I.C.s

$$\left(\frac{\phi}{1-\phi}\right)' = \frac{1-\phi-\phi(-1)}{(1-\phi)^2} = \frac{1}{(1-\phi)^2}$$

$$\frac{d\phi}{\phi(1-\phi)} = \frac{1}{\varepsilon}dx'.$$

further integration:

$$(*)\ \cdots\quad \varepsilon\frac{d\phi}{dx'} = \phi(1-\phi).$$

$$\int\frac{1}{\phi(1-\phi)}d\phi = \left(\frac{x'}{\varepsilon}\right) + C$$

$$\phi_{x'} = \frac{1}{\varepsilon}(\phi - \phi^2)$$

$$\phi = \frac{1}{\varepsilon}\int\phi(1-\phi)\,dx'.$$

@ $x' = 0$ & random x' ↓ ϕ preserves proparties.

$x' = x - Ut$

$x' = x$

$$\frac{\phi}{1-\phi} = D\exp\left(\frac{x'}{\varepsilon}\right).$$

$$\phi(x') = \frac{D\exp(x'/\varepsilon)}{1 + D\exp(x'/\varepsilon)}. = 0.5 + 0.5x$$

partial fraction for integration.

expanding $(*)$:

$$\frac{d\phi}{\phi(1-\phi)} = \frac{dx'}{\varepsilon}.$$

$$\frac{(1-\phi+\phi)\,d\phi}{\phi(1-\phi)} = \frac{dx'}{\varepsilon}.$$

$$\int\left[\frac{1}{\phi} + \frac{1}{1-\phi}\right]d\phi = \frac{1}{\varepsilon}\int dx'$$

$$\ln\phi + \ln(1-\phi) = \left(\frac{x'}{\varepsilon}\right) + C$$

$$\ln[\phi(1-\phi)] = \left(\frac{x'}{\varepsilon}\right) + C$$

$$\phi(1-\phi) =$$

Substituting ICs:

$$C_2 \exp\left(\frac{x}{\varepsilon}\right) = \left(\frac{1}{2}+\frac{1}{2}x\right)\left[1+C_2\exp\left(\frac{x}{\varepsilon}\right)\right]$$

$$= \frac{1}{2} + \frac{1}{2}C_2\exp\left(\frac{x}{\varepsilon}\right) + \frac{x}{2} + \frac{x}{2}C_2\exp\left(\frac{x}{\varepsilon}\right)$$

$$= \frac{1}{2}+\frac{x}{2} + C_2\left[\frac{1}{2}\exp\left(\frac{x}{\varepsilon}\right)+\frac{x}{2}\exp\left(\frac{x}{\varepsilon}\right)\right]$$

$$C_2 \cdot \exp\left(\frac{x}{\varepsilon}\right)\left[1-\frac{1}{2}-\frac{x}{2}\right] = \frac{1}{2}+\frac{x}{2}.$$

$$C_2 = \frac{1}{\exp(x/\varepsilon)} \cdot \frac{1+x}{1-x}.$$

→ interface solution:

$$\phi(x-Ut) = \frac{\frac{1}{\exp(x/\varepsilon)}\frac{1+x}{1-x}\exp[(x-Ut)/\varepsilon]}{1+\frac{1}{\exp(x/\varepsilon)}\frac{1+x}{1-x}\exp[(x-Ut)/\varepsilon]}$$

$$\phi(x-Ut) = \frac{(1+x)\exp\left(\frac{x-Ut}{\varepsilon}\right).}{(1-x)\exp\left(\frac{x}{\varepsilon}\right)+(1+x)\exp\left(\frac{x-Ut}{\varepsilon}\right)}$$

For the "non-interfacial" part:

$$\frac{C_2\exp(x/\varepsilon)}{1+C_2\exp(x/\varepsilon)} = 0 \quad \text{or} \quad 1$$

if $\phi \equiv 1$:

$$C_2\exp\left(\frac{x}{\varepsilon}\right) = 1 + C_2\exp\left(\frac{x}{\varepsilon}\right).$$

Equation Derivation for 3(c).

$$0.5 + 0.5\left\{x-\gamma\left[1-\frac{x-\gamma t+1}{2-2\gamma t}\right]\right\}.$$

$$0.5 + 0.5\left[x-\gamma+\gamma\cdot\frac{x-\gamma t+1}{1-\gamma t}\right]$$

$$0.5 + 0.5x - 0.5\gamma + 0.5\gamma\frac{(x-\gamma t+1)}{1-\gamma t}$$

$$0.5 + 0.5x + 0.5\gamma\left(\frac{x-\gamma t+1}{1-\gamma t} - \frac{2-2\gamma t}{1-\gamma t}\right)$$

$$0.5 + 0.5x + 0.5\gamma\left(\frac{x-\cancel{\gamma t}+1-2+\cancel{2\gamma t}}{1-\gamma t}\right).$$

$$0.5(1+x) + 0.5\gamma\left(\frac{\gamma t-1+x}{1-\gamma t}\right)$$

$$0.5(1+x) + 0.5\gamma\left(-1+\frac{x}{1-\gamma t}\right).$$

$$0.5(1+x) - \frac{\gamma}{2} + \frac{\gamma}{2}\cdot\frac{x}{1-\gamma t}$$

Simplifying 3(d).

$$0.5 + 0.5\left\{x - \gamma t - \gamma t\frac{x-\gamma t - Ut+1}{1-\gamma t} - Ut\right\}.$$

$$0.5 + 0.5x - 0.5\gamma t - 0.5\cdot\gamma t\cdot\frac{x-\gamma t - Ut+1}{1-\gamma t} - Ut.$$

$$\frac{x+1}{2} - 0.5\gamma t\left[\frac{1-\gamma t + x - \gamma t - Ut + 1}{1-\gamma t}\right] - Ut$$

$$\frac{x+1}{2} - 0.5\gamma t\cdot\left[\frac{2(1-\gamma t)+x-Ut}{1-\gamma t}\right] - Ut$$

$$\frac{x+1}{2} - \gamma t - \frac{0.5\gamma t(x-Ut)}{1-\gamma t} - Ut$$

$$x = \gamma t - \gamma t\frac{x-\gamma t+1}{1-\gamma t} + \frac{x}{1-\gamma t}$$

Lecture 5 1/23/2024.

$\varphi(x,t)$.

$$A\frac{\partial\varphi}{\partial t} + B\frac{\partial\varphi}{\partial x} + \tilde{C} = 0 \Rightarrow \frac{\partial\varphi}{\partial t} + \frac{B}{A}\cdot\frac{\partial\varphi}{\partial x} = -\frac{\tilde{C}}{A}.$$

$$\left.\frac{d\varphi}{dt}\right|_{\xi} = \frac{\partial\varphi}{\partial t} + \left.\frac{dx}{dt}\right|_{\xi}\frac{\partial\varphi}{\partial x} = -\frac{\tilde{C}}{A}.$$

$$\left.\frac{dx}{dt}\right|_{\xi} = \frac{B}{A}. \quad \& \quad \left.\frac{d\varphi}{dt}\right|_{\xi} = -\frac{\tilde{C}}{A}.$$

on characteristics: $\frac{dt}{A} = \frac{dx}{B} = \frac{d\varphi}{-\tilde{C}}$.

i.e., const.

$\varphi(x, y, z, t)$

$$A\frac{\partial\varphi}{\partial t} + B\frac{\partial\varphi}{\partial x} + C\frac{\partial\varphi}{\partial y} + D\frac{\partial\varphi}{\partial z} = -\tilde{E}$$

$$\downarrow$$

$$\frac{\partial\varphi}{\partial t} + \frac{B}{A}\cdot\frac{\partial\varphi}{\partial x} + \frac{C}{A}\cdot\frac{\partial\varphi}{\partial y} + \frac{D}{A}\cdot\frac{\partial\varphi}{\partial z} = -\frac{\tilde{E}}{A}.$$

$$\left.\frac{d\varphi}{dt}\right|_{\xi}\frac{\partial\varphi}{\partial t} + \left.\frac{dx}{dt}\right|_{\xi}\frac{\partial\varphi}{\partial x} + \left.\frac{dy}{dt}\right|_{\xi}\frac{\partial\varphi}{\partial y} + \left.\frac{dz}{dt}\right|_{\xi}\frac{\partial\varphi}{\partial z} = -\frac{\tilde{E}}{A}.$$

$$\left.\frac{dx}{dt}\right|_{\xi} = \frac{B}{A}, \quad \left.\frac{dy}{dt}\right|_{\xi} = \frac{C}{A}, \quad \left.\frac{dz}{dt}\right|_{\xi} = \frac{D}{A}.$$

$$\left.\frac{d\varphi}{dt}\right|_{\xi} = -\frac{\tilde{E}}{A}$$

Alternatively:

$$\frac{dt}{A} = \frac{dx}{B} = \frac{dy}{C} = \frac{dz}{D}. = \frac{d\varphi}{-\tilde{E}}$$

Conservation Laws.

"speed-up" of characteristics.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$ Primitive Form.

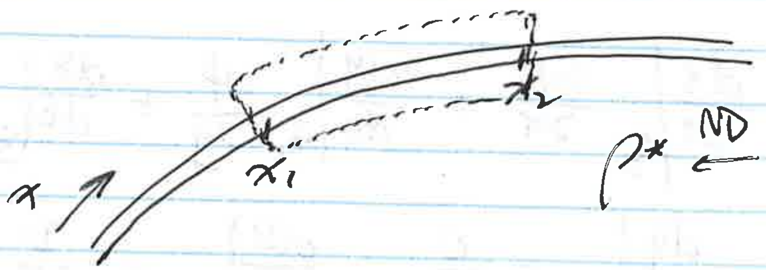
$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$$ Conservative Form.

flux term.

General conservation law. (1D).

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} F(u, x, t). = 0.$$

Example Traffic flow.
~~[illegible]~~



$\rho^* \xleftarrow{ND} \rho$ = traffic density

$$\frac{\# \text{cars}}{\text{length}}.$$

$F(\rho, x, t)$ = flux of cars:

$$\frac{\# \text{cars}}{\text{time}}.$$

F^*

% Conservation within the control volume.

$$\int_{x_1}^{x_2} \rho^*(x, t_2).dx - \int_{x_1}^{x_2} \rho^*(x, t_1)\, dx$$

$$= \int_{t_1}^{t_2} \left[F^*(x_1, t) - F^*(x_2, t)\right] dt.$$

Conservation law in integral form.

↓

*** always holds!!!

$$\left.\begin{array}{l} x_2 \to x_1 \\ t_2 \to t_1. \end{array}\right] \to \begin{array}{l} x_2 = x_1 + dx. \\ t_2 = t_1 + dt. \end{array}$$ @ lim dx & dt → 0

$$\frac{\partial \rho^*}{\partial t} . + \frac{\partial}{\partial x} F^* = 0.$$ differential conservation law of cars.

✓

Remark: F^* has to be differentiable, so this form may not hold universally :(

For cars: $F^*(\rho^*) = \rho^* C^*(\rho^*)$.

traffic density $\left(\frac{\text{cars}}{m}\right)$

traffic speed $\left(\frac{m}{s}\right)$.

Conservative form : $\frac{\partial \rho^*}{\partial t} + \frac{\partial}{\partial x}\left(\overset{\text{assumed.}}{C^*(\rho^*)}\,\rho^*\right) = 0.$

Primitive form : $\frac{\partial \rho^*}{\partial t} + \underbrace{\left(\frac{\partial C^*}{\partial \rho^*}\rho^* + C^*\right)}_{\frac{\partial F^*}{\partial \rho^*}} \frac{\partial \rho^*}{\partial x} = 0$

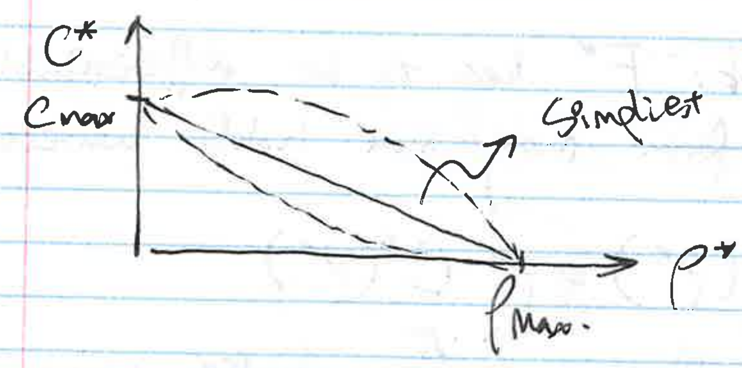
i.e., speed of characteristics.

$$\left.\frac{dx}{dt}\right|_{\xi} = \frac{dC^*}{d\rho^*}\rho^* + C^* \leftarrow \text{func.}(\rho^*)$$

$$\left.\frac{d\rho^*}{dt}\right|_{\xi} = 0$$

Characteristics are a family of straight lines.

$C^*(\rho^*)$



Simpliest mod.: straight line.

$$C^* = C_{max}\left(1 - \frac{\rho^*}{\rho_{max}}\right).$$

"linear model"

nondimensionalize.

$$\rho = \frac{\rho^*}{\rho_{max}} \qquad c = \frac{C^*}{C_{max}}.$$

$C(\rho) = 1 - \rho \rightarrow$ linear model

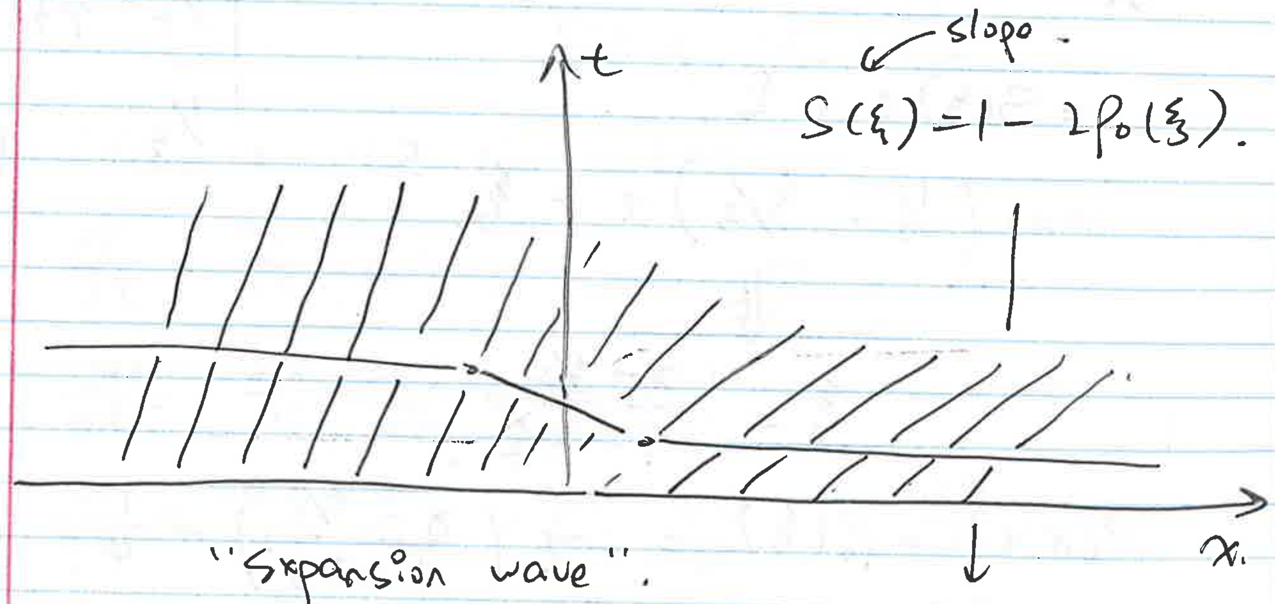
in conservative form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho C(\rho)\right) = 0.$$

$$\frac{\partial \rho}{\partial t} + \left(c + \rho\frac{dc}{d\rho}\right)\frac{\partial \rho}{\partial x} = 0.$$

$$\frac{\partial \rho}{\partial t} + (1 - 2\rho)\frac{\partial \rho}{\partial x} = 0.$$

$$\underbrace{\left.\frac{dx}{dt}\right|_{\xi} = 1 - 2\rho. \quad \left.\frac{d\rho}{dt}\right|_{\xi} = 0}$$



$$x = S(\xi)t + \xi.$$

$$\rho(x,t) = \rho_0(\xi).$$

ill.-defined later.

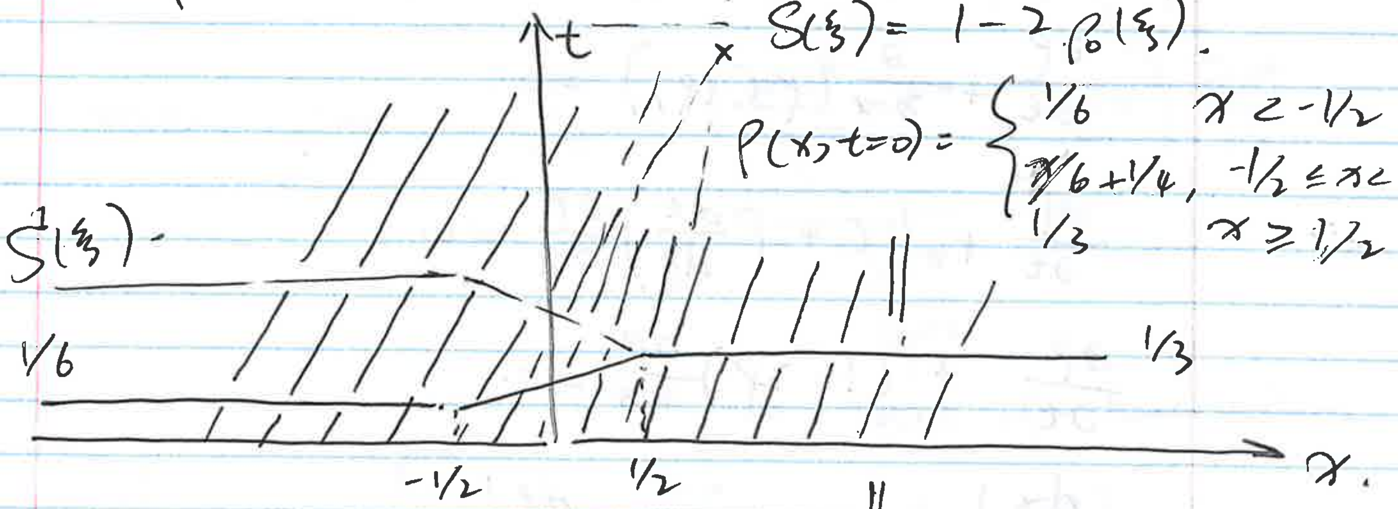
↑

"discontinuous form"

Compression wave.

$S(\xi) = 1 - 2\rho_0(\xi)$.

$$\rho(x, t=0) = \begin{cases} 1/6 & x < -1/2 \\ \cancel{x}/6 + 1/4, & -1/2 \le x < 1/2 \\ 1/3 & x \ge 1/2 \end{cases}$$



$$\Downarrow$$

$$S(\xi) = \begin{cases} 2/3. & \xi < -1/2 \\ 1/2 - \xi/3 & -1/2 \le \xi < 1/2 \\ 1/3 & 1/2 \le \xi. \end{cases}$$

$\frac{dx}{dt} = 1 - 2\rho(\xi)$.

$x = S(\xi)t + \xi$.

$x = (1/2 - \xi/3)t + \xi$.

$$\Downarrow$$

$$\xi = \frac{x - t/2}{1 - t/3}.$$

$$\rho(x,t) = \rho_0(\xi) = \frac{1}{6}\left(\frac{x - t/2}{1 - t/3}\right) + \frac{1}{4}.$$

For all char. @ $\underline{\underline{t=3}}$, $\longrightarrow \frac{\partial \rho}{\partial x} \to \infty$

↓

shock formation. time.

$$\left.\frac{dx}{d\xi}\right|_t = 0$$

⤳ look for a weak sol'n : Smooth sol'n + jumps.

We are seeking : shock speed. (shock location)

Sol'n in diff. parts

jump across shock.

Concentrate on one shock : $X_s(t)$.

Coordinate change : $z = x - X_s(t)$.

↗ Riding the shock.

$\rho(x,t) \to \rho(z,\tau) \quad \tau = t$.

$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho c) = 0$ ⤳ z & τ.

$-\dot{X}_s$

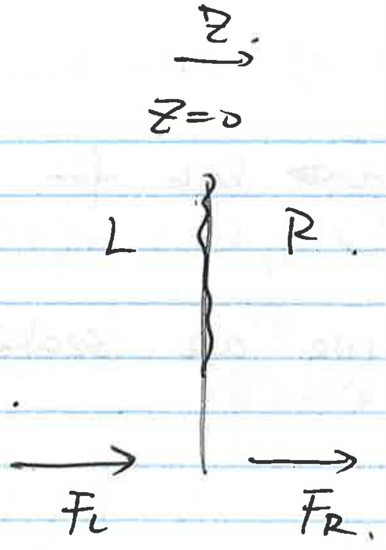
$$\frac{\partial(\cdot)}{\partial t} = \frac{\partial z}{\partial \tau} \cdot \frac{\partial(\cdot)}{\partial z} + \frac{\partial \tau}{\partial t} \cdot \frac{\partial(\cdot)}{\partial \tau}. \qquad \tau = t.$$

$$\frac{\partial(\cdot)}{\partial x} = \underset{1}{\underset{\parallel}{\frac{\partial z}{\partial x}}} \cdot \frac{\partial(\cdot)}{\partial z} + \underset{0}{\underset{\parallel}{\frac{\partial \tau}{\partial x}}} \cdot \frac{\partial(\cdot)}{\partial \tau}. \qquad \frac{\partial z}{\partial x} = 1.$$

$$\frac{\partial z}{\partial t} = -\dot{X}_s$$

Original cons. PDE $\Longrightarrow$ $\frac{\partial \rho}{\partial \tau} + \frac{\partial}{\partial z}\left[\underbrace{\rho c - \rho \dot{X}_s}_{\text{new flux in moving frame of reference}}\right] = 0$

Flux across the shock in frame of reference moving with the shock is continuous.



Principal of conservation:

$F_L = F_R$. ← frame of ref. of shock.

$\rho_L C_L(\rho_L) - \rho_L \dot{X}_s = \rho_R C_R(\rho_R) - \rho_R \dot{X}_s$

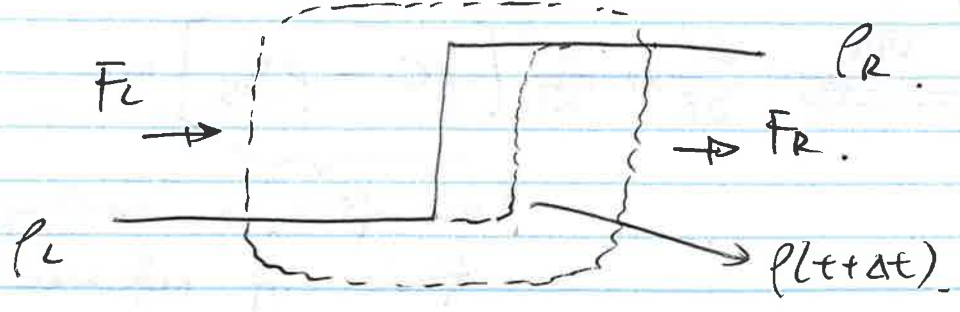
$$\dot{X}_s = \frac{\rho_R C_R(\rho_R) - \rho_L C_L(\rho_L)}{\rho_R - \rho_L}.$$

Hugoniot cond.

traffic problem

in general: $\dot{X}_s = \frac{F_R - F_L}{\rho_R - \rho_L}$. ← fluxes ← state variables.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} F = 0.$$



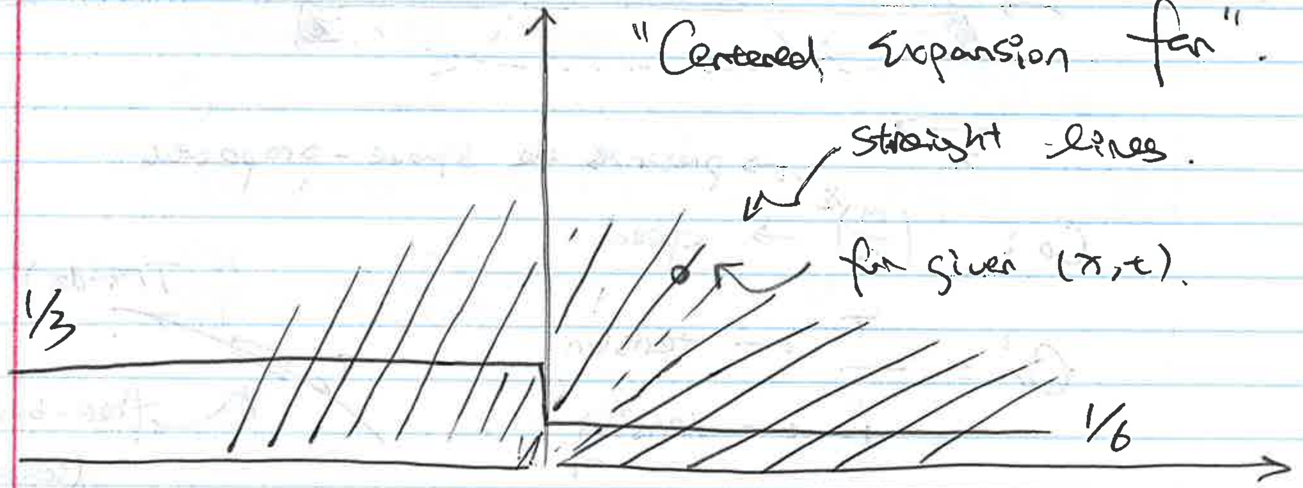
$$\int_{CV} \rho(t+\Delta t)dx - \int_{CV} \rho(t)\,dx = \dot{X}_s \Delta t(\rho_L - \rho_R)$$

$(F_L - F_R)\Delta t$ ← integration of fluxes w.r.t. time.

$$\dot{X}_s = \frac{\rho_R \overset{2/3}{C_R} - \rho_L \overset{5/6}{C_L}}{\underset{1/3}{\rho_R} - \underset{1/6}{\rho_L}} = 1/2.$$

traffic problem continued, with discontinuous I.C.:

"Centered Expansion fan".



$\frac{dx}{dt}\Big|_\xi = 1 - 2\rho$.

(x,t)

⇓

$\frac{d\rho}{dt}\Big|_\xi = 0$.

slope: $\frac{dx}{dt}\Big|_\xi = \frac{x}{t}$.

because char. passes (0,0) $= 1 - 2\rho$.

$\rho(x,t) = \frac{1}{2}\left(1 - \frac{x}{t}\right)$.

lecture 6 1/25/2024.

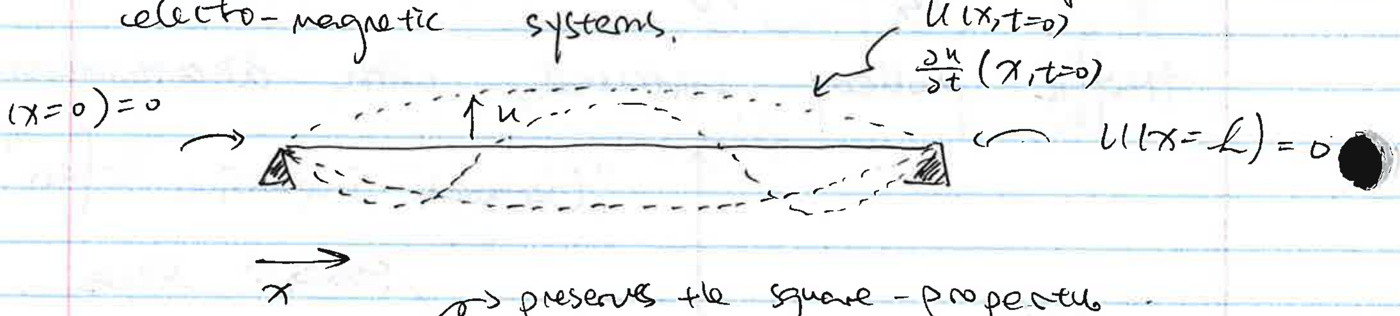
Chapter 3 (Lele). → 2nd-order ODE → 1st orders.

Second-order PDE:

wave-eqn. $u(x,t)$.

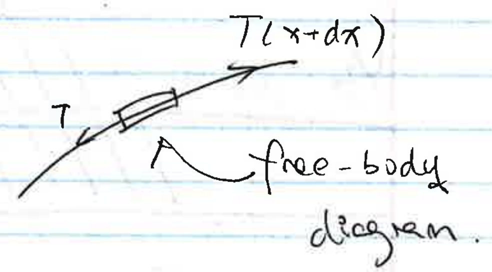
$$\frac{\partial^2 u}{\partial t^2} - C_0^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Sound waves. vibration of a string / membrane.
electro-magnetic systems.



C_0 : $\left(\frac{m}{s}\right)^{\frac{1}{2}}$ → speed
↳ preserves the square-property.

$C_0^2 = \frac{T}{\rho}$ ← tension
← density



D'Alembert's Soln. (holds for infinite domain).

$-\infty < x < \infty$.

$0 \le t < \infty$

I.C. : $u(x, t=0) = f(x)$.

$\frac{\partial u}{\partial t}(x, t=0) = g(x)$.

$u(x,t) = \frac{1}{2}\left[f(x - C_0t) + f(x + C_0t)\right] + \frac{1}{2C_0}\int^{x+C_0t} g(x')\,dx'$

in multi-dimensions, the PDE writes: (wave eqn.)

$$\frac{\partial^2 u}{\partial t^2} - C_0^2 \nabla^2 u = 0.$$

linear PDE : → transform methods.
eigenfunction expansions.

Today: Method of characteristics to convert to a system of 1st-order PDEs

Approach:

- Rewrite as a system of coupled 1st order PDEs
- Decouple the system
- Solve each decoupled ODE/PDEs.
- Construct Solution of original PDEs.

$$\rightarrow \frac{\partial^2 u}{\partial t^2} - C_0^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad (*).$$

$u_1 = \frac{\partial u}{\partial x}$, $u_2 = \frac{\partial u}{\partial t}$.

$$\vec{v} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial t} \end{Bmatrix}$$

$$(*) : \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) - C_0^2 \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = 0$$

$$\frac{\partial u_2}{\partial t} - C_0^2 \frac{\partial u_1}{\partial x} = 0.$$

$$\boxed{\frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial x} = 0}$$

$$\frac{\partial u_1}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x} u\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} u\right) = \frac{\partial u_2}{\partial x}$$

$\mathbb{B}.$

$$\vec{U}: \underset{\overset{\uparrow}{\frac{\partial}{\partial t}\begin{Bmatrix}u_1\\u_2\end{Bmatrix}}}{\frac{\partial \vec{U}}{\partial t}} + \underset{\downarrow}{\begin{bmatrix}0 & -1\\-c^2 & 0\end{bmatrix}} \underset{\overset{\uparrow}{\frac{\partial}{\partial x}\begin{Bmatrix}u_1\\u_2\end{Bmatrix}}}{\frac{\partial \vec{U}}{\partial x}} = \begin{bmatrix}0\\0\end{bmatrix}$$

decompose the matrix

$$\frac{\partial \vec{U}}{\partial t} + \mathbb{B}\frac{\partial \vec{U}}{\partial x} = 0.$$ ← coupled system of 1st-order PDEs.

We see a combination of u_1 & u_2 that decouples the system.

eigenvalues & eigenvectors of $\mathbb{B}$:

$$\mathbb{B}x = \lambda x.$$

$$\mathbb{B} = \begin{bmatrix}0 & -1\\-c_0^2 & 0\end{bmatrix}.$$

$$\det(\mathbb{B} - \lambda I) = 0 \rightarrow \begin{vmatrix}-\lambda & -1\\-c_0^2 & -\lambda\end{vmatrix} = 0.$$

$$\lambda^2 - \lambda_0^2 = 0 \Rightarrow \lambda^{(1)} = c_0 \ \& \ \lambda^{(2)} = -c_0.$$

$$\lambda_0^{(1)} = c_0, \quad \lambda^{(2)} = -c_0.$$

$$x^{(1)} = \begin{bmatrix}1\\-c_0\end{bmatrix}, \quad x^{(2)} = \begin{bmatrix}1\\c_0\end{bmatrix}.$$

$$Q = [x^{(1)}\ x^{(2)}] = \begin{bmatrix}1 & 1\\-c_0 & c_0\end{bmatrix}$$

DIC 2D.

$$\mathbb{B} = Q\Lambda Q^{-1}.$$

$$\Lambda = \begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix} \longrightarrow Q^{-1} = \begin{bmatrix}\frac{1}{2} & -\frac{1}{2c_0}\\\frac{1}{2} & \frac{1}{2c_0}\end{bmatrix}.$$

$$\vec{\mathbb{V}} = Q^{-1}\vec{U} \Rightarrow \vec{U} = Q\vec{\mathbb{V}}$$

$$\vec{V} = \begin{Bmatrix}v_1\\v_2\end{Bmatrix} = \begin{Bmatrix}\frac{1}{2}u_1 - \frac{1}{2c_0}u_2\\\frac{1}{2}u_1 + \frac{1}{2c_0}u_2\end{Bmatrix} \Rightarrow \begin{aligned}V_1 &= \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{1}{2c_0}\frac{\partial u}{\partial t}\right]\\V_2 &= \frac{1}{2}\left[\frac{\partial u}{\partial x} + \frac{1}{2c_0}\cdot\frac{\partial u}{\partial t}\right].\end{aligned}$$

Recall:

$$\frac{\partial \vec{U}}{\partial t} + \mathbb{B}\frac{\partial \vec{U}}{\partial x} = 0.$$

$$\frac{\partial \vec{U}}{\partial t} + Q\Lambda Q^{-1}\frac{\partial \vec{U}}{\partial x} = 0.$$

$$Q^{-1} *: \frac{\partial \vec{V}}{\partial t} + \Lambda\frac{\partial \vec{V}}{\partial x} = 0.$$

$$\frac{\partial}{\partial t}\begin{bmatrix}v_1\\v_2\end{bmatrix} + \begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix}\begin{bmatrix}\frac{\partial v_1}{\partial x}\\\frac{\partial v_2}{\partial x}\end{bmatrix} = 0.$$

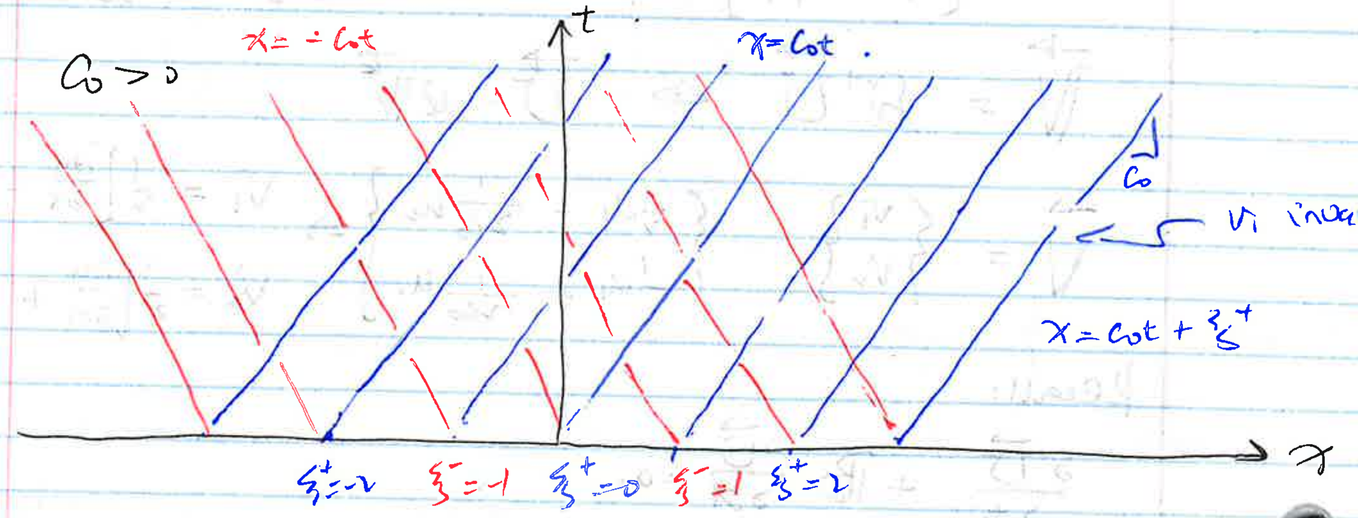
Decoupled: System.
$$\begin{cases}\frac{\partial v_1}{\partial t} + c_0\cdot\frac{\partial v_1}{\partial x} = 0\\[2ex]\frac{\partial v_2}{\partial t} - c_0\cdot\frac{\partial v_2}{\partial x} = 0\end{cases}$$ ← characteristic method.

$$V_1: \text{ on char.: } \left.\frac{dx}{dt}\right|_{\xi^+} = c_0 \ \& \ \left.\frac{dv_1}{dt}\right|_{\xi^+} = 0$$

$$x = c_0t + \xi^+ \Rightarrow V_1(x,t) = F^+(\xi^+)$$ ← just nomination!!

V_2 : on char. : $\left.\frac{dx}{dt}\right|_{\xi^-} = -C_0$ & $\left.\frac{dV_2}{dt}\right|_{\xi^-} = 0.$

$x = -C_0 t + \xi^- \Rightarrow V_2(x,t) = F^-(\xi^-)$



I.C. for V_1 & V_2 :

$V_1 = \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{1}{C_0}\frac{\partial u}{\partial t}\right] \rightarrow V_1(x, t=0) = \frac{f'(x)}{2}$

$V_2 = \frac{1}{2}\left[\frac{\partial u}{\partial x} + \frac{1}{C_0}\frac{\partial u}{\partial t}\right] \rightarrow V_2(x, t=0) = \frac{f'(x)}{2}.$

Sol'n for $u(x, t=0) = f(x).$

$\frac{\partial u}{\partial t}(x, t=0) = 0$

[illegible] [illegible]

$V_1(x,t) = F^+(\xi^+) - F^+(x - C_0 t).$

$V_1(x, t=0) = F^+(x) = \frac{1}{2} f'(x)$

$V_1(x,t) = F^+(x - C_0 t) = \frac{1}{2} f'(x - C_0 t)$

$V_2(x,t) = F^-(\xi^-) = F^-(x + C_0 t).$

$V_2(x, t=0) = F^-(x) = \frac{1}{2} f'(x).$

$V_2(x,t) = F^-(x + C_0 t) = \frac{1}{2} f'(x + C_0 t)$

V_1 & V_2 .

$\vec{U} = Q\vec{V}.$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -C_0 & C_0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \rightarrow \begin{array}{l} u_1 = V_1 + V_2 \\ u_2 = C_0(V_2 - V_1) \end{array}$$

$u_1 = \frac{1}{2}\left[f'(x - C_0 t) + f'(x + C_0 t)\right].$

$\frac{\partial u}{\partial x}$ $\quad$ $\frac{\partial}{\partial x} f'(x - C_0 t).$

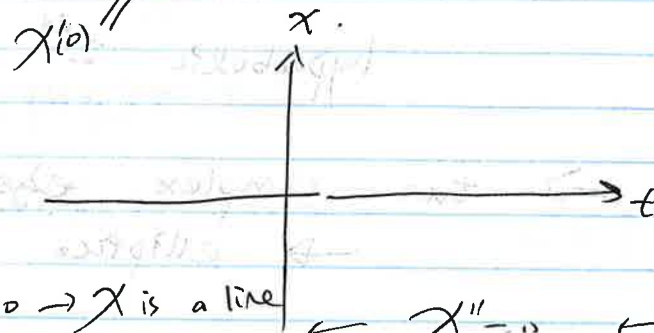
$\xRightarrow{\int dx} u = \frac{1}{2}\left[f(x - C_0 t) + f(x + C_0 t)\right] + \chi(t).$

$u(x, t=0) = f(x) + \chi(0) = f(x).$

$\chi(0) = 0.$

$\cancel{\frac{\partial^2 u}{\partial t^2}}$. $\quad \frac{\partial u}{\partial t}(x, t=0) = (-C_0 f' + C_0 f')/2 = 0.$

$\chi(0)''$



$\frac{\partial^2 u}{\partial t^2} - C_0^2 \cdot \frac{\partial^2 u}{\partial x^2} = 0$

$\downarrow$

$u = \frac{1}{2}\left[f(x + C_0 t) + f(x - C_0\right.$

$\chi(0) = \chi'(0) = 0 \rightarrow \chi$ is a line $\leftarrow \chi'' = 0 \leftarrow \chi'' - C_0^2 \cdot 0 = 0$

Generalization for 2nd-order PDEs in x & y:

$$A\frac{\partial^2 u}{\partial x^2} + B\cdot\frac{\partial^2 u}{\partial x\partial y} + C\frac{\partial^2 u}{\partial y^2} = D$$ ← first order term.

$$u_1 = \frac{\partial u}{\partial x} \quad \& \quad u_2 = \frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial x}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \begin{bmatrix} \frac{B}{A} & \frac{C}{A} \\ -1 & 0 \end{bmatrix}\frac{\partial}{\partial y}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} D/A \\ 0 \end{Bmatrix}$$

$$\frac{\partial \vec{U}}{\partial x} + \mathbb{B}\frac{\partial \vec{U}}{\partial y} = \vec{d}$$

* eigenvalues

if $\mathbb{B}$ can be diagonalized →

$$\frac{\partial \vec{V}}{\partial x} + \Lambda\frac{\partial \vec{V}}{\partial y} = \vec{\tilde{d}}$$ decoupled.

$$\mathbb{B} = \begin{bmatrix} \frac{B}{A} & \frac{C}{A} \\ -1 & 0 \end{bmatrix}$$

$\det(\cancel{\mathbb{B}}) = 0$

$\downarrow$

$\mathbb{B} - \lambda I$

∼ eigenvalues.

$B^2 - 4AC > 0$ → two distinct real eigenvalues.

hyperbolic 2nd order PDE.

$B^2 - 4AC < 0$ ⇒ two complex eigenvalues.

→ elliptice system

uplace eqn.

$$\nu\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$$ Heat eqn.

$B^2 - 4AC = 0$ ⇒ repeated eigenvalues.

→ parabolic 2nd-order PDE.

$$\frac{\partial^2 u}{\partial x^2} - C_0^2\frac{\partial^2 u}{\partial y^2} = 0 \qquad A: C_0^2 > 0.$$ (Example).

Problem Session 3

Q1. $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$ $u(x,t=0) = \begin{cases} 0, & x \le 0 \\ x/t, & 0 < x \le \varepsilon \\ 1, & x > \varepsilon. \end{cases}$

Find Solution. when $\varepsilon \to 0$.

$$\left.\frac{dx}{dt}\right|_{\xi} = u, \quad \left.\frac{du}{dt}\right|_{\xi} = 0 \rightarrow u = F(\xi).$$

soln ← $u(x,t) = \begin{cases} 0 & x \le 0. \\ x/t+\varepsilon & 0 < \frac{\varepsilon x}{t+\varepsilon} \le \varepsilon \\ 1. & x - t > \varepsilon. \end{cases}$

$$\xi = \frac{\varepsilon x}{t+\varepsilon}$$

$$x = \frac{\xi}{\varepsilon}t + \xi$$

$$= F(\xi)t + \xi.$$

$$x = u(\xi,t)\,t + \xi.$$

$\rightarrow u(\xi,t) = F(\xi).$

$u(\xi(x,t),t) = F(\xi)$

$u(\xi(x,t=0),t=0) = F(\xi)$

ξ

$u(x,t=0) = F(\underbrace{\xi(x,t=0)}_{x}).$

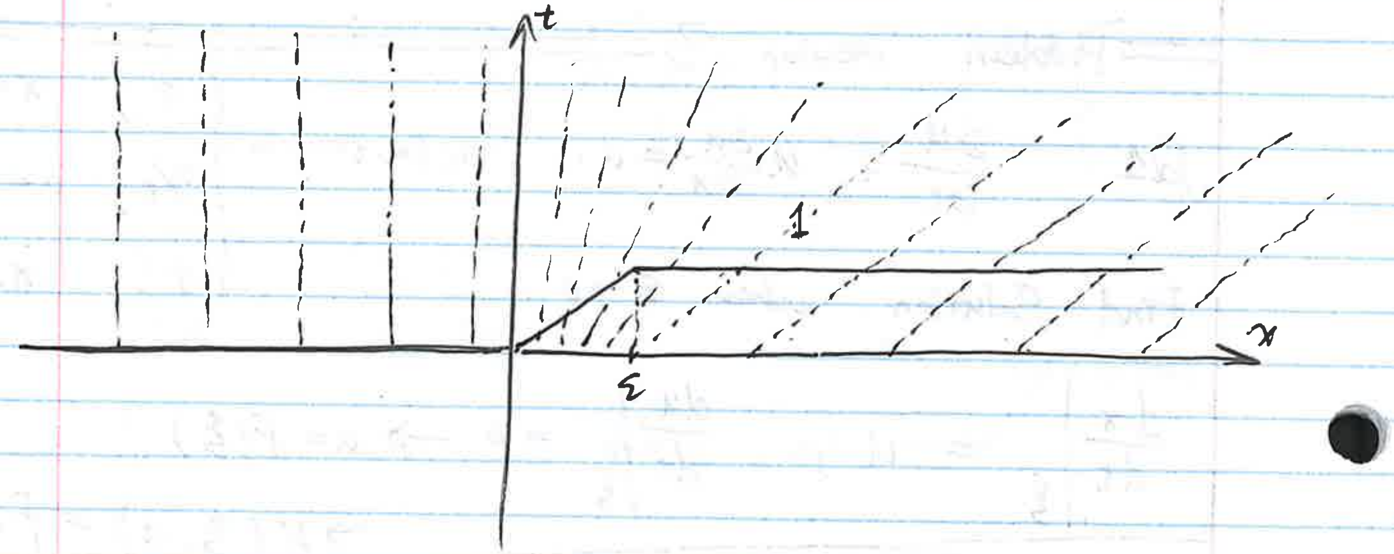
$u(x,t=0) = F(x).$

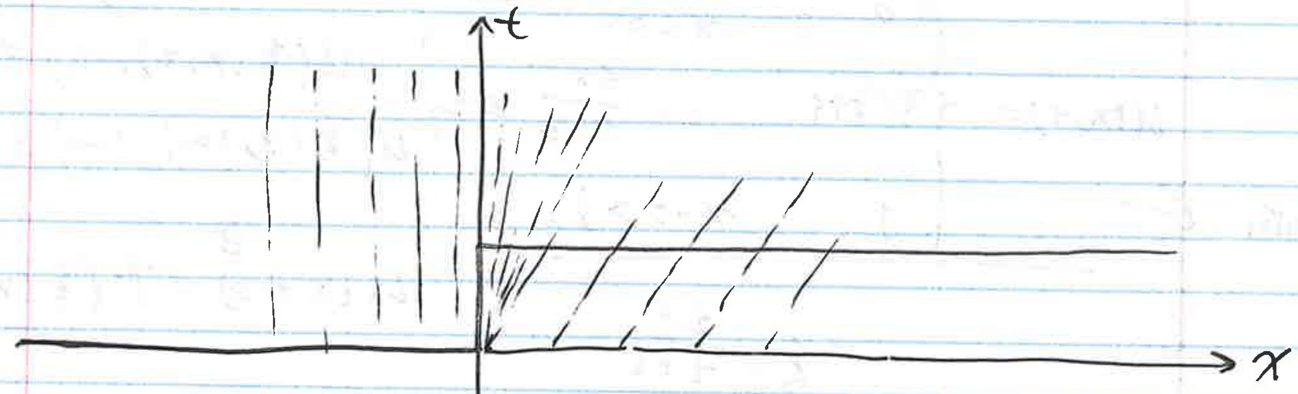
$$F(x) = \begin{cases} 0 & x \le 0. \\ x/\varepsilon & 0 < x \le \varepsilon \\ 1. & x > \varepsilon \end{cases}$$

$$u = F(\xi) = \begin{cases} 0 & x \le 0. \\ \xi/\varepsilon & 0 < x \le \varepsilon \\ 1. & x \ge \varepsilon \end{cases}$$

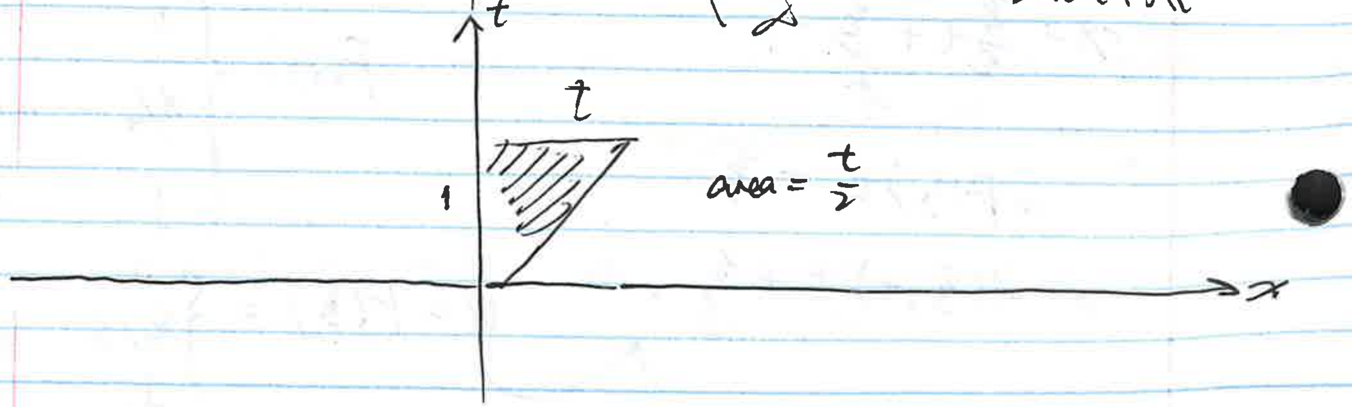
Now, letting $\varepsilon \to 0$.

$$u(x,t) = \begin{cases} 0, & x \le 0. \\ x/t, & 0 < x \le t. \\ 1 & x > t. \end{cases}$$





"physical solution"



Multi-Lane Traffic Transport Problem

$$\frac{\partial A\rho}{\partial t} + \frac{\partial Ac\rho}{\partial x} = 0.$$

"flux"

ρ = density of cars per unit length, $\frac{\# \text{ cars}}{\text{length}}$

$$A = \begin{cases} 1. & x \le 0 \\ 2. & x > 0. \end{cases}$$

c = Speed of traffic = $1 - \frac{1}{4}\rho$.

$$\rho(x, t=0). = \begin{cases} 2. & x \le -3. \\ 1. & x > -3. \end{cases}$$

$\rho(x,t)$ at $t > 0$,

$m = A\rho$.

$\hookrightarrow m(x,t)$.

"switch the form for I.C.s

$$m(x,t=0) = \begin{cases} 2 & x \le - \\ 1. & -3 < x \\ 2. & x > 0 \end{cases}$$

$\underbrace{}_{A\rho}$

$$\frac{\partial m}{\partial t} + \frac{\partial m \cdot c(m/A)}{\partial x}. = 0.$$

$$\frac{\partial m}{\partial t} + \left[\frac{\partial c(m/A)}{\partial x}\right] \cdot m + \frac{\partial m}{\partial x} \cdot c(m/A) = 0.$$

$$\frac{\partial m}{\partial t} + \frac{\partial c(m/A)}{\partial (m/A)} \cdot \frac{\partial (m/A)}{\partial x} \cdot m + c(m/A) \cdot \frac{\partial}{\partial} = 0$$

$$\frac{\partial m}{\partial t} + c'(m/A)\cdot\frac{m}{A}\cdot\frac{\partial m}{\partial x} + c(m/A)\frac{\partial m}{\partial x} = 0.$$

$$\frac{\partial m}{\partial t} + \left[1 - \frac{m}{4A} - \frac{m}{4A}\right]\frac{\partial m}{\partial x} = 0.$$

$$\frac{\partial m}{\partial t} + \left[1 - \frac{m}{2A}\right]\cdot\frac{\partial m}{\partial x} = 0 \quad \cdots \text{ (1st-order).}$$

$$\Rightarrow \left.\frac{dx}{dt}\right|_{\xi} = 1 - \frac{m(\xi,t)}{2A(x(\xi,t))}.$$

$$\left.\frac{dm}{dt}\right|_{\xi} = 0$$

$\hookrightarrow m = M_0(\xi)$

Real I.C.s:

$$m(x,t=0) = \begin{cases} 2 & x \le -3 \\ 1 & -3 < x \le 0 \\ 2 & x > 0 \end{cases}$$

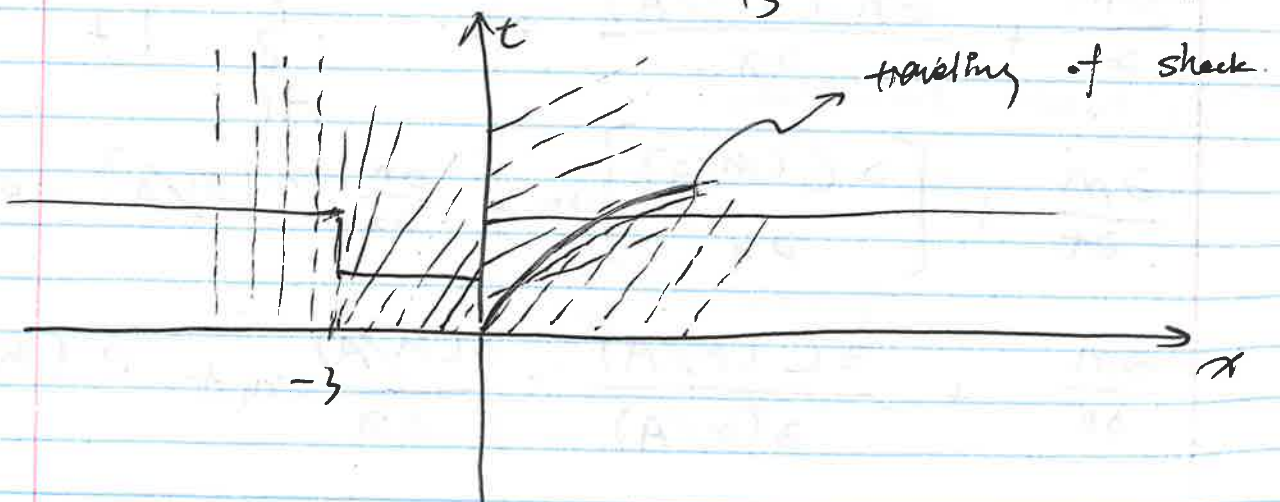
Consider 2 cases:

$x < 0$.

$$\left.\frac{dx}{dt}\right|_{\xi} = 1 - \frac{m}{2}$$

$x \geq 0$.

$$\left.\frac{dx}{dt}\right|_{\xi} = 1 - \frac{M}{4\times2}.$$



Basic formula for shock speed.

$$= \frac{F_L - F_R}{m_L - m_R}.$$

$$f = mc\left(\frac{m}{A}\right).$$

$$F_L = \left(1 - \frac{m}{4A}\right)m, \quad F_R = \left(1 - \frac{m}{4A}\right)m.$$

$$\downarrow \qquad\qquad \downarrow$$

$$\left(1 - \frac{1}{4\times2}\right) = \frac{7}{8} \qquad \left(1 - \frac{2}{4\times2}\right)\times2 = \frac{6}{4}.$$

$$= \frac{7/8 - 6/4}{1-2}$$

$$= 5/8.$$

expansion fan.

$$\frac{3}{S_2} = t - \frac{x}{S_1}$$

$$\frac{3}{1-\frac{m}{2}} = t - \frac{x}{1-\frac{m}{4}}.$$

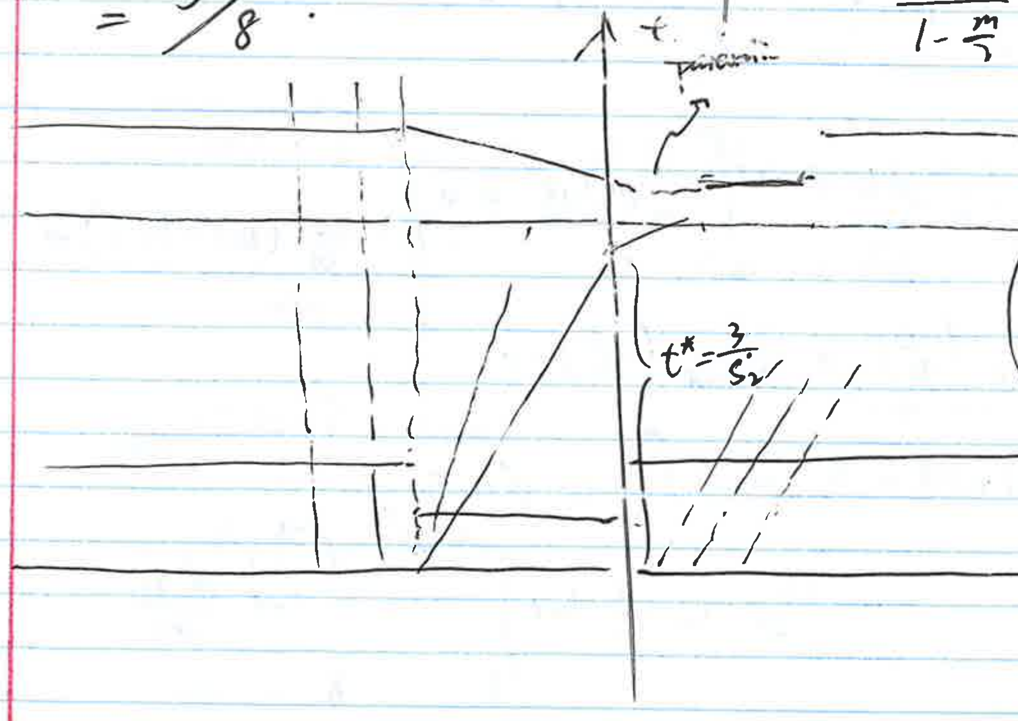
$\Downarrow$

m^2

$$\left(\frac{4x-6t+6}{-t}\right)m$$

$$+\left(\frac{-8x+8t-}{-t}\right.$$

$= 0$



$u_1 = \frac{\partial u}{\partial x}$

$u_2 = \frac{\partial u}{\partial t}$

Homework Problem. Derivation

3. $\left.\frac{dx}{dt}\right|_{\xi} = \frac{ux}{t+1}$ $\left.\frac{du}{dt}\right|_{\xi} = \frac{-ut}{t+1}$

Convert to matrix-vector format.

Recall Lecture 5: on characteristics:

$$\frac{dt}{t+1} = \frac{dx}{ux} = \frac{du}{-ut}.$$

Integration on three sides:

$$\ln(t+1)+\xi = \int \frac{1}{ux}dx = \int \frac{1}{-ut}dt.$$

From $\int \frac{1}{ux}dx = -\int \frac{1}{ut}dt$

$$\rightarrow \int \frac{1}{ux}dx + \int \frac{1}{ut}dt = 0$$

$$\rightarrow. \int \frac{1}{u} d\ln x + \int \frac{1}{u} d\ln t = 0$$

$$-\frac{1}{u}(\ln t - \ln\xi) + \int \ln t \cdot \frac{-u'}{u^2}dt$$

$$\frac{\partial}{\partial \ln t}\left(\frac{1}{u}\right) + \frac{\partial}{\partial \ln x}\left(\frac{1}{u}\right) = 0$$

$$\rightarrow \ln(t+1) + \xi = \int -\frac{1}{u} d\ln t$$

$$= -\frac{1}{u}\ln t\Big|_{\xi}^{t} + \int_{\xi}^{t} \ln t \frac{d\frac{1}{u}}{dt}$$

Converting to systems of ODEs:

$$\int \frac{1}{u} d\ln x + \int \frac{1}{u} d\ln t = 0$$

$$\begin{cases} U_1 = ux \\ U_2 = ut. \end{cases}$$

$$\begin{bmatrix} -\frac{t}{t+1} - \lambda & \frac{u}{t+1} \\ -\frac{t}{t+1} & \frac{1}{t} - \lambda \end{bmatrix} \Rightarrow. \frac{-ut}{(t+1)^2} + \left(\lambda - \frac{1}{t}\right)($$

$$\frac{d}{dt}\begin{bmatrix} ux \\ ut \end{bmatrix} = \begin{bmatrix} -\frac{t}{t+1} & \frac{u}{t+1} \\ -\frac{t}{t+1} & \frac{1}{t} \end{bmatrix}\begin{bmatrix} ux \\ ut \end{bmatrix}$$

$$(\dot{ux}) = \dot{u}x + u\dot{x} = \frac{-ut}{t+1}\cdot x + u\cdot\frac{ux}{t+1}.$$

$$= \frac{ux}{t+1}(-t+u).$$

$$(\dot{ut}) = \dot{u}t + u = \frac{-ut}{t+1}\cdot t + u$$

$$= ut\left(-\frac{t}{t+1} + \frac{1}{t}\right).$$

$$u = \exp(-t)\cdot(t+1)\exp(F(\xi))$$

$$= \frac{(t+1)\exp(F(\xi))}{\exp(t)}. \rightarrow U_0(x) = F(x).$$

$$\exp(F(\xi)) = F(x). \rightarrow u = \frac{(t+1)F(x)}{\exp(t)}.$$

$$F(\xi) = \ln[F(x)]$$

ln5

Lecture 7. 1/30/2024.

"last session on Methods of Characteristics".

Wave Equation.

$u(x,t)$. $\rightarrow \frac{\partial^2 u}{\partial t^2} - C_0^2 \frac{\partial^2 u}{\partial x^2} = 0$.

$-\infty < x < \infty$. I.C.s: $u(x,t) = f(x)$.

$0 \le t < \infty$ $\frac{\partial u}{\partial t}(x, t=0) = g(x)$.

-- Solution: D'Alembert solution

$$\frac{1}{2}\left[f(x - C_0 t) + f(x + C_0 t)\right]$$

$$+ \frac{1}{2C_0}\int_{x-C_0t}^{x+C_0t} g(s)\, ds$$

right-going wave

left-going wave

Recall: $\frac{\partial^2 u}{\partial t^2} - C_0^2 \cdot \frac{\partial^2 u}{\partial x^2} = 0$.

$$\underbrace{\left(\frac{\partial}{\partial t} - C_0 \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + C_0 \cdot \frac{\partial}{\partial x}\right)} u = 0$$

Variable: needs of advection.

... "double-advection".

$\xi^+ = x - C_0 t$ const.

$\xi^- = x + C_0 t$. const.

$(x,t) \rightarrow (\xi^+, \xi^-)$.

→ wave equation in ξ^+ & ξ^-:

$$\frac{\partial}{\partial \xi^+}\frac{\partial}{\partial \xi^-} \cdot u = 0$$

$u_{\xi^+\xi^-} = u_{\xi^-\xi^+} = 0$.

$f(x,y)$, for which $f_{xy} = 0 \rightarrow f = G(x) + H($

$u = G^+(\xi^+) + G^-(\xi^-)$. $\leftarrow u(x,t)$.

$= G^+(x - C_0 t) + G^-(x + C_0 t)$.

I.C.: $u(x, t=0) = f(x)$.

$\frac{\partial u}{\partial t}(x, t=0) = 0$

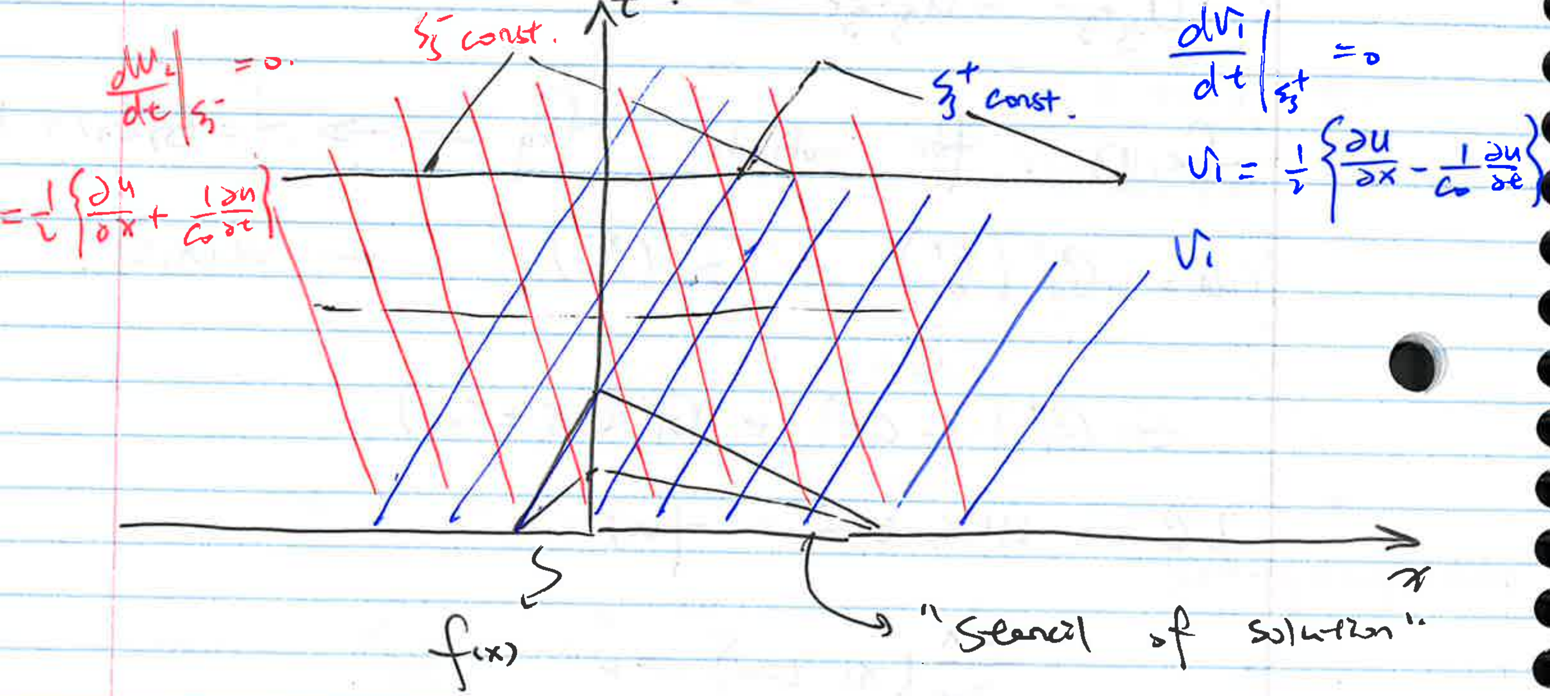
$$\begin{cases} G^+(x) + G^-(x) = f(x). \\ -C_0 G^{+\prime}(x) + C_0 G^{-\prime}(x) = 0. \end{cases}$$

$$\begin{cases} G^+(x) + G^-(x) = f(x). \\ G^+(x) - G^-(x) = C. \end{cases} \rightarrow \begin{matrix} G^+ = \frac{1}{2} f + \frac{C}{2} \\ G^- = \frac{1}{2} f - \frac{C}{2} \end{matrix}$$

$$u(x,t) = \frac{1}{2}\left[f(x - c_0 t) + f(x + c_0 t)\right]$$

Interpreting the sol'n.

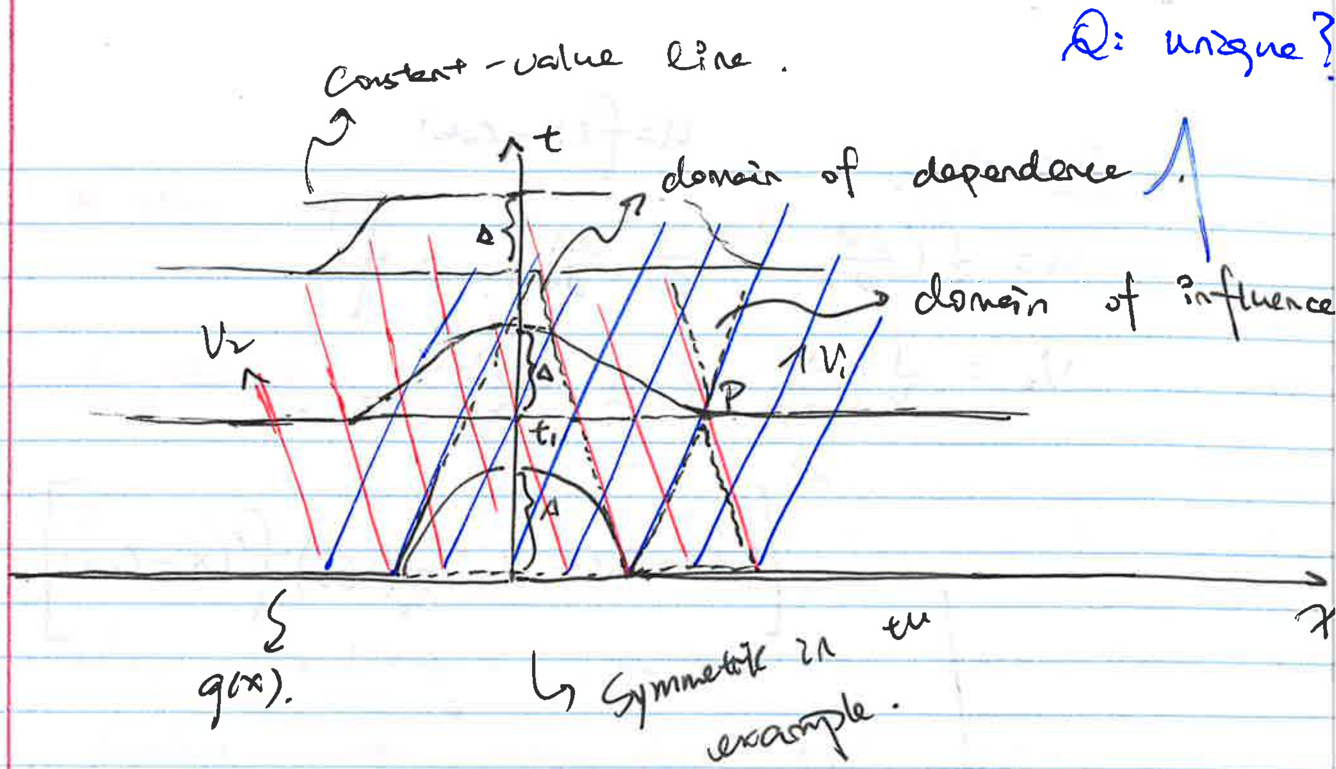
Ex 1 $\left.\begin{array}{l} f(x) \neq 0. \\ g(x) = 0. \end{array}\right\}$.



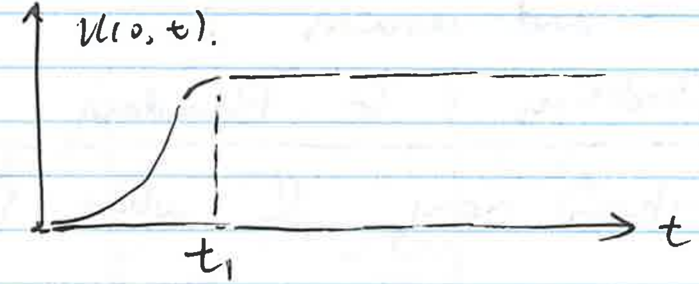
↓

the initial wave shape is preserved during wave-propagation

Ex 2 $\left.\begin{array}{l} f(x) = 0 \\ g(x) \neq 0 \end{array}\right\}$



domain of dependence.



Ex 3: One way waves?

(Wave actuators, wave absorbers, boundary con in numerical simulations ...)

$u(x,t) = f(x - c_0 t)$. is a sol'n.

$$\frac{\partial u}{\partial t} = -c_0 \cdot f'(x - c_0 t).$$

$$\text{I.C.}: u(x,0) = f(x).$$

$$\frac{\partial u}{\partial t}(x,0) = -c_0 f'(x)$$

Example. $u = f(x - c_0 t)$.

$$V_1 = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{1}{c_0} \cdot \frac{\partial u}{\partial t}\right). = f'$$

$$V_2 = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{1}{c_0} \cdot \frac{\partial u}{\partial t}\right) = 0.$$

$$\left[f'(x - c_0 t) + \frac{1}{c_0}(-c_0) f'(x - c_0 t)\right] \overset{\shortparallel}{0}$$

$V_2 = 0$ @ $t = 0$

and remains 0.

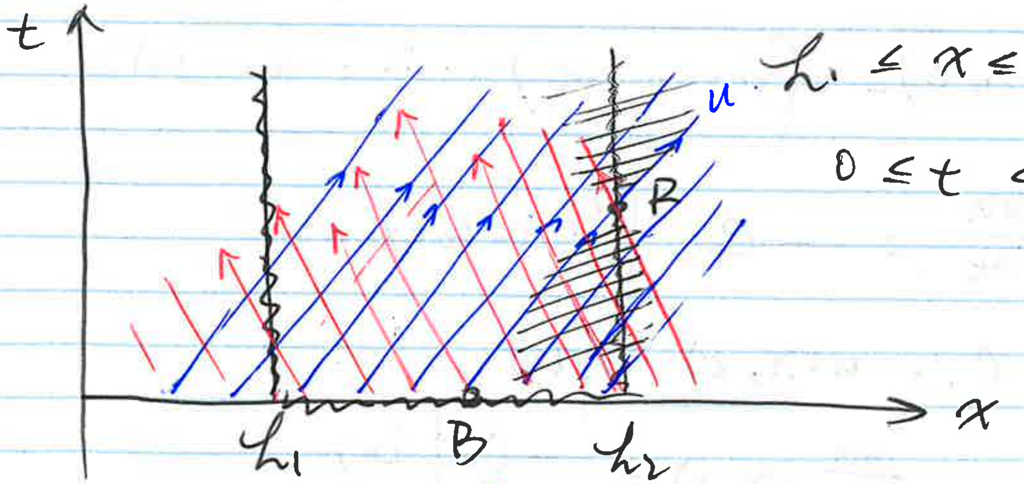
Initial Conditions & Boundary Conditions

where, how many & what ?

↳ well-posed PDE problem.

Finite domain.

(Domain of interest).



$h_1 \le x \le h_2$

$0 \le t < \infty$

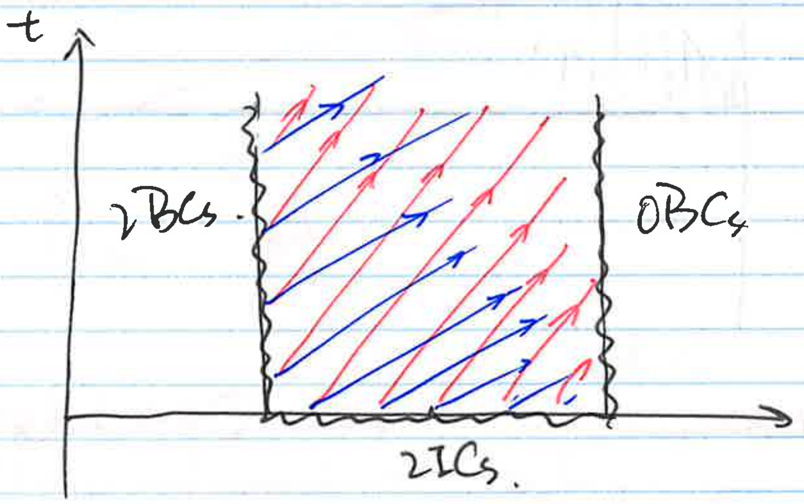
need 2 I.C.s

* How many I.C.s for B? 2. I.C.s

* How many B.C.s for R? 1. B.C.s

* How many B.C.s for h? 1. B.C.s.

Edge conditions specify the char. that are coming into the domain of interest. (from outside



"Supersonic sys."

i.e., information travels with speed of sound relative to flow.

Sol'n to wave eqn. in Finite Domain.

D'Alambert → $-\infty < x < \infty$

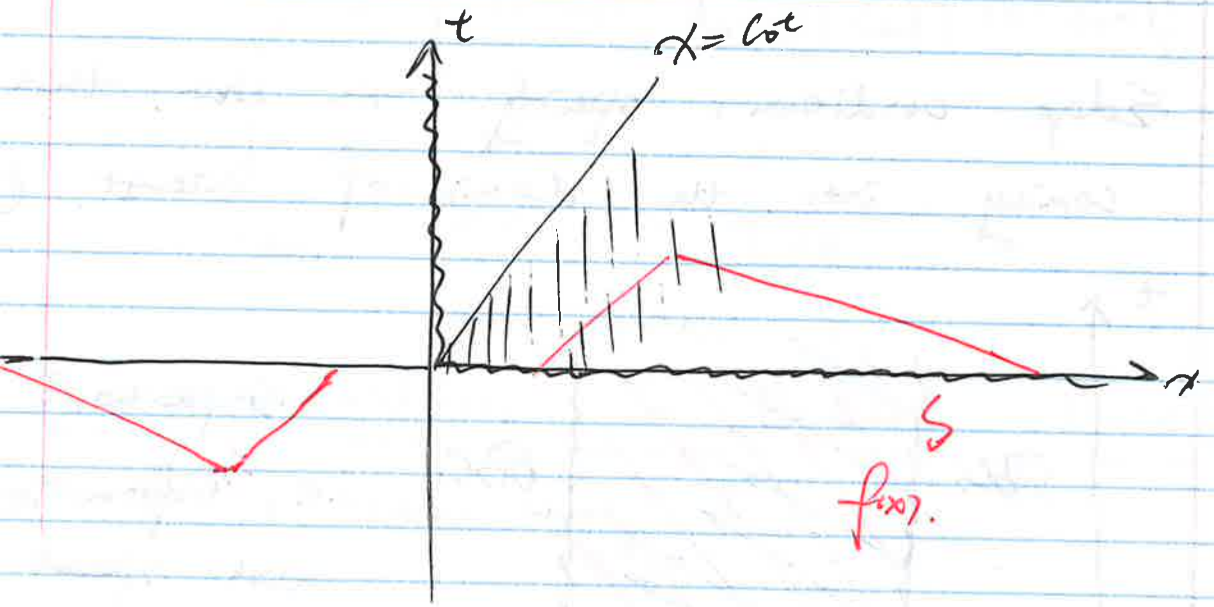
Semi-infinite domain

- $0 \le x < \infty$
- $0 \le t < \infty$

Recall: $\frac{\partial^2 u}{\partial t^2} - C_0^2 \frac{\partial^2 u}{\partial x^2} = 0$

I.C.s : $u(x, t=0) = f(x)$.

$\frac{\partial u}{\partial t}(x, t=0) = 0$



Case (1) : $u(x=0, t) = 0$ ⟵ Dirichlet. B.C.s

"Method of Images".

Embed into a problem with $-\infty < x < \infty$

$$\tilde{f}(x, t=0) = \begin{cases} f(x), & 0 \le x < \infty \\ -f(-x)., & -\infty < x < 0 \end{cases}$$

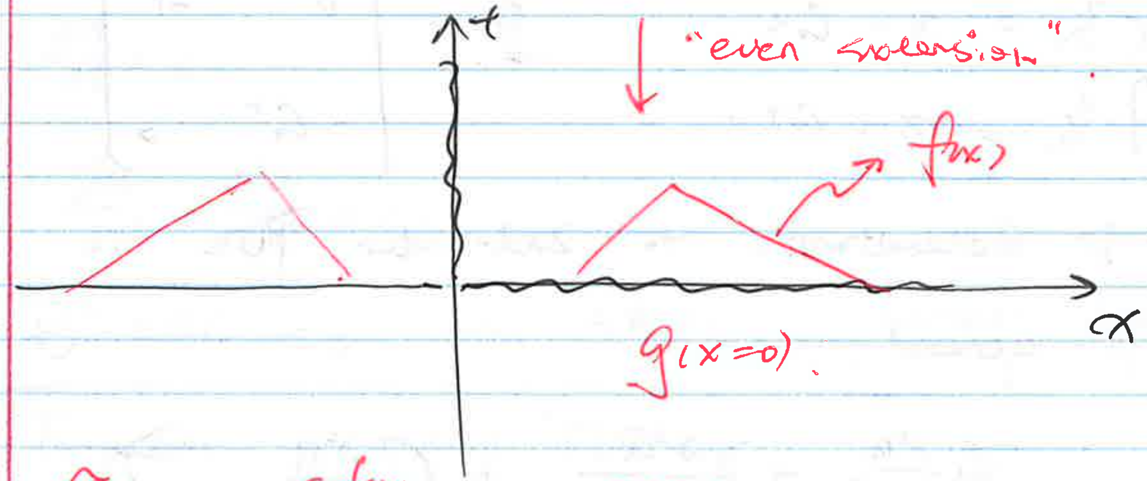
odd extension for f

In other words, $u(x,t) = \frac{1}{2}\left[\tilde{f}(x - C_0 t) + \tilde{f}(x + C_0 t)\right]$

· for $x > C_0 t$: $\frac{1}{2}\left[f(x + C_0 t) - f(C_0 t - x)\right]$

· for $x < C_0 t$: $\frac{1}{2}\left[f(x + C_0 t) + f(x - C_0 t)\right]$.

Case (2). $\frac{\partial u}{\partial x}(x=0, t) = 0$. ⟵ Neumann. B.C.s

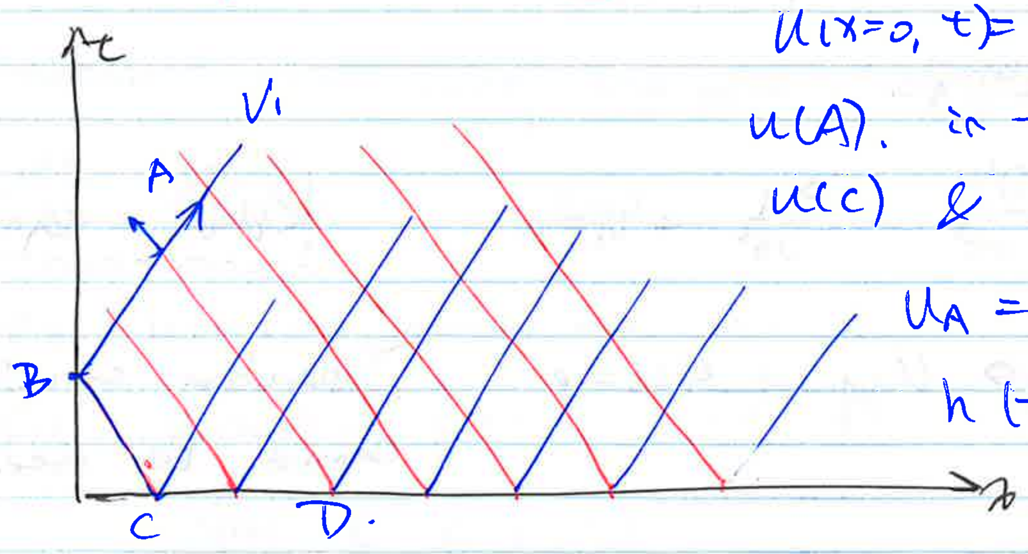


$$\tilde{f}(x) = \begin{cases} f(x) & 0 \le x < \infty \\ f(-x). & -\infty < x < 0. \end{cases}$$

i.e., time-dep.

Time-dependent. B.C.s

$u(x=0, t) = h(t)$.



$u(A)$. in terms of $u(C)$ & $u(D)$.

$u_A = u(x_A, t_A)$.

$h\left(t_A - \frac{x_A}{C_0}\right) + \frac{1}{2} f(x_A + C_0 t_A)$

→ think back to B.C.s & I.C.s

$- \frac{1}{2} f(C_0 t_A - x_A)$

Lecture 8 2/1/2024

$$\frac{\partial^2 u}{\partial t^2} - C_0 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial x}{\partial t} = \pm C_0 \qquad \det |\underline{\underline{B}} - \lambda \underline{\underline{I}}| = 0.$$

$$\leftarrow \begin{cases} \xi_+ = x - C_0 t \\ \xi_- = x + C_0 t. \end{cases} \qquad \underline{\underline{B}} = \begin{bmatrix} 0 & -1 \\ -C_0^2 & 0 \end{bmatrix}$$

$\Rightarrow$ Generalization to 2nd-order PDE ...

In general.

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial y \partial x} + C\frac{\partial^2 u}{\partial y^2} = D$$

$$\frac{\partial y}{\partial x} = \frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A}$$

$\Rightarrow$ $u_{\xi\eta} = 0$

Example

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = 0. \qquad \text{KdV eqn.}$$

$\hookrightarrow$ $u_{\xi\eta} + 4u = 0$ D'Alembert solution cannot be used anymore.

Example heat equation.

$$\frac{\partial u}{\partial t} - K\frac{\partial^2 u}{\partial x^2} = 0.$$

$B^2 - 4AC$?. $\rightarrow$ parabolic

$$\frac{\partial x}{\partial t} = 0.$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$B^2 - 4AC < 0$ $\rightarrow$ Elliptic. complex characteristics.

Formulate a new PDE:.

$$\frac{\partial^2 u}{\partial x^2} - x\frac{\partial^2 u}{\partial y^2} = 0.$$

$x > 0$: hyperbolic eqn.

$x < 0$: elliptic eqn.

wave eqn.

"Euler-Tricomi Eqn".

Construct sol'n using superposition. (linearized system).

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \qquad 0 < x < 1, \quad t > 0.$$

B.C.s: $u(0,t) = u(1,t) = 0$, $t > 0$.

I.C.s: $u(x,0) = f(x)$. $0 < x < 1$.

$$f(x) \in C^2[0, 1).$$

Start with an Ansatz.

$u(x,t) = \Phi(t) \cdot \Psi(x).$

↑ assumption.

Goal: find a non-trivial sol'n.

if $u \neq 0$, ..

$\Phi(t)$ & $\Psi(x) \neq 0$.

$\Phi'(t)\Psi(x) = \Phi(t)\Psi''(x)$ ... plug in original eqn.

$$\equiv \frac{\Phi'(t)}{\Phi(t)} = \frac{\Psi''(x)}{\Psi(x)}$$

↑ as long as the function is not a zero-func, we good :).

$= -\lambda$

↑ defined as a "separation const."

$\Phi'(t) + \lambda\Phi(t) = 0$

$\Psi''(x) + \lambda\Psi(x) = 0$ ... connected via λ.

Q: what's going on w/ our B.C.s?

A: $\Psi(0) = \Psi(1) = 0. \longrightarrow$

Ex: Harmonic Oscillator.

General sol'n: $\Psi(x) = C_1 \sin(\sqrt{\lambda}x) + C_2 \cos(\sqrt{\lambda}x).$

plugging in the B.C.s:

$\rightarrow 0 + C_2 = 0$

Second B.C.s: $\Psi(1) = 0 \rightarrow C_1 \sin(\sqrt{\lambda}) = 0.$

$\downarrow$

$\sin(\sqrt{\lambda}) = 0.$

$\sqrt{\lambda} = \pi, 2\pi, \ldots$

$\lambda = k\pi,$

$\forall\ k = 1, 2, \ldots \infty$

.. General Sol'n:

$\Psi_k(x) = C_{1k} \sin(k\pi x).$

→ what if $\lambda < 0$? (Aside).

$\Psi(x) = C_1 e^{kt} + C_2 e^{-kt}.$

prove it: this will only be working if $\lambda = 0$.

→ Second ODE

$\Phi(t) + \lambda\Phi(t) = 0.$

$$\frac{d\Phi'(t)}{\Phi(t)} = dt.$$

$\Phi(t) = C_{k\phi}\, e^{-\lambda t}.$

$\Rightarrow \Phi(t) = e^{-k^2\pi^2 t}.$

$\Rightarrow U_k = C_{k\phi}\, e^{-k^2\pi^2 t}\, C_{1k} \sin(k\pi t)$

$U_k = C_k e^{-k^2\pi^2 t} \sin(k\pi x).$

General sol'n. $u(x,t) = \sum_{k=1}^{\infty} A_k u_k(x,t)$.

↳ Q: is it going to converge to a finite number?

$\Downarrow$

$A_k \to 0$.

"sufficiently rapidly"

$$f(x) = \sum_{k=1}^{\infty} A_k \sin(k\pi x).$$

↑ function can be represented as this "series".

$$\int_0^1 \sin(m\pi x) f(x) = \int_0^1 \sum_{k=1}^{\infty} A_k \sin(k\pi x)\sin(m\pi x)$$

$$= A_m \int_0^1 \sin^2(m\pi x)$$

$$= \frac{A_m}{2}$$

Since, $\int_0^1 \sin(k\pi x)\sin(m\pi x)\,dx = 0, \ \forall\, k \neq m$.

... why? because $A_k \to 0$ "sufficiently rapidly"

k^{th} coefficient.

$$A_k = 2\int_0^1 f(x)\sin(k\pi x)\,dx.$$

the solution:

$$u(x) = \sum_{k=1}^{\infty} A_k u_k(x,t)$$

... we use these "sin" functions due to "orthogonality".

Problem Session 4

2/2/2024. → Coupled system of 1st-order PDEs. Methods of Images.

$$\begin{cases} \rho_0 \dfrac{\partial u}{\partial t} + \dfrac{\partial P}{\partial x} = 0. \\ \dfrac{\partial P}{\partial t} + \rho_0 C_0 \dfrac{\partial u}{\partial x} = 0. \end{cases}$$

I.C.s: $u(x,t=0) = f(x)$ → pressure disturbances.

$P(x,t=0) = g(x)$

Step 1. Matrix form

$$\begin{bmatrix} \partial u/\partial t \\ \partial P/\partial t \end{bmatrix} + \begin{bmatrix} 0 & 1/\rho_0 \\ \rho_0 C_0^2 & 0 \end{bmatrix}\begin{bmatrix} \partial u/\partial x \\ \partial P/\partial x \end{bmatrix} = 0$$

Defn: $\vec{\Phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} u \\ P \end{bmatrix}$.

$$\frac{\partial \vec{\Phi}}{\partial t} + A \cdot \frac{\partial \vec{\Phi}}{\partial x} = 0$$

2. $A = Q \Lambda Q^{-1}$.

uple.

$Q = [\vec{q}_{1}, \vec{q}_{2}]$. $\Lambda = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$,

$\det(A - \lambda I) = 0$.

$$\det\left(\begin{bmatrix} -\lambda & 1/\rho_0 \\ \rho_0 C_0^2 & -\lambda \end{bmatrix}\right) = 0 \rightarrow \begin{cases} \lambda_1 = C_0 \\ \lambda_2 = -C_0 \end{cases}$$

Find eigenvecs. $(A - \lambda_1 I)\vec{q}_1 = 0$.

$$\begin{bmatrix} -C_0 & 1/\rho_0 \\ \rho_0 C_0^2 & -C_0 \end{bmatrix} \begin{bmatrix} q_1^{(1)} \\ q_2^{(2)} \end{bmatrix} = 0.$$

$-C_0 q_1^{(1)} + \frac{1}{\rho_0} \cdot q_1^{(2)} = 0$.

$\rho_0 C_0^2 q_1^{(1)} - C_0 q_1^{(2)} = 0$.

$\rightarrow q_1^{(2)} = \rho_0 C_0 \, q_1^{(1)}$.

$\vec{q}_1 = \begin{bmatrix} 1 \\ \rho_0 C_0 \end{bmatrix}$,

$(A - \lambda_2 I)\vec{q}_2 = 0. \rightarrow \vec{q}_2 = \begin{bmatrix} 1 \\ -\rho_0 C_0 \end{bmatrix}$

$\Lambda = \begin{bmatrix} C_0 & 0 \\ 0 & -C_0 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 1 \\ \rho_0 C_0 & -\rho_0 C_0 \end{bmatrix}$.

$Q^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1/\rho_0 C_0 \\ 1 & -1/\rho_0 C_0 \end{bmatrix}$.

$$\frac{\partial \vec{\phi}}{\partial t} + A \cdot \frac{\partial \vec{\phi}}{\partial x} = 0.$$

$$\frac{\partial \vec{\phi}}{\partial t} + Q \Lambda Q^{-1} \frac{\partial \vec{\phi}}{\partial x} = 0.$$

$$Q^{-1}\frac{\partial \vec{\phi}}{\partial t} + Q^{-1} Q \Lambda Q^{-1} \cdot \frac{\partial \vec{\phi}}{\partial x} = 0.$$

$$\frac{\partial}{\partial t} Q^{-1}\vec{\phi} + \Lambda \frac{\partial}{\partial x} \cdot Q^{-1}\vec{\phi} = 0.$$

$$\vec{\psi} = Q^{-1}\vec{\phi} \xrightarrow[C_\varphi]{} \frac{\partial \vec{\psi}}{\partial t} + \Lambda \frac{\partial \vec{\psi}}{\partial x} = 0.$$

$$\frac{\partial}{\partial t}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \underset{\underset{-C_0}{\downarrow}}{\lambda_2} \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = 0$$

$$\begin{cases} \dfrac{\partial \psi_1}{\partial t} + C_0 \dfrac{\partial \psi_1}{\partial x} = 0. \\ \dfrac{\partial \psi_2}{\partial t} - C_0 \cdot \dfrac{\partial \psi_2}{\partial x} = 0 \end{cases}$$

... Decoupling step.

Step 3. Methods of Characteristics

Eq. 1. $x = C_0 t + \xi_1$, $\psi_1(\xi_1, t) = \psi_1(\xi_1, 0)$.

Eq. 2. $x = -C_0 t + \xi_2$.

$$\psi_2(\xi_2, t) = \psi_2(\xi_2, 0).$$

4. Given ICs: $\begin{cases} u(x, t=0) = f(x) \\ p(x, t=0) = g(x). \end{cases}$

Translate ICs. $x = \xi_1 = \xi_2 = \xi$.

$u(\xi, 0) = f(\xi)$,

$p(\xi, 0) = g(\xi)$.

$$\vec{\psi} = Q^{-1}\vec{\phi} \Rightarrow \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1/\rho_0 C_0 \\ 1 & -1/\rho_0 C_0 \end{bmatrix}\begin{bmatrix} u \\ p \end{bmatrix}$$

$$\psi_1(\xi_1, 0) = \frac{1}{2}\overbrace{u(\xi_1, 0)}^{f(\xi_1)} + \frac{1}{2\rho_0 C_0}\overbrace{p(\xi_1, 0)}^{g(\xi_1)} \rightarrow \psi_1(\xi_1, t)$$

$$\psi_2(\xi_2, 0) = \frac{1}{2}\underbrace{u(\xi_2, 0)}_{f(\xi_2)} + \frac{1}{2\rho_0 C_0}\underbrace{p(\xi_2, 0)}_{g(\xi_2)} \rightarrow \psi_2(\xi_2, t).$$

Final-form solution.

$\xi_1 = x - C_0 t$, $\xi_2 = x + C_0 t$.

$$\psi_1(x, t) = \frac{1}{2} f(x - C_0 t) + \frac{1}{2\rho_0 C_0} g(x - C_0 t).$$

$$\psi_2(x, t) = \frac{1}{2} f(x + C_0 t) - \frac{1}{2\rho_0 C_0} g(x + C_0 t).$$

→ Final Step: $\vec{\phi} = Q\vec{\psi}$.

$$\begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \rho_0 C_0 & -\rho_0 C_0 \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}.$$

$$\boxed{\begin{aligned} u(x, t) &= \frac{1}{2} f(x - C_0 t) + \frac{1}{2\rho_0 C_0} g(x - C_0 t) \\ &\quad + \frac{1}{2} f(x + C_0 t) - \frac{1}{2\rho_0 C_0} g(x + C_0 t). \\ p(x, t) &= \frac{\rho_0 C_0}{2} f(x - C_0 t) + \frac{1}{2} g(x - C_0 t) \\ &\quad - \frac{\rho_0 C_0}{2} f(x + C_0 t) + \frac{1}{2} g(x + C_0 t) \end{aligned}}$$

Only right-going waves.

$$\psi_2(\xi_2) = \frac{1}{2} f(\xi_2) - \frac{1}{2\rho_0 C_0} g(\xi_2) = 0.$$

$$\rightarrow g(\xi_2) = \rho_0 C_0 f(\xi_2).$$

$$\rightarrow g(x) = \rho_0 C_0 f(x)$$

try to satisfy the BCs.

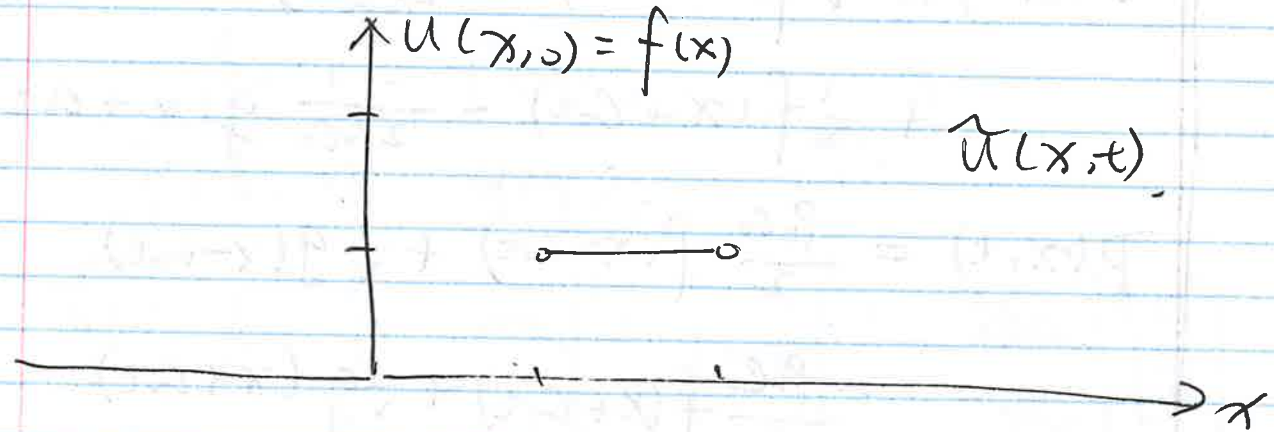
\# Methods of Images

▷ Wave equation on semi-infinite domain.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \quad 0 \leq x < \infty, \quad t \geq 0.$$

BC: $\frac{\partial u}{\partial x}(0,t) = \quad f(x) = \begin{cases} 0, & 0 \leq x < 1, \\ 1, & 1 \leq x < 2 \\ 0, & 2 \leq x < \infty. \end{cases}$

$\frac{\partial u}{\partial t}(x,0) = 0$

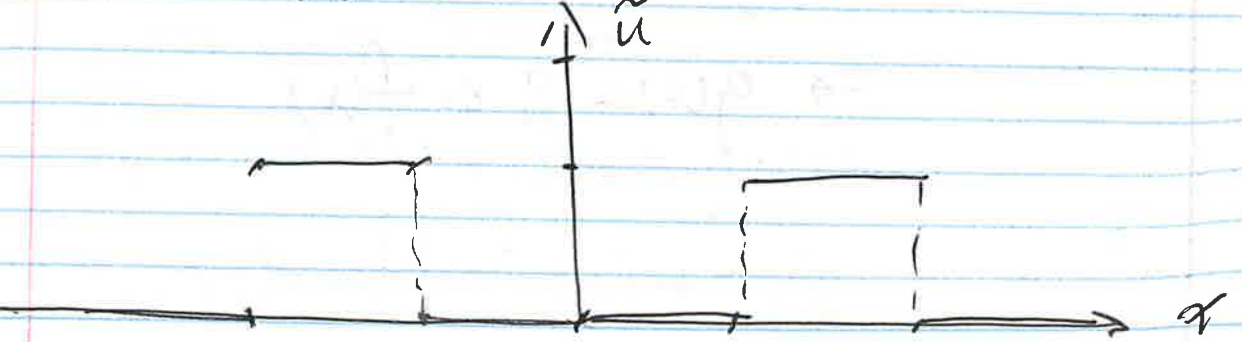


$\hat{u}(x,t)$

$$\frac{\partial^2 \hat{u}}{\partial t^2} = c^2 \cdot \frac{\partial^2 \hat{u}}{\partial x^2}. \quad -\infty < x < \infty. \quad t \geq 0$$

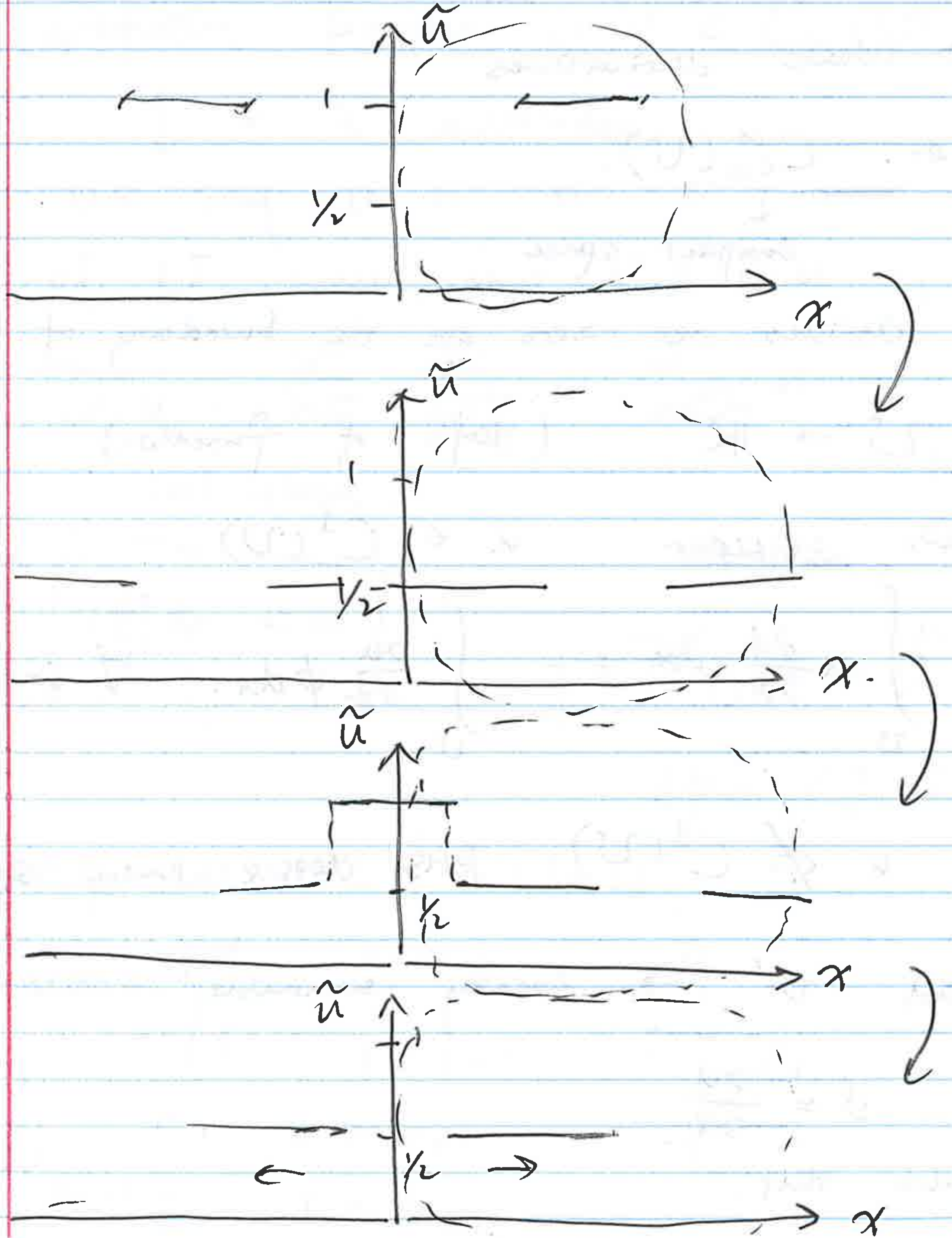
I.C.s: $\hat{u}(x,0) = \hat{f}(x) = ?$.

$\frac{\partial \hat{u}}{\partial t}(x,0) = 0.$



$$\hat{f}(x) = \begin{cases} f(x), & 0 \leq x < \infty \\ f(-x), & -\infty < x \leq 0. \end{cases}$$

$$\hat{u}(x,t) = \frac{1}{2}\hat{f}(x-ct) + \frac{1}{2}\hat{f}(x+ct).$$



Lecture 9 2/6/2024.

Review: { Separation of variables.
ODEs + B.Cs.
general sol'n → coeff.
series sol'n.

→ Weak derivatives.

Let. $C_c^\infty(U)$.
↓
compact space.

vanishes to zero at the boundary of domain

$\phi : U \to \mathbb{R}$. (def'n of function)

Now, consider $u \in C^1(U)$.

$$\int_U u \frac{\partial \phi}{\partial x_i} dx = - \int_U \frac{\partial u}{\partial x_i} \phi \, dx. \quad \forall \, i = 1, 2, \dots, n.$$

if $u \notin C^1(U)$ RHS doesn't make sense

find 'v' → locally summable

$v = \frac{\partial u}{\partial x_i}$.

Such that.

$$\int_U u \frac{\partial \phi}{\partial x_i} dx = - \int_U v \phi \, dx$$

↑ this form: weak derivatives

Definition: Suppose $u, v \in L^1_{loc}(U)$.
↓ α is multiindex

we say v is the α^{th} weak derivatives of U such that $v = D^\alpha u$,

$$D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u$$

$|\alpha|$ times.

L_1 Defn: $\boxed{\int_U |f| \, dx < \infty}$

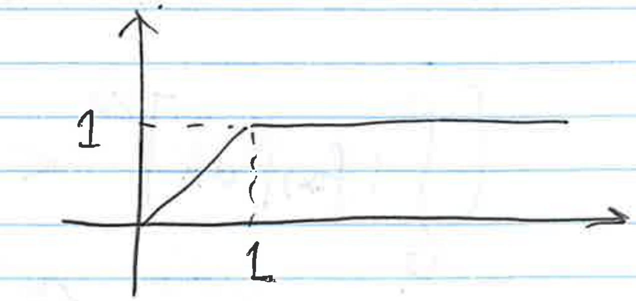
provided,

$$\int_U u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U v \phi \, dx$$

$\forall \phi \in C_c^\infty(U)$.

e.g. let $n = 1$, $U = (0, 2)$.

$$u(x) = \begin{cases} x & 0 < x \le 1. \\ 1. & 1 \le x \le 2. \end{cases}$$



Define:
$$v(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & 1 \le x \le 2. \end{cases}$$

$$\int_0^2 u\phi' dx = \int_0^1 x\phi' dx + \int_1^2 \phi' dx$$

$$= -\int_0^1 \phi dx + \phi(1) - \cancelto{0}{\phi(0)} + \cancelto{0}{\phi(2)} - \phi(1)$$

$$= - \int_0^2 v\phi \, dx$$

$W^{k,p}(U)$. → Sobolev spaces.

$u: U \to \mathbb{R}$.

S.t., for each α, with $|\alpha| \le k$.

$D^\alpha u$ exists, in the weak sense & belongs to

$L^p(U)$ $\left(\int_{U} |f(x)|^p\right)^{1/p}$ $< \infty$.

A special case. if $p=2$. then

$W^{k,2}(U) = H^k(U)$. $(k=0, 1, \dots)$

↖ Hilbert.

$\left[\int_U |f(x)|^2 dx\right]^{1/2} < \infty$

↖ the "norm" of the space

Def'n if $u \in H^k(U)$.

$$\|u\|_{H^k} = \sum_{|\alpha|\le k} \left(\int_U |D^\alpha u|^p dx\right)^{1/p}.$$

↑ if $\alpha=0$. it gives

$\left(\int_U |f(x)|^2 dx\right)^{1/2} < \infty$

"norm is bounded".

e.g. $\|u\|_{H^2} \left(\int_U |u|^2\right)^{1/2} + \left(\int_U \left|\frac{\partial u}{\partial x}\right|^2 dx\right)^{1/2}$

$+ \left(\int_U \left|\frac{\partial^2 u}{\partial x^2}\right|\right)^{1/2} < \infty$

* Properties of Hilbert spaces.

Def'n For every $f, g \in H$. ← Hilbert sp.

we can define a scalar

"Scalar" not'n → (f, g).

"inner product"

1). $(f, g) \ge 0$., for all $f, g \in H$.

2) $(f, f) = 0$, iff $f = 0$

can be complex

3). $(\lambda f, g) = \lambda (f, g)$

4). $(f, g) = (g, f)$.

5). $(f+g, h) = (f, h) + (g, h)$.

example $L^2(\Omega)$ with $\Omega \subset \mathbb{R}^N$.

$\downarrow$ define.

$$(f, g) = \int_\Omega f(x) \overline{g(x)} \, dx$$

for real-valued funcs.

$$\int_\Omega f(x) g(x) \, dx$$

example $L^2_w(\Omega)$

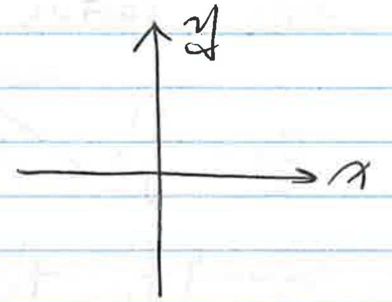
weighting func.

$$(f, g)_w = \int_\Omega f(x) . g(x) w(x) \, dx$$

$\forall \; w(x) > 0$

Theorem: For H being Hilbert, if we set $\|f\| = \sqrt{(f, f)}$ then $\|\cdot\|$ is a norm.

Orthogonality:



example: $\mathbb{R}^2$

$(x, y) = \|x\| \|y\| . \cos\theta$

and $x \perp y$, $(x, y) = 0$ (IFF)

Defn. if H is Hilbert, & f & $g \in H$, then f & g are orthogonal iff $(f, g) = 0$.

e.g.: $f_n(x) = \sin(nx)$.

then $\{f_n\}_{n=1}^{\infty}$. is orthogonal in $L^2(0, \pi)$.

☆ Infinite orthogonal sequences

- equipped with projection operators. P_E.
- if $\{f_n\}_{n=1}^{\infty}$ is a countable orthogonal set in H. & $f_n \neq 0$ for all "n".
- we expect, simply $n \to \infty$.

$$H = \text{span}\left(\{f_n\}_{n=1}^{\infty}\right).$$

then $P_E g = \sum_{n=1}^{\infty} \frac{(g, f_n)}{(f_n, f_n)} f_n$.

if this series converges.

let $e_n = f_n / \|f_n\|$; $C_n = (g, e_n)$.

$E_N = \text{Span}\{f_1, f_2, \dots, f_n\}$

s.t. $\{e_n\}_{n=1}^{\infty}$ is an orthogonal set.

$P_E g = \sum_{n=1}^{N} C_n e_n$.

Bessel's inequality. "just state it"

$$\sum_{n=1}^{\infty} |C_n|^2 = \sum_{n=1}^{\infty} |(g, e_n)|^2 \leq \|g\|^2$$

Consequently, $\lim_{n\to\infty} C_n = 0$

Theorem Riesz - Fischer

$\sum_{n=1}^{\infty} C_n e_n$ is convergent in H,

to $P_E g = g$

Lecture 10 2/8/2024.

Recap: 1). weak derivatives.

2) Sobolev - Hilbert.

3) general properties of Hilbert space.

Summarizing: Thm. Let $\{e_n\}_{n=1}^{\infty}$ $\frac{f_n}{\|f_n\|}$ $\downarrow$

be an orthonormal set in H. form the basis.

→ a). $\{e_n\}_{n=1}^{\infty}$ form a basis in H.

→ b). $\underbrace{g = \sum_{n=1}^{\infty} (g, e_n).\, e_n}_{\text{generalized Fourier series}}$. $\forall\, g \in H$.

→ c). $\|g\|^2 = \sum_{n=1}^{\infty} |(g, e_n)|^2$ → Bessel's equality.

→ d). $\{e_n\}_{n=1}^{\infty}$ is complete in H.

★ Bounded Linear Operators.

⇝ e.g., derivatives, integrals, ...

$$B(X, Y) = \{T : X \to Y \text{ is linear } \|T\|_{X,Y} < \infty\}$$

where $\|T\|_{X,Y} = \sup_{f \neq 0} \frac{\|Tf\|_Y}{\|f\|_X}$

$\to$ on $[a, b]$

Let $Lu = a_2(x)u'' + a_1(x)u' + a_0(x)u$.

a general second-order differential operator.

$a_j(x) \in C([a,b])$ ← compact support.

& $a_2(x) \neq 0$.

Boundary condition.

$B_1u = C_1u(a) + C_2u'(a)$.

$B_2u = C_3u(b) + C_4u'(b)$.

also. $|C_1| + |C_2| \neq 0$, $|C_3| + |C_4| \neq a$ (*)

One can then write a general problem of the form, $Lu = f(x)$. $a < x < b$.

$B_1u = 0$

$B_2u = 0$

Intuition of the conjugate function ψ.

$\lambda_1 = a_1 + ib_1$, $\lambda_2 = a_2 - ib_2$

Adjoint problems

$$(L\phi, \psi) = \int_a^b (a_2\phi'' + a_1\phi' + a_0\phi)\bar{\psi}\,dx$$

$$\xrightarrow{IBP} = \int_a^b \phi\,\overbrace{((a_2\bar{\psi})'' - (a_1\bar{\psi})' + a_0\bar{\psi})}^{L^*\bar{\psi}}\,dx$$

---

then we can say.

$(L\phi, \psi) = (\phi, L^*\psi)$.

if we expand the forms.

$$L^*\psi = a_2\psi'' + (2a_2' - a_1)\psi' + (a_2'' - a_1' + a_0)\psi.$$

if $a_1 = a_2'$, $L^*\psi = L\psi$.

$\Rightarrow (L\phi, \psi) = (\phi, L\psi)$.

"self-adjoint"

eg.

operator for 1D heat eqn. $L = \frac{d^2u}{dx^2}$

$$\to \int \frac{d^2u}{dx^2} v\,dx$$

$$\to \int u \frac{d^2\bar{v}}{dx^2}\,dx.$$

Integral by parts.

"no boundary terms"

$(Lu, v) = (u, Lv)$.

$$(L\phi, \psi) - (\phi, L^*\psi) = J(\phi, \psi)\Big|_a^b.$$

Some algebra, we can show:

$$J(\phi, \psi) = a_2(\phi'\bar{\psi} - \phi\bar{\psi}') + (a_1 - a_2')\phi\bar{\psi}$$

if $J(\phi, \psi)\Big|_a^b = 0$, then T is self-adjoint

▷ Let $L_0\phi = a_2(x)\phi'' + a_1(x)\phi' + a_0(x)\phi$.

in addition to ⊛, $a_2(x) < 0$.

A special case

$$p(x) = \exp\left(\int_a^x \frac{a_1(s)}{a_2(s)}\, ds\right).$$

← a function of x.

$p(x) > 0$.

$w(x) > 0$.

$\in C([a,b])$.

$$w(x) = -\frac{p(x)}{a_2(x)}.$$ ← "weight function".

$$q(x) = a_0(x) \cdot w(x).$$

then $L_0\phi = \lambda\phi$. → is an eigenvalue problem.

$$-(p\phi')' + q_0\phi = \lambda w(x)\phi.$$

functions.

this ODE is called the Sturm-Liouville equation.

" the distribution of λs are to be discussed

Let $L_1\phi = (p\phi')' + q\phi$.

$$L\phi = \frac{L_1\phi}{w(x)}.$$

Notice, for a test func. ψ.

$$(L_1\phi, \psi) = (\phi, L_1\psi).$$

$$\Rightarrow \int_a^b [(p\phi')'\psi + q\phi\psi]\, dx.$$

$$\Rightarrow -\int_a^b \psi'(p\phi')\, dx + \int q\phi\psi\, dx.$$

$$\Rightarrow -\int_a^b (\psi' p)'\phi\, dx + \int_a^b q\phi\psi\, dx.$$

$$\Rightarrow \int [(p\psi')' + q\psi]\phi\, dx$$

$$\Rightarrow (\phi, L\psi).$$

$$L\phi = \frac{L_1\phi}{w(x)}.$$

→ claim is that

$$L_0\phi = \lambda\phi.$$

if & only if

$$L\phi = \lambda\phi.$$

In order to show that $L\phi$ is saf-adjoint.

$$(\phi, \psi)_w = \int_a^b \phi(x)\, \overline{\psi(x)}\, w(x)\, dx$$

$$\|\phi\|_{Lw}^2 = \left(\int_a^b |\phi(x)|^2 w(x)\, dx \right)^{1/2}$$

$$= \sqrt{(\phi, \phi)}$$

Problem Session #5

2/9/2024.

Unsteady Heat Conduction. in an annular heat pipe.

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T = \alpha \overset{\uparrow}{\Delta} T.$$

Laplacian operator.

inner radius βR. $0 < \beta < 1$.

outer radius R.

a). PDE in polar coordinates.

$$\frac{\partial T}{\partial t} = \alpha \left[\frac{1}{R} \cdot \frac{\partial}{\partial R}\left(R \frac{\partial T}{\partial R} \right) + \frac{1}{R^2} \cdot \underbrace{\cancel{\frac{\partial^2 T}{\partial \theta^2}}}_{\text{axisymmetric.}} + \underbrace{\cancel{\frac{\partial^2 T}{\partial z^2}}}_{\text{infinitely long}} \right]$$

b).

$$\frac{\partial T}{\partial t} = \alpha \cdot \frac{1}{R} \cdot \frac{\partial}{\partial R}\left(R \frac{\partial T}{\partial R} \right).$$

"2nd-order in R, 1st-order in t!"

→ we need 1 I.C.s. & 2 B.C.s.

c). $-\frac{\partial T}{\partial R} = Qq_t$ at $R = \beta R$ $q \neq 0$ → dimensionless constant

$$\frac{\partial T}{\partial R} = -h(T - T_\infty) \quad \text{at } R = \mathcal{R}.$$

$$T = T_\infty. \quad \text{at } t = 0.$$

$R, t,$ → ind. var

$T.$ → dep. var

$h, T_\infty, Q, q. \beta, \mathcal{R}. \alpha$ → parameters.

→ So we need to non-dimensionalize the system.

$$\textcircled{H} = \frac{T - T_\infty}{QR}, \qquad r^* = \frac{R}{\mathcal{R}}, \; N = hR. \quad \tau = \frac{t}{t_c}$$

Dimensionless variables.

Dimensionless PDE.

$$\frac{1}{t_c}\frac{\partial \theta}{\partial \tau} = \frac{\alpha}{\mathcal{R}^2}\frac{1}{r^*}\cdot\frac{\partial}{\partial r^*}\left(r^* \cdot \frac{\partial \textcircled{H}}{\partial r^*}\right).$$

$$\frac{1}{t_c} \sim \frac{\alpha}{R^2} \rightarrow t_c = \frac{R^2}{\alpha}$$

"letting the const = 1"

$$\boxed{\tau = \frac{\alpha t}{R^2}}$$

the final dimensionless PDE looks like:

$$\frac{\partial \textcircled{H}}{\partial \tau} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial \textcircled{H}}{\partial r^*}\right).$$

$$-\frac{\partial \textcircled{H}}{\partial r^*} = q, \quad r^* = \beta.$$

$$\frac{\partial \textcircled{H}}{\partial r^*} = -N\textcircled{H}, \quad r^* = 1.$$

$$\textcircled{H} = 0, \quad \tau = 0$$

∴ # Sturm-Liouville problems require the B.C.s to be homogenized.

→ decompose: $\textcircled{H} = \tilde{\textcircled{H}} + \textcircled{H}_{ss} \rightarrow \frac{\partial \textcircled{H}_{ss}}{\partial \tau} = 0.$

General sol'n

particular sol'n

∴ analogy in ODE:

$$0 = \frac{1}{r^*}\frac{d}{dr^*}\left(r^* \cdot \frac{d\textcircled{H}_{ss}}{dr^*}\right) \Rightarrow \textcircled{H}_{ss} = k_1 \ln(r^*) + k_2$$

GOAL: $\frac{\partial \tilde{\textcircled{H}}}{\partial \tau} = \frac{1}{r^*}\cdot\frac{\partial}{\partial r^*}\left(r^* \cdot \frac{\partial \tilde{\textcircled{H}}}{\partial r^*}\right).$

$$\frac{\partial \tilde{\textcircled{H}}}{\partial r^*} = 0, \quad \frac{\partial \tilde{\textcircled{H}}}{\partial r^*} = -N\tilde{\textcircled{H}}, \quad \tilde{\textcircled{H}} = -\textcircled{H}_{ss}$$

$r^* = \beta$ $\qquad r^* = 1$ $\qquad \tau = 0$

$$\frac{d\textcircled{H}_{ss}}{dr^*} = -q, \quad \text{at} \quad r^* = \beta.$$

$$\frac{d\textcircled{H}_{ss}}{dr^*} = -N\textcircled{H}_{ss}. \quad \text{at} \quad r^* = 1.$$

After solving the constants:

$$\rightarrow \textcircled{H}_{ss} = \beta q \left[\frac{1}{N} \cdot - \ln(r)\right]$$

S.o.V.:

$$\tilde{\textcircled{H}}(r^*, \tau) = T(\tau). X(r^*).$$

$$\frac{T'}{T} = \underbrace{\frac{1}{X'} \cdot \frac{1}{r^*} \cdot \frac{\partial}{\partial x^*}\left(r^* \frac{\partial X}{\partial r^*}\right) = -\lambda^2}_{\text{separation constant}}$$

definition

↓

the Bessels equation

$$\mathcal{L}(X) = -\lambda^2 X.$$

→ get the B.C.s terms:

$X'(r^* = \beta) = 0.$

$X'(r^* = 1) = -NX.$

> both are homogeneous B.C.s

is this eigenfunction universal ?

$$X_m = A_m J_0(\lambda_m r^*) + B_m Y_0(\lambda_m r^*)$$

↑ the eigenfunctions.

→ eigenvalue condition

at $r^* = \beta$, $X_n' = 0$.

$$\rightarrow A_m \lambda_m J_0'(\lambda_m \beta) + B_m \lambda_m Y_0'(\lambda_m \beta) = 0$$

$$B_m = -A_m . \frac{J_0'(\lambda_m \beta)}{Y_0'(\lambda_m \beta)}.$$

$$X_m(r^*) = A_m \left[J_0(\lambda_m r^*) - \frac{J_0'(\lambda_m \beta)}{Y_0'(\lambda_m \beta)} \cdot Y_0(\lambda\right.$$

$$\frac{T'}{T} = -\lambda^2. \quad \Rightarrow T(\tau) = \exp(-\lambda^2 \tau).$$

$$\tilde{\textcircled{H}}(r^*, \tau) = \sum_{m=1}^{\infty} A_m \exp(-\lambda^2 \tau) X_m(r^*).$$

→ 2nd boundary conditions:

$x' = -Nx$, at $r^* = 1$.

$$A_m \lambda_m J_0'(\lambda_m) + B_m \lambda_m Y_0'(\lambda_m) = -N\left[A_m J_0(\lambda_m) + B_m Y_0\right.$$

i.e., the eigenvalue condition

2 $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -x$.

I.C. $u(x,0) = f(x)$.

$$\left.\frac{dx}{dt}\right|_{\xi} = u. \qquad \left.\frac{du}{dt}\right|_{\xi} = -x \quad \Bigg\} \text{ Set up systems of ODEs}$$

$$\frac{d}{dt}\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}. \rightarrow \vec{v} = \begin{bmatrix} x \\ u \end{bmatrix}.$$

Next step(s): find eigenvalues, eigenvectors, ...

$$\frac{d\vec{v}}{dt} = A\vec{v} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\vec{v} \quad \lambda^2+1=0 \downarrow \lambda = \pm i$$

$\lambda = i$, $\vec{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$.

$\lambda = -i$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$. $\Big\} \vec{v} = \begin{bmatrix} x \\ u \end{bmatrix}$

$\downarrow$

$$C_1\vec{v}_1 \exp(\lambda_1 t) + C_2\vec{v}_2 \exp(\lambda_2 t)$$

$$\begin{bmatrix} x \\ u \end{bmatrix} = C_1 \exp(it)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \exp(-it)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Subs. the I.C.s $\rightarrow \begin{cases} u(\xi,0) = f(\xi) \\ x(t=0) = \xi. \end{cases}$

$\downarrow$

$$C_1 = \frac{\xi - i f(\xi)}{2}, \quad C_2 = \frac{\xi + i f(\xi)}{2}.$$

$\exp(it) = \cos(t) + i\sin(t)$.

$$\vec{v} = \begin{bmatrix} x \\ u \end{bmatrix} = k_1\cos(t) + k_2\sin(t).$$

$\downarrow$

$k = \xi \quad k_2 = -f(\xi)$

Once we see the purely imaginary roots, the solution takes the form: $k_1\cos(t) + k_2\sin(t)$

$x = \xi\cos(t) - f(\xi)\sin(t)$

$u = -k_1\sin(t) + k_2\cos(t)$

$= -\xi\sin(t) - f(\xi)\cos(t)$

Shocks & expansion fan.

3. $\frac{\partial u}{\partial t} + u^2 \cdot \frac{\partial u}{\partial x} = 0$.

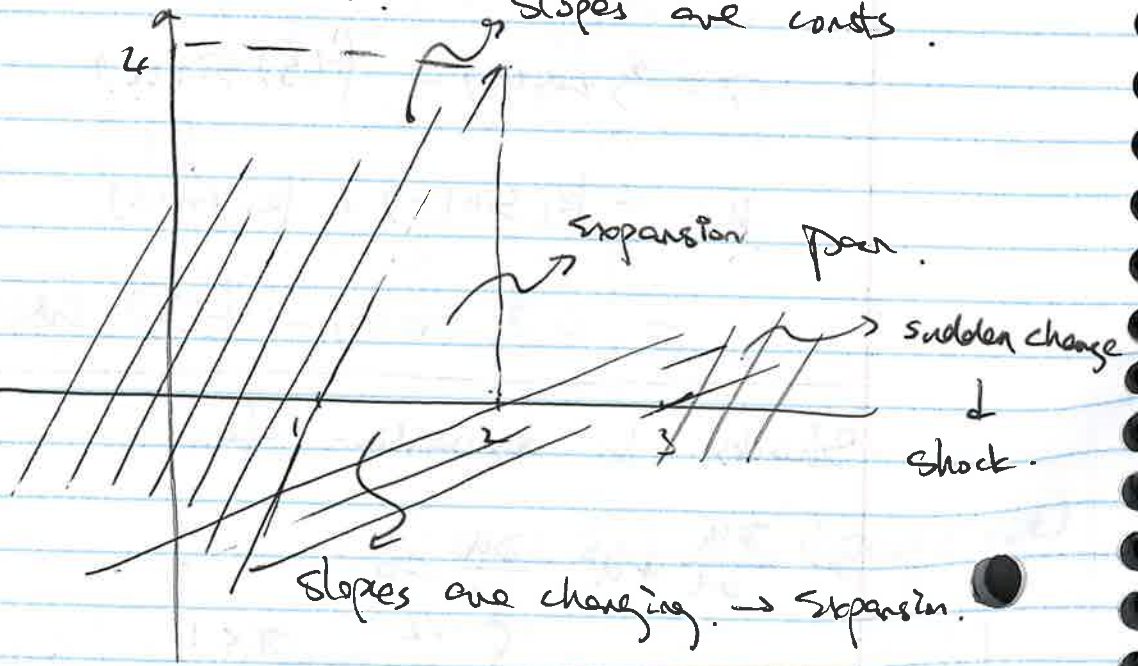
$$\rightarrow u(x,0) = \begin{cases} 1/2 & x < 1. \\ x, & 1 \leq x \leq 2 \\ 1, & 2 < x < 3 \\ 1/ & x \geq 3 \end{cases}$$

$$\left.\frac{du}{dt}\right|_{\xi} = 0. \to \quad u = F(\xi). = \begin{cases} 1/2, & \xi < 1. \\ \xi/2, & 1 \le \xi \le 2 \\ 1 . & 2 < \xi < 3 \\ 1/2, & \xi \ge 3. \end{cases}$$

$$\frac{dx}{dt} = u^2 = (F(\xi))^2.$$

$$x = [F(\xi)]^2 t + \xi.$$

$$\begin{cases} \frac{1}{4}t + \xi. & \xi < 1. \\ \left(\frac{\xi}{2}\right)^2 t + \xi. & 1 \le \xi \le 2. \\ t + \xi, & 2 < \xi < 3. \\ \frac{t}{4} + \xi. & \xi \ge 3. \end{cases}$$



Shock location & time: $(X_{shock}, t_{shock}) = (3, 0)$

Equation of shock → find shock speed

convert PDE to conservative form.

↓

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^3}{3}\right) = 0$$

$$F = \frac{u^3}{3}.$$

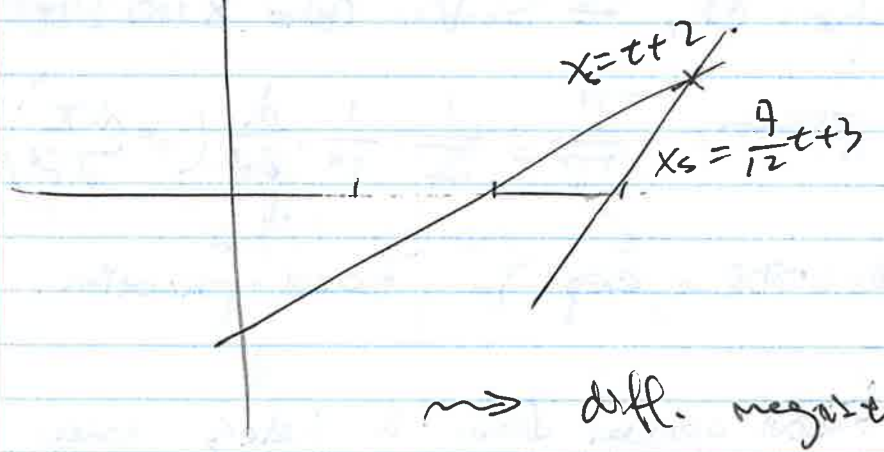
$$\dot{x}_{shock} = \frac{u_R^3 - u_L^3}{3(u_R - u_L)}.$$

$$u_L(\xi = 3^-) = 1.$$

$$u_R(\xi = 3^+) = 1/2$$

$$\dot{x}_{shock} = \frac{7}{12}.$$

$$X_{shock} = \frac{7}{12}t + 3$$



u_L & u_R

⇝ diff. magnitudes. order matters

If $u_L > u_R$ → expansion fan.

$u_L < u_R$ → shock.

#Problem Session 5.

Problem 1 (d).

$$\Theta = \tilde{\Theta} + \Theta_{ss}$$

$\tilde{\Theta} \downarrow$ Unsteady. $\Theta_{ss} \downarrow$ Steady-state

For steady-state: $\frac{d\Theta_{ss}}{d\tau} = 0$

Recall the dimensionless PDE:

$$\underbrace{\frac{1}{t_c}\frac{\partial \Theta_{ss}}{\partial \tau}}_{0} = \frac{\alpha}{R^2}\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \cdot \frac{\partial \Theta_{ss}}{\partial r^*}\right).$$

$\frac{d\Theta_{ss}}{dr^*} = 0$ → the steady state soln is inhomogeneous → Form SL problem.

Second-order ODE for Θ_{ss}: $k_1 \ln(r^*) + k_2$.

→ solve for consts. for Θ_{ss}.

To solve for $\tilde{\Theta}$, → SoV: $\Theta = X(r^*)T(\tau)$.

Form SL problem: $\frac{T'}{T} = \frac{1}{X} \cdot \frac{1}{r^*} \cdot \frac{d}{dr^*}\left(r^* \frac{dX}{dr^*}\right) = -\lambda^2$.

1st order ODE → exp(·) Bessel function.

↓

Recall general form in Lele's notes:

$$R(r) = C_{1n} J_m(\lambda_r r) + C_{2n} Y_m(\lambda_r r).$$

Practice Problems: → Shock & expansion fan.

PDE: $\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} = 0$. $\Rightarrow$ $-\infty < x < \infty$.
$0 \le t < \infty$

$$\text{I.C.s: } u(x, t=0) = f(x) = \begin{cases} 1/2, & x < 1 \\ x/2, & 1 \le x \le 2. \\ 1, & 2 < x < 3 \\ 1/2, & x > 3. \end{cases}$$

→ the PDE in flux form (conservative): $\frac{\partial u}{\partial t} + \frac{\partial(\frac{1}{3}u^3)}{\partial x} = 0$

(a). ▶ the characteristic line: $\left.\frac{\partial x}{\partial t}\right|_{\xi} = u^2$.

▶ the characteristic solution curve: $\left.\frac{\partial u}{\partial t}\right|_{\xi} = 0$

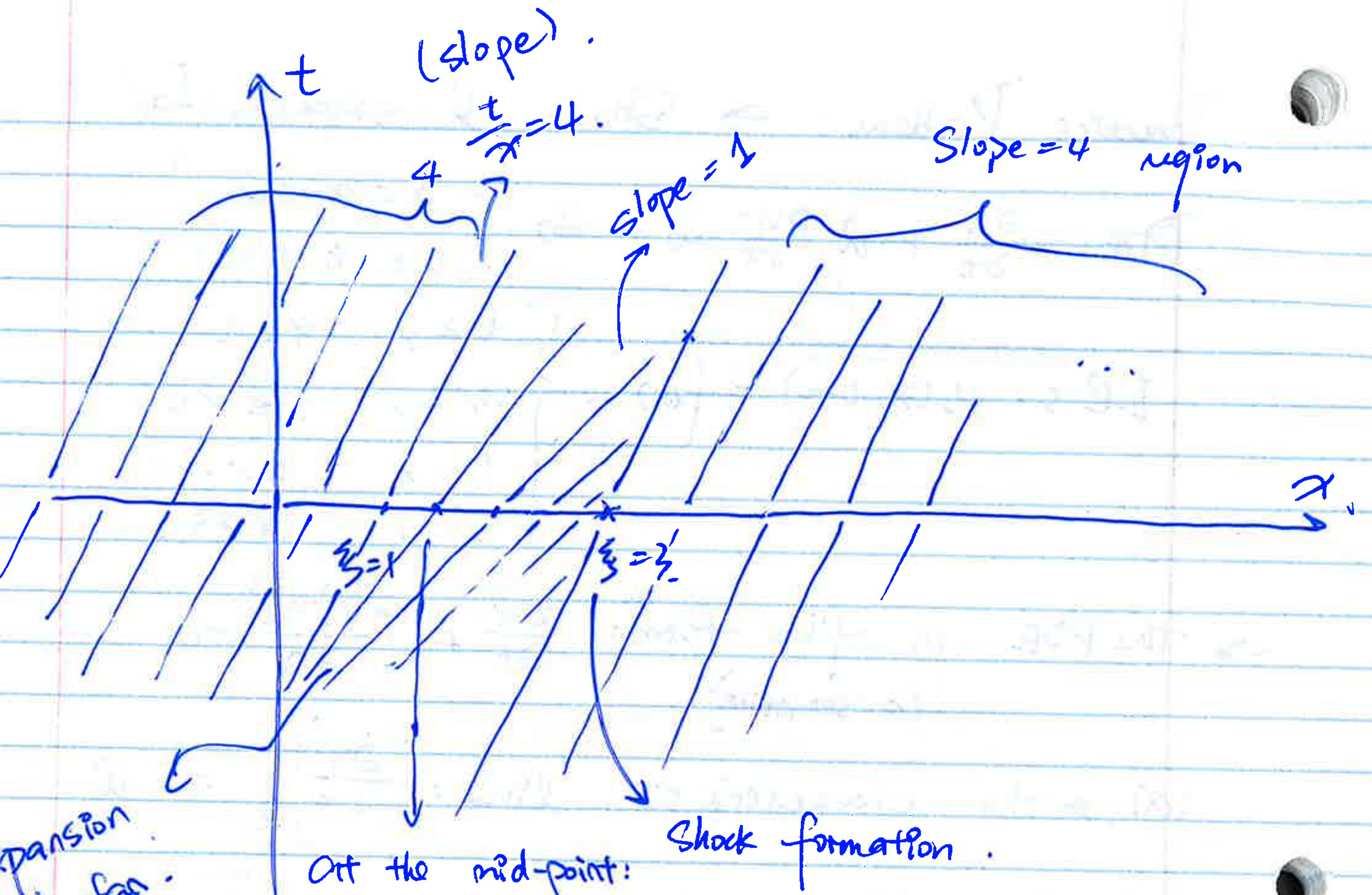
→ $x = u^2 t + \xi$. $\rightsquigarrow$ $\xi = x - u^2(\xi)t$.

$u = F(\xi)$.

$$\text{Applying the ICs: } F(\xi) = \begin{cases} 1/2, & \xi < 1. \\ \xi/2, & 1 \le \xi < 2. \\ 1, & 2 < \xi < 3 \\ 1/2, & \xi > 3. \end{cases}$$

$$\rightarrow t = \frac{x - \xi}{[F(\xi)]^2} = \begin{cases} 4(x-\xi), \xi < 1. \\ \frac{4x}{\xi^2} - \frac{4}{\xi}, & 1 \le \xi < 2. \\ x - \xi. & 2 < \xi < 3. \\ 4(x-\xi), & \xi > 3. \end{cases} \quad \cdots (*).$$

Based on (*), we can sketch the characteristics,



At the mid-point:

$\xi = 1.5$

$$\frac{4x}{\left(\frac{3}{2}\right)^2} - \frac{4}{\left(\frac{3}{2}\right)} = \frac{16x}{9} - \frac{8}{3}$$

(b) Determine the shock position.

Recall the equation for calculating the shock speed:

$$\dot{X}_s = \frac{F_L - F_R}{u_L - u_R} = \frac{\frac{1}{3}u_L^2 - \frac{1}{3}u_R^2}{u_L - u_R}$$

Since shock is being formed at $\xi = 3$.

$$\dot{X}_s = \frac{\frac{1}{3} - \frac{1}{3}\cdot\frac{1}{8}}{\frac{1}{2}} = \frac{7}{24}\cdot 2 = \frac{7}{12}.$$

$$X_s = \frac{7}{12}t + 3.$$

(c). Determine time & location for $t_{shock/fan}$.

We may begin with deriving the formula for the position of the expansion fan.

Recall: $t = \frac{4x}{\xi^2} - \frac{4}{\xi}$. → $\xi \in [1, 2)$.

↓

$$x = \frac{\xi^2}{4}t + \xi.$$

$$\frac{\xi^2}{4}t + \xi = \frac{7}{12}t + 3$$ ⟵ solve for position.

$$\left(\frac{\xi^2}{4} - \frac{7}{12}\right)t = 3 - \xi.$$

$$t = \frac{3 - \xi}{\frac{\xi^2}{4} - \frac{7}{12}} = \frac{36 - 12\xi}{3\xi^2 - 7}.$$

Recall: $\xi = x - \frac{\xi^2}{4}t$.

$$\frac{t}{4}\xi^2 + \xi - x = 0.$$

$$\xi = \frac{-1 \pm \sqrt{1 + t}}{\frac{t}{2}}$$

$$= \frac{1}{t}\left(-2 \pm 2\sqrt{1+t}\right).$$

↑

$$\xi^2 = \frac{1}{t^2}\left[4 \pm 8\sqrt{1+t} + 4(1+t)\right] = \frac{1}{t^2}\left[8 + 4t \pm 8\sqrt{1+t}\right]$$

From the sketch, one can tell that the expansion wave hits the shock at $\xi = 2$.

$$\xi = 2 \downarrow \quad x-2=t \quad x=t+2$$

$$t+2 = \frac{7}{12}t+3$$

$$\frac{5}{12}t = 1$$

$$t = \frac{12}{5} \rightarrow x = \frac{12}{5}+2 = \frac{22}{5}$$

(d). determine the solution of the expansion fan.

→ Recall the characteristic for the expansion fan:

$$t = \frac{4x}{\xi^2} - \frac{4}{\xi} \rightarrow \xi^2 t + 4\xi = 4x$$

$$\hookrightarrow \xi = \frac{-4 \pm \sqrt{16+16xt}}{2t} = \frac{-2 \pm 2\sqrt{1+xt}}{t}$$

→ solution: $u = F(\xi). = \frac{\xi}{2}$

$$\hookrightarrow u = \frac{-1 \pm \sqrt{1+xt}}{t}.$$

Since ξ is defined in the region $[1, 2]$

$$\rightarrow \xi > 0 \rightarrow u = \frac{-1+\sqrt{1+xt}}{t}$$

(e). determine the expression for the shock speed after it hits the fan.

→ After the expansion fan hits the shock, u_L is the expression we derived, $u_R = 1/2$.

$$\dot{X}_s = \frac{F_L - F_R}{u_L - u_R} = \frac{\frac{1}{3}\left(\frac{-1+\sqrt{1+xt}}{t}\right)^3 - \frac{1}{3}\left(\frac{1}{2}\right)^3}{\sim}$$

4. when forming the matrix-vector ODE:

1). original equation

2). consistency condition.

$$\rightarrow \vec{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \downarrow \begin{cases} \phi_1 = \frac{\partial P}{\partial t} \\ \phi_2 = \frac{\partial P}{\partial x} \end{cases}$$

$$\frac{\partial \phi_1}{\partial t} + 2U\frac{\partial \phi_1}{\partial x} + (U^2 - c^2)\frac{\partial \phi_2}{\partial x} = 0.$$

$$\frac{\partial \phi_2}{\partial t} - \frac{\partial \phi_1}{\partial x} = 0$$

3) separate the "∂_t" and "∂_x"

4) → focus on the ∂_x part.

$$\frac{\partial}{\partial x}\begin{bmatrix} 2U & U^2-c^2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \rightarrow \text{original system.}$$

5) the two rows of the matrix stand for the two equations

Practice Midterm 2019-2020.

$$\frac{\partial u}{\partial t} + (1 - \frac{u}{2} + u^2)\frac{\partial u}{\partial x} = 0 \quad \begin{cases} -\infty < x < \infty \\ 0 \le t < \infty \end{cases}$$

$$u(x,0) = F(x) = \begin{cases} 2, & -\infty < x < 0. \\ 2-x, & 0 \le x < 1. \\ 1, & 1 \le x < \infty \end{cases}$$

To determine the characteristics, ⇒ Equation of characteristics and the characteristic solution curves:

$$\left.\frac{\partial x}{\partial t}\right|_{\xi} = 1 - \frac{u}{2} + u^2 \rightarrow \left[1 - \frac{u}{2} + u^2\right]t + \xi = x.$$

$$\left.\frac{\partial u}{\partial t}\right|_{\xi} = 0 \rightarrow u = F(\xi). \quad \cdots (*)$$

Recall the initial condition:

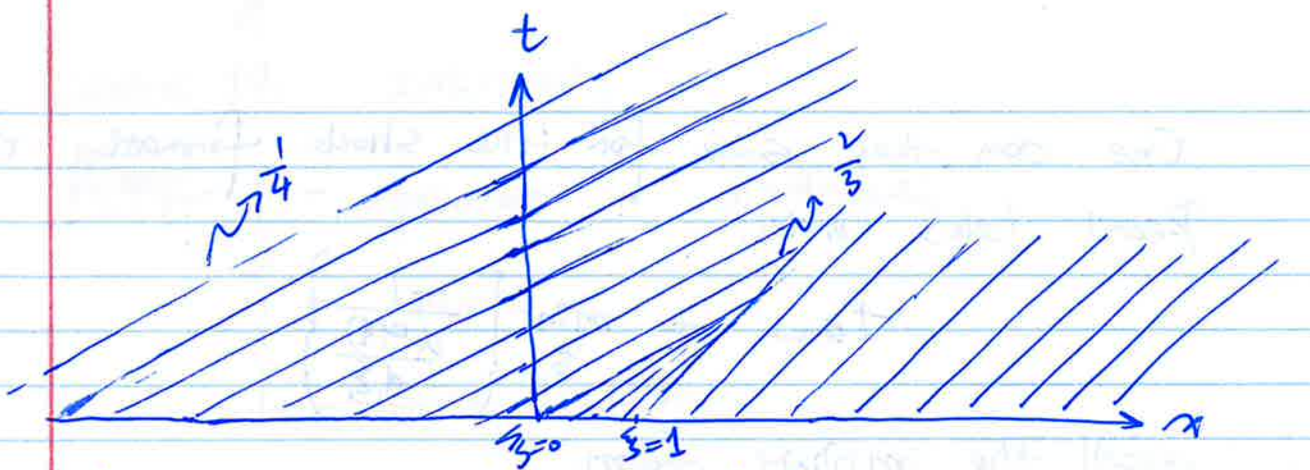
$$F(\xi) = \begin{cases} 2, & -\infty < \xi < 0. \\ 2-\xi, & 0 \le \xi < 1. \\ 1, & 1 \le \xi < \infty \end{cases}$$

modifying Eqn. (*).

$$t = \frac{x - \xi}{1 - \frac{u}{2} + u^2} = \frac{x - \xi}{1 - \frac{F(\xi)}{2} + [F(\xi)]^2}$$

$$t = \begin{cases} \frac{x-\xi}{4}, & -\infty < \xi < 0. \\ \frac{x-\xi}{\xi^2 - \frac{7}{2}\xi + 4}, & 0 \le \xi < 1 \\ \frac{x-\xi}{3/2}, & 1 \le \xi < \infty. \end{cases}$$

We can then sketch the characteristics



2. the solution contains a shock.

3. to find the explicit solution, recall the equation of the characteristics.

$$\xi = x - \left[1 - \frac{u}{2} + u^2\right]t = \underline{\underline{x - t + \frac{ut}{2} - u^2 t}}.$$

$$u = F(\xi) = F\left(x - t + \frac{ut}{2} - u^2 t\right).$$

Substitute into the shock region:

$\xi=0$: $\xi = x - t + t - 4t = x - 4t$

$\xi=1$: $\xi = x - t + \frac{t}{2} - t = x - \frac{3}{2}t$.

"transition region"!

combine $u = 2 - \xi$

$$2 - u = x - t + \frac{ut}{2} - u^2 t$$

$$tu^2 - \left(1 + \frac{t}{2}\right)u + 2 - x + t = 0$$

$$u = \frac{\left(1+\frac{t}{2}\right) \pm \sqrt{\left(1+\frac{t}{2}\right)^2 - 4t(2-x+t)}}{2t}$$

One can then solve for the shock formation time.

Recall Lele's notes,

$$t_{shock} = \min_{\xi}\left\{\frac{-1}{\frac{dF(\xi)}{d\xi}}\right\}$$

recall the implicit form,

$$u = F\left(x - t + \frac{ut}{2} - u^2 t\right).$$

$$\frac{\partial u}{\partial x} = \left[1 - t(2u - \tfrac{1}{2})\frac{\partial u}{\partial x}\right]F'(\xi).$$

$$\frac{\partial u}{\partial x} = F'(\xi) - t(2u - \tfrac{1}{2})\frac{\partial u}{\partial x}\cdot F'(\xi).$$

$$F'(\xi) = \frac{\frac{\partial u}{\partial x}}{1 - t(2u - \frac{1}{2})\frac{\partial u}{\partial x}}$$

$$t = \min\left\{\frac{t(2u - \frac{1}{2})\frac{\partial u}{\partial x} - 1}{\underbrace{\frac{\partial u}{\partial x}}}\right\}.$$

$$t(2u - \tfrac{1}{2}) - \cancel{\frac{1}{u}}$$

Problem 2.

$$u(x,0) = G(x) = \begin{cases} 1, & -\infty < x < 0. \\ 2, & 0 \le x < \infty \end{cases}$$

$$\downarrow$$

$$F(\xi) = \begin{cases} 1, & -\infty < \xi < 0. \\ 2, & 0 \le \xi < \infty \end{cases}$$

11
↑
Lecture 12. 2/15/2024.

Recap
- Operators → differential.
- Adjoints.
- Sturm-Liouville.

Setting $L_0\phi = a_2(x)\phi'' + a_1(x)\phi' + a_0(x)\phi,$

$\forall\ x \in [a,b]$

$a_j \in C([a,b])$ & $a_2 < 0.$

$$p(x) = \exp\left(\int_a^x \frac{a_1(s)}{a_2(s)}\,ds\right).$$

$\vee$ 0.

$$w(x) = \frac{-p(x)}{a_2(x)}$$

$$q_1(x) = a_0(x)\,w(x)$$

then $L_0\phi = \lambda\phi$ is equivalent to

$$\underbrace{-(p\phi')' + q\phi} = \lambda w\phi.$$

$L_1\phi$ is self-adjoint

$$(\psi, L_1\phi) = (\phi, L_1\psi)$$

$$(f,g) = \int_a^b f\bar{g}\,dx$$

Let $L\phi = \frac{L_1\phi}{w(x)}$.

$\Rightarrow L\phi = -\frac{(p\phi')'}{w} + \frac{q\phi}{w}$.

$L\phi = \lambda\phi$ ← Some eigenvalue problem.

... this is not self-adjoint

Recall: A new weighted space.

$$L^2_w(a,b) = \left\{\phi : \|\phi\|_w = \left(\int_a^b |\phi(x)|^2 w(x)\, dx\right)^{1/2} < \infty\right\}$$

↓

"bounded"

$$(\phi, \psi)_w = \int_a^b \phi(x)\bar\psi(x)\, w(x)\, dx$$

$$\underbrace{(L\phi, \psi)_w = (\phi, L\psi)_w}$$

in our setting, Boundary conds.

$$B_1\phi = C_1\phi(a) + C_2\phi'(a) = 0$$

$$B_2\phi = C_3\phi(b) + C_4\phi'(b) = 0$$

$$|C_1| + |C_2| \neq 0 \quad \& \quad |C_3| + |C_4| \neq 0$$

For such B.C.s. → char. along B.C.s self-adjoint

for self-adjoint.

⇓

$$J(\phi, \psi)\Big|_a^b = p(x)\left(\phi'\bar\psi - \phi\bar\psi'\right)\Big|_a^b = 0$$

if ψ also has the same B.C.s as ϕ.

then $J(\phi, \psi)\Big|_a^b = 0$

$T \equiv \{B_1, B_2, L\}$. is self-adjoint.

Theorem:

a). $L_0\phi = \lambda\phi$ is equivalent to $L\phi = \lambda\phi$.

b). $T = \{L, B_1, B_2\}$ is self-adjoint.

c). T has a countable sequence of real distinct eigenvalues. $\{\lambda_n\}_{n=1}^{\infty}$ with $\lambda_n \to \infty$.

d) the corresponding eigenfunctions $\{\phi_n\}$ may be chosen to form a basis of $L^2_w(a,b)$

e). These eigenfunctions are orthogonal in the weighted space, $L^2_w(a,b)$, s.t.

$$\frac{\int_a^b \phi_n \bar\phi_m\, w\, dx}{\int_a^b \phi_n \bar\phi_n\, w\, dx} = \begin{cases} 0, & n \neq m \\ 1, & n = m. \end{cases} \to \delta_{nm}$$

↖ normalization const.

f). Any $f \in L^2_w(a,b)$ can be written as

$$f = \sum_{n=1}^{\infty} C_n \phi_n .$$

$$\& \quad C_n = (f, \phi_n)_w .$$

✱ Normalization of orthogonal Eigenfunctions.

$\{\phi_n\}_{n=1}^{\infty}$ are not orthonormal.

they can be normalized using the inner product: e.g., $\psi_n = \dfrac{\phi_n}{(\phi_n, \phi_n)_w}$

⇝ handling Periodic B.C.s

$$\phi(a) = \phi(b) \quad \& \quad \phi'(a) = \phi'(b)$$

$$\equiv \quad \phi(a) - \underbrace{\phi(b) = 0 \quad \& \quad \phi'(a)}_{\text{not separable !!!}} - \phi'(b) = 0$$

→ admits $\lambda = 0$ and $\phi = 1$

Therefore, $J_w(\phi, \psi) = 0$ still holds

$\{L, P_1, P_2\}$ is still self-adjoint

$$\begin{tikzcd} P_1 \arrow[d] & P_2 \arrow[d] \\ \text{PBC1} & \text{PBC2} \end{tikzcd}$$

✱ Additionally, eigenvalues are not simple.

i.e., for each eigenvalue, there are two eigenfunctions.

Consider the heat equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

B.C.s: $u(0,t) = u(1,t) < 0, \quad \forall t > 0.$

I.C.s: $u(x,0) = f(x), \quad 0 < x < 1$

the solution u_k writes.

$$u_k = C_k e^{-k^2\pi^2 t} \sin(k\pi x)$$

the general sol'n:

$$u = \sum_{k=1}^{\infty} A_k u_k(x,t).$$ ← generalized Fs series

1). as long as $f \in L^2_n(0,1)$.

2). $\{\sin k\pi x\}_{k=1}^{\infty}$ are orthogonal

3). $u(x,0) = f(x)$.

$$f(x) = \sum_{n=1}^{\infty} A_k \sin(k\pi x)$$ just the Fourier Series

$$A_k = \frac{(f, \phi_k)}{(\phi_k, \phi_k)}$$ ← projections along each basis

$\phi_k = \sin(k\pi x)$ i.e., normalization.

$$\frac{\int_0^1 f \sin(k\pi x) dx}{\int_0^1 \sin(k\pi x)\sin(k\pi x)\, dx}$$ ← weight = 1.

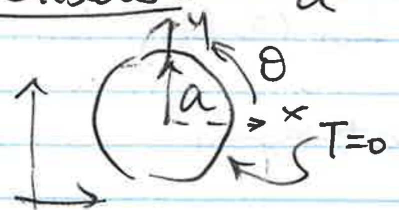
Corresponding ODE:

$X'' + \lambda X = 0.$ ← harmonic oscillator

$X(0) = 0$

$X(1) = 0$ i.e., Sturm Liouville problem.

Consider: a heated cylindrical rod



$$\frac{\partial T}{\partial t} = \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$ ← nondim. ~

$x = a\cos\theta$, $y = a\sin\theta$

Using chain rule,

$$\frac{\partial T}{\partial t} = \frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right).$$ ← modified PDE.

I.C.s $T(r,0) = f(r)$.

$T(a,t) = 0$

$T(0,t) = $ finite.

Using the SoV ansatz, we have

$$T(r,t) = R(r), T(t)$$

Plugging in the solution Ansatz:

$$\frac{1}{R(r)}\frac{1}{r}\cdot\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) = \frac{1}{T(t)}\frac{dT(t)}{dt} = -\lambda^2$$

↑ separation const.

→ $R(r)$ equation.

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\lambda^2 R(r).$$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\cdot\frac{dR}{dr} = -\lambda^2 R(r)$$

B.C.s: $T(a,t) = R(a)T(t) = 0$

$\Rightarrow R(a) = 0$

multiply by r^2 : $r^2R''(r) + rR'(r) + r^2\lambda^2R(r) = 0$

→ General Bessel Eqn.:

$$x^2\phi'' + x\phi' + (x^2-\nu^2)\phi = 0$$

of order ν.

$$\phi(x) = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

↑ Bessel's func of 1st kind

Bessel's func of 2nd kind.

- Problem Session 6

Sturm-Liouville Problem

$$\mathcal{L}y = -(p(x)y')' + q(x)y = \lambda r(x) y.$$

$$\mathcal{L}y = \lambda r(x) y.$$

$$\frac{\mathcal{L}y}{r(x)} = \lambda y.$$

new operator → $\mathcal{L}_1 y = \lambda y$.

p, q, r are real-valued, continuous funcs

$p(x) > 0, \quad r(x) > 0, \quad x \in [a, b]$

→ finite domain

B.C.s $\quad c_1 y(a) + c_2 y'(a) = 0. \quad$ (B.C.1)

$c_3 y(b) + c_3 y'(b) = 0. \quad$ (B.C.2)

①. All eigenvalues are real, distinct, & can be ordered.

$$\lambda_1 < \lambda_2 < \cdots < \infty.$$

② the eigenfunctions corresponding to distinct eigenvalues are orthogonal to each other.

$\lambda_m \to \phi_m, \Rightarrow \mathcal{L}_1(\phi_m) = \lambda_m \phi_m.$

$\lambda_n \to \phi_n,$

$$\langle \phi_m, \phi_n \rangle = \int_a^b r(x)\,\phi_m(x)\,\phi_n(x) = N_m \delta_{mn}.$$

$N_m = \langle \phi_m, \phi_m \rangle.$

③. Any function in L_r^2 and satisfy B.C.s:

↳ is bounded.

$$f(x) = \sum_{m=1}^{\infty} C_m \phi_m(x).$$

where $C_n = \dfrac{\langle f, \phi_m \rangle}{\langle \phi_m, \phi_m \rangle}.$

→ lies in L_r^2 space

↓

$f(x) \in L_r^2.$

$$\left\{ f : \left(\int_a^b |f(x)|^2 r(x)\,dx \right)^{1/2} < \infty \right\}$$

Singular SL problem.

$p(a)$ and/or $r(a) = 0 \rightarrow a$ singular point

$p(b)$ and/or $r(b) = 0 \rightarrow b$ singular point.

Let's say a is a singular pt.

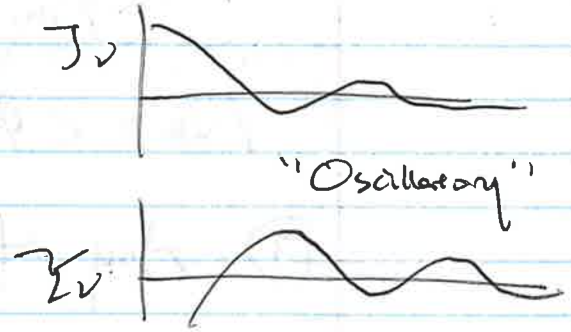
1). B.C.s on b.

2). regularity requirement on a.

i.e., $|y(a)| < \infty$.

Example on Singular Sturm-Liouville Problem.

Bessel's function



$$t^2 y'' + t y' + (t^2 - \nu^2) y = 0$$

2 ind. solns:

1) $J_\nu \Rightarrow$ Bessel functions of first kind. of order ν,

2). $Y_\nu \Rightarrow$ Bessel function of second kind of order ν,

$$y = C_1 J_\nu(t) + C_2 J_\nu(t)$$

Modified Bessel Equation

$$t^2 y'' + t y' - (t^2 + \nu^2) y = 0.$$

① $I_\nu(x) = i^{-\nu} J_\nu(ix)$. $\rightarrow$ ① exponentially increasing ② well-behaved @ 0.

② $K_\nu = \frac{\pi}{2}\left\{\frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)}\right\}$ ① exponentially decaying. ② singular at $t=0$

$$y = C_1 I_\nu(t) + C_2 K_\nu(t).$$

$\rightarrow$ Not Oscillatory. in nature.

Recall Sturm-Liouville Problem.

$$-(py')' + q(x) y = \lambda r(x) y \qquad (**)$$

$$t^2 y'' + t y' + (t^2 - \nu^2) y = 0 \quad t = \sqrt{\lambda} x.$$

$$\Rightarrow x^2 y'' + x\frac{dy}{dx} + (\lambda x^2 - \nu^2) y = 0$$

$\frac{d^2y}{dx^2}$

$$\Rightarrow \underbrace{x\frac{d^2y}{dx^2} + \frac{dy}{dx}}_{} + \left(\lambda x + \frac{\nu^2}{x}\right) y = 0$$

$$\Rightarrow d\left(x \cdot \frac{dy}{dx}\right) - \frac{\nu^2}{x} y = -\lambda x y.$$

$$\Rightarrow -d\left(x\frac{dy}{dx}\right) + \frac{\nu^2}{x} y = \lambda x y. \quad (*)$$

Compare eqn $(*)$ & $(**)$:

$p(x) = x, \quad q(x) = \frac{\nu^2}{x}, \quad r(x) = x$

domain ⇒ $x \in [0, 1]$.

Impose B.C.s on the domain.

① $y(1)=0$

② y is well-behaved at $x=0$.

In general, the solution writes:

$$y(x) = C_1 J_\nu(\sqrt{\lambda} x) + C_2 Y_\nu(\sqrt{\lambda} x)$$

$\nu = 0$.

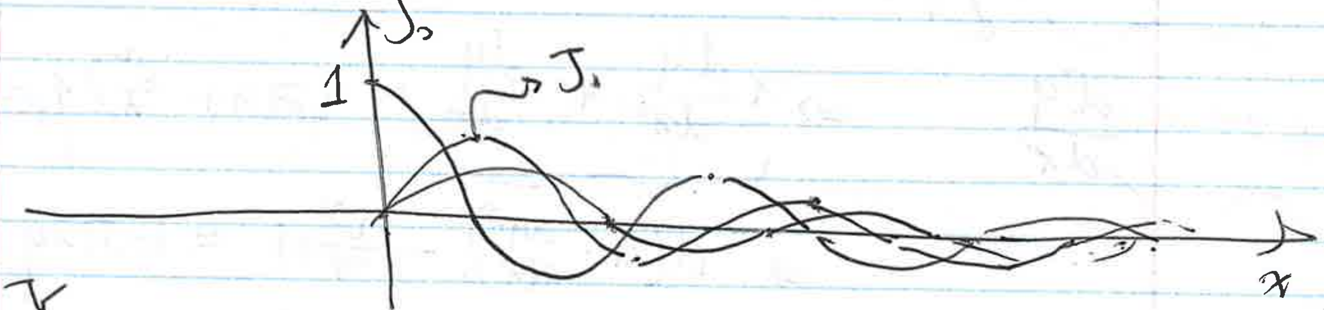
$$\Rightarrow y(x) = C_1 J_0(\sqrt{\lambda} x) + C_2 Y_0(\sqrt{\lambda} x)$$

$$y(0) = C_1 J_0(0) + C_2 Y_0(0) \Rightarrow C_2 = 0$$

$$y(1) = C_1 J_0(\sqrt{\lambda}) + C_2 Y_0(\sqrt{\lambda}) = 0$$

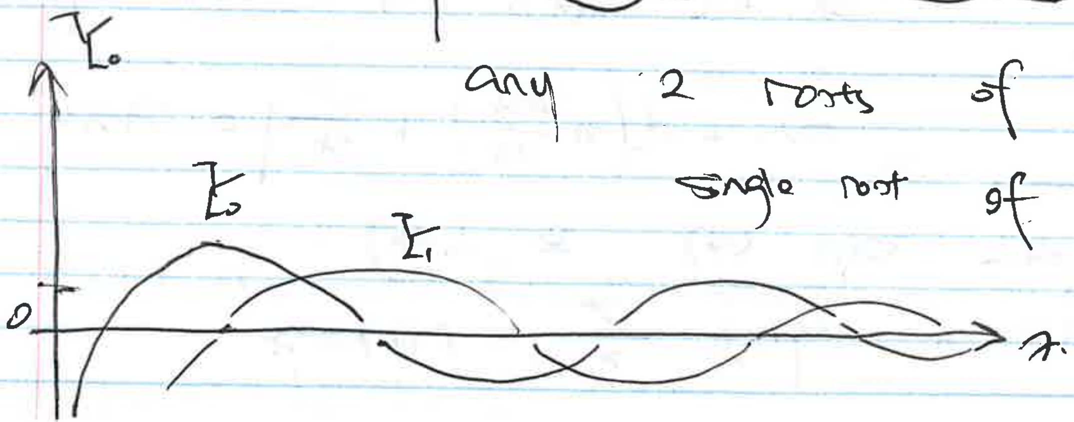
$$y(1) = C_1 J_0(\sqrt{\lambda}) = 0$$

In general,



any 2 roots of J_{n+1}.

single root of J_n.



$J_0(\sqrt{\lambda}) = 0$

$\sqrt{\lambda_n} = j_{0,n}$

$\lambda_n = j_{0,n}^2$.

$\phi_n = J_0(j_{0,n} x)$.

$$\int_0^1 x J_0(j_{0,n} x) J_0(j_{0,m} x)\, dx = 0$$

$m \neq n$.

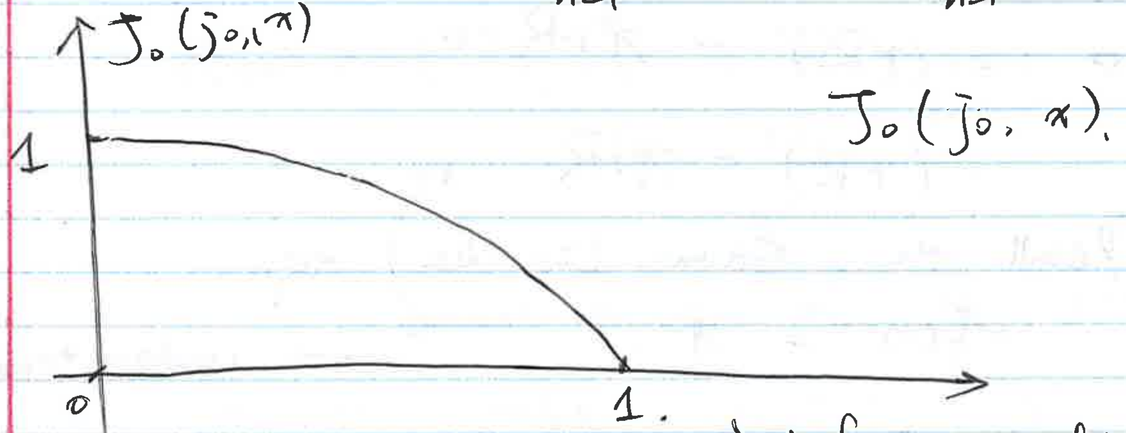
↓

"orthogonality property"

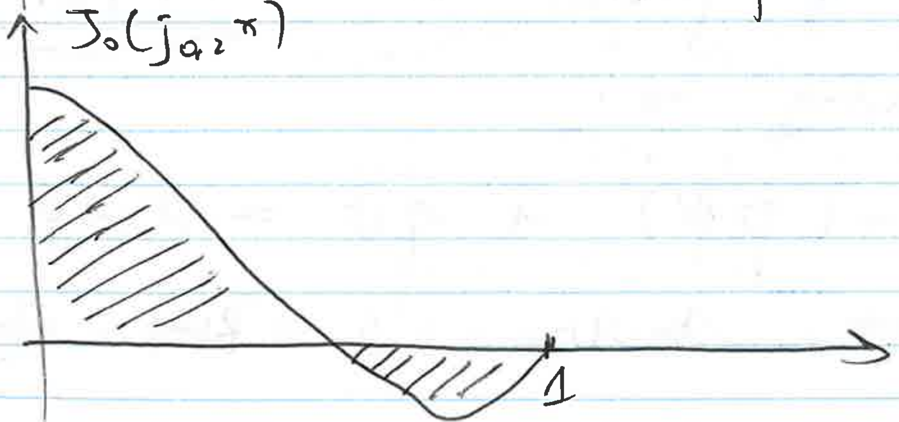
Solution is the linear combination of the eigenfunctions (orthogonal)

↑

Final soln: $y = \sum_{n=1}^{\infty} C_n \phi_n = \sum_{n=1}^{\infty} C_n J_0(j_{0,n} x)$



↳ first eigenfunction

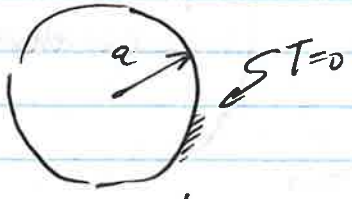


Lecture 13. 2/20/2024.

Recap: Formalize S.L. properties.

↳ 1D heat equation.

↳ 2D heat equation.



solving $\frac{\partial T}{\partial t} = \frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial T}{\partial r}\right)$.

↑ "non-dimensionalized".

For space. $r^2 R'' + rR' + \lambda^2 r^2 R$ (★)

↓ "dividing by r".

$$- rR'' + R' - \lambda^2 r R = 0$$

$$\rightsquigarrow \; -(rR')' - \lambda^2 r R = 0.$$

$$-(rR')' = \lambda^2 r R.$$

Recall the Sturm-Liouville eqn.

$p(r) = r$. → understand singular behavior

weighting function w → $w(r) = r$. → orthogonality

comparing to

$$-(p\phi')' + q\phi = \lambda w \phi$$

Substitute $x = \lambda r$, & $R(r) = \Phi(\lambda r)$.

then $\frac{dR}{dr} = \lambda \frac{d\Phi}{dr}$

$$\frac{d^2R}{dr^2} = \lambda^2 \frac{d^2\Phi}{dr^2}.$$

then $r^2\lambda^2\Phi'' + r\lambda\Phi' + \lambda^2 r^2 \Phi = 0$.

$$x^2\Phi'' + x\Phi' + x^2\Phi = 0.$$

Bessel equation of 0th order.

General form:

$$x^2\Phi'' + x\Phi' + (x^2 - \nu^2)\Phi = 0$$

the sol'n to this eqn:

$$\Phi(x) = C_1 J_\nu(x) + C_2 Y_\nu(x)$$

⇓

Bessel function of the "νth" order.

Note: for our problem, $x = \lambda r$, $\nu = 0$.

Sol'n: ⇒ $C_1 J_0(\lambda r) + C_2 Y_0(\lambda r)$.

Recall the cond.: $T(r=0, t)$ = finite.

$Y_0 \to -\infty$ as $r \to 0$.

Note: we can handle only one inhomogeneity.

therefore, the sol'n writes:

$$R(r) = C_1 J_0(\lambda r).$$

$$R(1) = C_1 J_0(\lambda) = 0. \quad \leftarrow \text{B.C.}$$

Either $C_1 = 0$ or $J_0(\lambda) = 0$.

$$\Rightarrow J_0(\lambda) = 0.$$

$\lambda = Z^n \rightsquigarrow$ zeros of the Bessel functions.

The eigenfunctions.

$$\{J_0(Z_n r)\}_{n=1}^{\infty}$$

✩ Orthogonality

$$\int_\Omega J_0(Z_m r)\, J_0(Z_n r)\, r\, dr = N_n \delta_{nm}.$$

$$\rightsquigarrow N_n = \int_\Omega J_0(Z_n r)\, J_0(Z_n r)\, r\, dr$$

General sol'n to this problem:

Now, $T(r,t) = \sum_{n=1}^{\infty} C_n J_0(Z_n r)\, e^{-Z_n^2 t}$

$$T(r, t=0) = \sum_{n=1}^{\infty} C_n J_0(Z_n r) = f(x)$$

$$C_n = \frac{(f(x), J_0(Z_n r))_w}{(J_0(Z_n r) \cdot J_0(Z_n r))_w} \quad \leftarrow w = r$$

Schrödinger's Eqn.

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi, \quad \hbar = \frac{h}{2\pi}.$$

↳ Hamiltonian operator (total energy).

✩ in classical mechanics,

$$H = \frac{p^2}{2m} + V,$$

↳ potential. ener.

$$p = mv.$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V} \qquad \text{"a formulation"}$$

$$\hat{p} \equiv \frac{\hbar}{i}\underline{\nabla} \Rightarrow \frac{\hat{p}^2}{2m} = \frac{-\hbar^2}{2m}\underline{\nabla}\cdot(\underline{\nabla}).$$

Ψ is wave function.

$$\int_\Omega |\Psi|^2 \, d\Omega = 1 \quad \text{"normalized"}.$$

$\Psi(\underline{x}, t) \rightsquigarrow$ separate in to products of functions (Goal).

Recall the Ansatz: $\Psi(\underline{x}, t) = X(\underline{x})\, T(t)$.

Subs. the Ansatz back to Schrödinger Eqn.

$$i\hbar \frac{1}{\Psi(t)} \cdot \frac{d\Psi(t)}{dt} = -\frac{\hbar^2}{2m} \cdot \frac{1}{\Psi(x)} \nabla^2 \Psi + \mathcal{V}.$$

↳ only when = Const.

$\downarrow$

E

Recall: $i\hbar \frac{df}{dt} = Ef.$

$$\left.\begin{aligned} \frac{df}{dt} &= -i\frac{Ef}{\hbar} \\ f(t) &= \exp\left(-\left(\frac{iE}{\hbar}\right)t\right). \end{aligned}\right\} \text{time.}$$

$$-\frac{\hbar^2}{2m} \cdot \nabla^2\Psi + \mathcal{V}\Psi(x) = E\Psi(x) \quad \text{space.}$$

↳ time-independent Schrödinger eqn.

Sol'ns are called "Stationary States".

$$\Psi(x,t) = \sum_{n=1}^{\infty} C_n \Psi_n(x) \cdot e^{-\frac{iEt}{\hbar}}.$$

$$\hat{H}\Psi(x) = E\Psi(x)$$

Transform ∇^2 to spherical coordinates.

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta} \cdot \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right).$$
$$+ \frac{1}{r^2\sin\theta}\left(\frac{\partial^2}{\partial\phi^2}\right)$$

polar coor. expansion.

$\uparrow$

$$\frac{\hbar^2}{2m}\nabla^2 = \frac{\hbar^2}{2m}\left[\quad \sim \quad\right]\Psi$$
$$+ \mathcal{V}(x)\Psi(x) = E\Psi.$$

$$\Psi(r,\theta,\phi) = R(r)Y(\theta,\phi).$$

$$-\frac{\hbar^2}{2m}\left[\frac{Y}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R(r)}{r^2\sin\theta} \cdot \frac{\partial}{\partial\theta}\left(\sin\theta \cdot \frac{\partial Y}{\partial\theta}\right) + \frac{R}{r^2\sin^2\theta}\frac{\partial^2 Y}{\partial\theta^2}\right].$$

$$+\mathcal{V}RY = ERY$$

$\uparrow$

... Subs. the SoV formulation.

dividing by YR.

$$-\frac{\hbar^2}{2m}\left\{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + (\mathcal{V} - E)\right\}$$
$$-\frac{\hbar^2}{2m}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \cdot \frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta} \cdot \frac{\partial^2 r}{\partial\phi^2}\right\} = 0.$$

$\uparrow$

☆ Sol'n is the spherical harmonics.

Lecture #14 2/20/2024.

ev.
- Heat eqn. in polar coordinates.
- Schrödingers eqn.
  - ↳ SoV, time-independent S.E.
    - → Stationary states soln.
    - ↳ Spatial S.E.
      - ↳ radial eqn.
      - ↳ Angular eqn.

Ansatz: $\psi(r, \theta, \phi) = R(r)\,Y(\theta, \phi)$.

$$-\frac{\hbar^2}{2m}\left[\frac{Y}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial Y}{\partial\theta}\right) + \frac{R}{r^2\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right]$$

n.

$$+ V(r)RY = ERY$$

$$-\frac{\hbar^2}{2m}\left\{\frac{1}{R}\cdot\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)\right\} + (V(r) - E)r^2.$$

$$= \frac{\hbar^2}{2m}\frac{1}{Y}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\cdot\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right\} \Rightarrow$$

Postive const. ← $l(l+1)$

eigenvalues vibrational modes

2 eqns. ... eigenvalues are the "so-called" "quantum sts."

$$-\frac{\hbar^2}{2m}\left\{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)\right\}$$

$$+ (V(r) - E)r^2 = l(l+1)$$

$$\& \quad \frac{1}{Y}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi}\right\}$$

$$= -l(l+1)$$

$$\sin\theta\frac{\partial}{\partial\theta}\cdot\left(\sin\theta\cdot\frac{\partial Y}{\partial\theta}\right) + \frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)\sin^2\theta\, Y$$

Ansatz $Y(\theta, \phi) = \Theta(\theta)\Phi(\phi)$

$$\rightsquigarrow \left\{\frac{1}{\Theta}\left[\sin\theta\frac{d}{d\theta}\left(\sin\theta\cdot\frac{d\Theta}{d\theta}\right)\right] + l(l+1)\sin^2\theta\right\}$$

$$+ \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$$

$$\frac{1}{\Theta}\left[\sin\theta\frac{d}{d\theta}\left(\sin\theta\cdot\frac{d\Theta}{d\theta}\right)\right] + l(l+1)\sin^2\theta = m^2$$

$$\& \quad \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2$$

another separation constant

$$\rightsquigarrow \Phi(\phi) = e^{im\phi}$$

Since the Ansatz $e^{im\phi}$ is periodic.

$$\Phi(\phi + 2\pi) = \Phi(\phi).$$

$$\exp\left[ing(\phi + 2\pi)\right] = \exp(im\phi)$$

$$\Rightarrow \exp(2\pi i m) = 1.$$

$\forall$ m being an integer, $m=0, \pm 1, \pm 2.$

$$\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{\Theta}{d\theta}\right) + \left[\ell(\ell+1)\sin^2\theta - M^2\right]\Theta = 0$$

$\rightsquigarrow$ A form of associated Legendre eqn.

Let $x = \cos\theta$, $\frac{d}{d\theta} = \frac{dx}{d\theta}\cdot\frac{d}{dx} = -\sin\theta\frac{d}{dx}.$

ugging in Legendre eqn. :

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -\sin\theta\frac{d}{dx}\left[-\sin^2\theta\cdot\frac{d\Theta}{dx}\right]$$

$$\Rightarrow \sin\theta\cdot\frac{d}{dx}\left((1-\cos^2\theta)\frac{d\Theta}{dx}\right).$$

$$\Rightarrow \sin\theta\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right]$$

Eventually we have :

$$\frac{d}{dx}\left((1-x^2)\frac{d\Theta}{dx}\right) + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)\Theta = 0$$

the solution:

$$P_\ell^m(x). \equiv (1-x^2)^{|m|/2}\left(\frac{d}{dx}\right)^{|m|} \underbrace{P_\ell(x)}_{\text{Legendre polynomials.}}$$

$$P_\ell(x) = \frac{1}{2^\ell \ell!}(x^2-1)!.$$

... Rodrigues formula.

$$\Theta(\theta) = AP_\ell^m(x) = AP_\ell^m(\cos\theta).$$

Associated Legendre polynomials are orthogonal.

ℓ must be non-negative integers.

$|m| > \ell$, then $P_\ell^M = 0.$

→ for any given "ℓ", there are $(2\ell+1)$ possible values of m.

Second solution blows up. → "eliminated".

↓

at $\Theta = 0$ & $\theta = \pi$

associated Legendre functions of

$Q_\ell^m(x)$

$$\int |\psi|^2 \underbrace{r^2 \sin\theta \, dr \, d\theta \, d\phi}_{d\Omega} = 1 .$$

... in polar coordinates.

$$\int_0^\infty |R|^2 r^2 \, dr = 1 \quad \& \quad \int_0^{2\pi}\int_0^{\pi} |Y|^2 \sin\theta \, d\theta \, d\phi = 1$$

By formulating Sturm-Liouville,

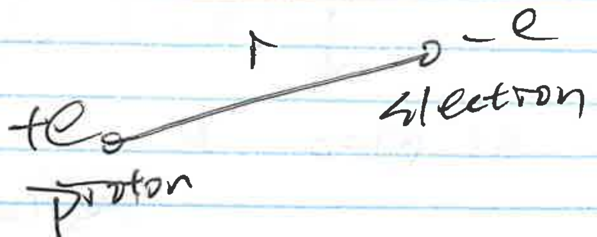
$$Y_l^m(\theta,\phi) = N_{lm}^{-1} e^{(im\phi)} P_l^m(\cos\theta).$$

$$N_{lm}^{-1} = \epsilon \sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}} \Rightarrow \epsilon = \begin{cases} (-1)^m & m \geq 0. \\ 1 & m < 0 \end{cases}$$

$$\int_0^{2\pi}\int_0^{\pi} \left[Y_l^m(\theta,\phi)\right]^* \left[Y_{l'}^{m'}(\theta,\phi)\right] \overbrace{\sin\theta \, d\theta \, d\phi}^{d\Omega} = \delta_{ll'}\delta_{mm'}$$

Y_l^m are called spherical harmonics.

Hydrogen atom.



Using Coulomb's law, $V(r) = \frac{-e^2}{4\pi\epsilon_0} \cdot \frac{1}{r}$

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]R = l(l+1)R$$

Let $u = rR(r)$. $R = u/r$.

$$\frac{dR}{dr} = \frac{\left[r\frac{du}{dr} - u\right]}{r^2} \Rightarrow r^2 \frac{dR}{dr} = \left[r \frac{du}{dr} - u\right]$$

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = r \frac{d^2u}{dr^2} + \cancel{\frac{du}{dr}} - \cancel{\frac{du}{dr}}$$

$$\Rightarrow r\frac{d^2u}{dr^2} - \frac{2mr}{\hbar^2}\left[V(r) - E\right]u = l(l+1)R.$$

Subs. u into the original PDE.

$$\Rightarrow -r\frac{d^2u}{dr^2} + \frac{2mr}{\hbar^2}\left[V(r) - E\right]u = -l(l+1)R$$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V(r) + l(l+1)\frac{Rr}{r}\frac{\hbar^2}{2mr} = Eu.$$

Eventually, we land in the following eqn:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V(r) + \frac{\hbar^2}{2m} \cdot \frac{l(l+1)}{r^2}\right]u = Eu$$

↳ mass of the particle.

Problem Session #7.

Legendre Eqn.

$-1 \le x \le 1$

$\Updownarrow$

$|x| \le 1$

$(1-x^2)y'' - 2xy' + \lambda^2 y = 0.$ (general form.)

$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}y\right] = -\lambda^2 y.$ (Standard form).

$p(x) = 1-x^2$ $r(x) = 1.$ $q(x) = 0.$

$x = \pm 1.$ are the singular points of this ODE.

∴ no more a regular Sturm-Liouville problem.

$$\lim_{\substack{x\to+1\\ x\to-1}}\left[\psi_n'\psi_m - \psi_n\psi_m.\right] \to 0$$

$\lambda^2 = m(m+1).$ where $m = 0, 1, 2, \ldots.$

$P_m(x) \longrightarrow$ Legendre Polynomials

$m=0,$ $P_0(x) = 1.$

$m=1,$ $P_1(x) = x.$

$m=2,$ $P_2(x) = \frac{1}{2}(3x^2-1)$

Inner Product.

$$\int_{-1}^{1} 1 \cdot P_n(x) P_m(x)\, dx = \frac{2}{2m+1}\cdot\delta_{nm}$$

Let $x = \cos\theta$, $dx = -\sin\theta\, d\theta$

$y(x) \Leftrightarrow \psi(\theta)$ $\hookrightarrow$ $-1 \le x \le 1.$ $0 \le \theta \le \pi.$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\cdot\frac{d\psi}{d\theta}\right) = -\lambda^2\psi$$

$p(\theta) = \sin\theta.$

$r(\theta) = \sin\theta.$ ... weighting function

$q(\theta) = 0$

$\hookrightarrow \lambda^2 = m(m+1),\ m = 0, 1, 2, \ldots$

$\Downarrow$

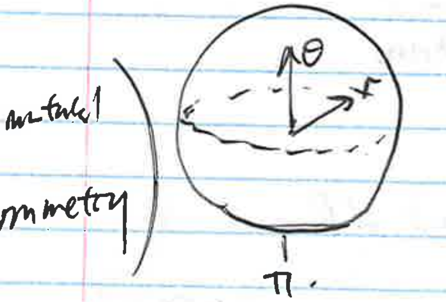
Eight function

$\psi_m = P_m(x) = P_m(\cos\theta).$

"finite soln at $x = \pm 1$"

$$\int_0^{\pi} \sin\theta \cdot P_n(\cos\theta) P_m(\cos\theta)\, d\theta = \frac{2}{2n+1}\cdot\delta_{nm}$$

Example cooling of a sphere.



I.C.: $\rightarrow T(r, \theta, t=0) = \Delta T f(r,\theta)$

B.C. $\rightarrow T(r=R, \theta, t) = 0$.

$T(t) \rightarrow$ finite. @ $r=0$, $\theta=0$ or π

$$\frac{\partial T}{\partial t} = \alpha \cdot \nabla^2_{(r,\theta,\phi)} T = \alpha\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2\left(\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)\right]$$

unsteady heat conduction (cooling).

Non-dimensionalization:

$$\textcircled{H} = \frac{T}{\Delta T}, \quad \xi = \frac{r}{R}, \quad \tau = \frac{\alpha t}{R^2}.$$

Dimensionless Problem:

$$\frac{\partial \textcircled{H}}{\partial \tau} = \frac{1}{\xi^2}\frac{\partial}{\partial \xi^2}\left(\xi^2 \cdot \frac{\partial \textcircled{H}}{\partial \xi}\right) + \frac{1}{\xi^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \cdot \frac{\partial \textcircled{H}}{\partial\theta}\right)$$

I.C: $\textcircled{H}(\xi, \theta, \tau=0) = f(\xi, \theta)$

B.C.s: $\textcircled{H}(\xi=1)=0$, $\textcircled{H}(\tau) \rightarrow$ finite, @ $\begin{cases}\xi=0\\ \theta=0 \text{ or } \pi\end{cases}$

$\textcircled{H} = \mathcal{T}(\tau) H(\xi, \theta)$. $\quad \frac{\partial \textcircled{H}}{\partial\tau} = \nabla^2_{(\xi,\theta)} \textcircled{H}$

$$H\mathcal{T}' = \mathcal{T}\nabla^2_{(\xi,\theta)}(H)$$

$$\rightarrow \frac{\mathcal{T}'}{\mathcal{T}} = \frac{1}{H}\cdot\underbrace{\nabla^2_{(\xi,\theta)}(H) = -\lambda^2}_{\text{Sturm-Liouville Problem (eventually).}}$$

time

first order, recall Ansatz.

$$\frac{\mathcal{T}'}{\mathcal{T}} = -\lambda^2$$

Further expand the Laplacian portion:

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2 \cdot \frac{\partial H}{\partial \xi}\right) + \frac{1}{\xi^2 \sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta \cdot \frac{\partial H}{\partial\theta}\right) = -\lambda^2 H$$

homogeneous, can separate H.

$\psi \rightarrow$ finite @ $\begin{cases}\theta=0\\ \theta=\pi\end{cases}$

$$\frac{\partial}{\partial\xi}\left(\xi^2 \cdot \frac{\partial H}{\partial\xi}\right) + \lambda^2 H \xi^2 = \frac{-1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \cdot \frac{\partial H}{\partial\theta}\right).$$

S.o.V: $H(\xi, \theta) = R(\xi)\psi(\theta)$.

$$-\frac{1}{R}\left[\frac{d}{d\xi}\xi^2\frac{dR}{d\xi} + \lambda^2 R\xi^2\right] = \underbrace{\frac{1}{\psi}\frac{1}{\sin\theta}\frac{d}{d\theta}\cdot\left(\sin\theta\frac{d\psi}{d\theta}\right) = -\beta^2}_{\text{SL-problem.}}$$

$$\begin{cases} R(1)=0 \\ R(0) \to \text{finite} \end{cases}$$

... Eigenfunctions in θ-space:

$$\Upsilon(\theta) = P_m(\cos\theta), \quad \beta^2 = m(m+1), \quad m = 0, 1, 2, \dots$$

ξ-space:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\left(\frac{dR}{d\xi}\right) - \frac{\overset{\to \beta^2}{\overbrace{m(m+1)}}}{\xi^2}R = -\lambda^2 R.$$

les is the eigenvalue

B.C.s in ξ-space:

$R(\xi=1)=0, \quad R(\xi=0) \to$ finite.

... Spherical Bessel's equation of n^{th} order.

↳ Solution: spherical Bessels function

⇓

$$R = A\, j(\lambda\xi) + B\, \cancel{y_m(\lambda\xi)} \to \text{finite}$$

$$R_m(\xi) = j_m(\lambda_{nm}\xi)$$

Applying the B.Cs

$R(\xi=1)=0 \longrightarrow j_m(\lambda_{nm}\xi) \Rightarrow$ eigenvalue condition.

$H(\xi,\Theta) = R(\xi)\,\Psi(\Theta)$ ... spherical harmonics

$\Theta(\xi,\theta,\tau)$.

↑

$$\Theta = \sum_{m=0}^{\infty}\sum_{m=1}^{\infty} A_{nm} \exp\left[-(\lambda_{nm})^2\tau\right] \cdot j_m(\lambda_{nm}\xi)\, P_m(\cos\theta).$$

↑ in Bessels functions

How to find the Fourier coefficients?

Applying I.C.s

$$\Theta(\xi,\theta,\tau=0) = f(\xi,\theta) \Rightarrow f(r,\theta)$$

$$= \sum_{m=0}^{\infty}\sum_{m=1}^{\infty} A_{nm} \exp\left(-(\lambda_{nm})^2\tau\right) j_0(\lambda_{nm}\xi)\, P_m(\cos\theta)$$

double summation, double inner-product.

Integrate w.r.t. θ,

$$\int_{\theta=0}^{\pi} f(\xi,\theta)\, P_s(\cos\theta)\, d\theta = \sum_{M=1}^{\infty} A_{sm}\, j_s(\lambda_{sm}) \frac{2}{(2s+1)}$$

Integrate w.r.t. ξ. (weighting function ξ^2)

$$\int_0^1 \xi^2_{js}(\lambda_{sm})\, j_s(\lambda_{sp})\, d\xi = N_p\, \delta_{mp}$$

$$\boxed{A_{sp} N_p \frac{2}{2s+1}}$$

↳ ξ goes from 0 to 1

⇑

$$\Rightarrow \int_0^1 \xi^2 j_s(\lambda_{sp}\xi) \int_{\theta=0}^{\pi} \sin\theta\, f(\xi,\theta)\, P_s(\cos\theta)\, d\theta$$

Lecture #15 2/27/2024

"last lecture on SoV."

Review. HW #5 (Pb. 3).

$\Delta u = 4.$ $(x,y) \in \Omega.$

$$\begin{cases} u(x,y) = 1, & x^2+y^2=1, \quad y>0. \\ u(x,y) = 0, & x^2+y^2 = 1, \quad y<0. \end{cases}$$

$v = u - (x^2+y^2).$

↳ subs. the "ansatz". getting rid of the inhomogeneity.

$\nabla^2 v = 0$

$v(x,y) = 1-1=0$, $y>0.$ $x^2+y^2=1.$

$v(x,y) = 0-1=-1$, $y<0.$ $x^2+y^2=1.$

"Needs to understand the B.C.s.

$v(r,\theta) = R(r)\,\textcircled{H}(\theta)$ ↘ subs. back to B.C.s.

$R(1)\,\textcircled{H}(\theta) = 0.$ $\forall$ $0 \le \theta < \pi.$

$R(1)\,\textcircled{H}(\theta) = -1$ $\forall$ $\pi < \theta \le 2\pi.$

$\textcircled{H}(\theta) = 0,$ $\forall$ $0 \le \theta \le \pi.$

Exam Prob.

$\textcircled{H}(\theta) = -\frac{1}{R(1)},$ $\forall \pi < \theta \le 2\pi.$

↳ piecewise constant $\textcircled{H}(\theta)$.

→ So your sol'n does not dep. on $\textcircled{H}$.

So we need to look for $\lambda=0$ sol'ns.

$\lambda=0.$ → $\textcircled{H}$ ~ piecewise const.

$\lambda>0$: $\textcircled{H}(\theta) = K_1 \sin(\sqrt{\lambda}\theta) + K_2 \cos(\sqrt{\lambda}\theta).$

↳ trivial sol'n. *** IMPORTANT

~~Recap~~

→ Sol'n to spherical harmonics. (S.E).

→ Radial eqn.

recall:

$$-\frac{\hbar^2}{2mr^2}\left\{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + [V(r) - E]\right\}.$$

$$-\frac{\hbar^2}{2mr^2}\frac{1}{Y}\left\{\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\cdot\frac{\partial^2 Y}{\partial\phi^2}\right\} = 0$$

Multiplying by $\left(-\frac{2mr^2}{\hbar^2}\right)$ ...

$$\frac{1}{R}\frac{d}{dr}\left[r^2\frac{dR}{dr}\right]-\frac{2mr^2}{\hbar^2}\left[\mathcal{V}(r)-E\right]R$$

$$=\ell(\ell+1)R$$

Recall:

H-atom:

$$\mathcal{V}(x)=\frac{-e^2}{4\pi\varepsilon_0}\frac{1}{r}$$

Let. $u=rR(r)$. $R=\frac{u}{r}$.

$$\frac{dR}{dr}=\left[r\frac{dR}{dr}-u\right]/r^2.$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\frac{d}{dr}\left(r\frac{du}{dr}-u\right).$$

$$=r\frac{d^2u}{dr^2}+\frac{du}{dr}-\frac{du}{dr}$$

$$=r\cdot\frac{d^2u}{dr^2}$$

$$\Rightarrow r\frac{d^2u}{dr^2}-\frac{2mr}{\hbar^2}\left[\mathcal{V}(r)-E\right]u=\ell(\ell+1)R.$$

$$r\frac{d^2u}{dr^2}-\frac{2mr}{\hbar^2}\mathcal{V}(r)+\frac{2mr}{\hbar^2}Eu=\ell(\ell+1)R.$$

$$-r\frac{d^2u}{dr^2}+\frac{2mr}{\hbar^2}\mathcal{V}(r)+\underbrace{\ell(\ell+1)R}_{Rr/r}=\frac{2mr}{\hbar^2}Eu.$$

... multiplying by $\frac{\hbar^2}{2mr}$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+\mathcal{V}(r)+\ell(\ell+1)\frac{Rr}{r}\frac{\hbar^2}{2mr}=Eu.$$

~ find forms of eigenvalue problem.

this implies:

$$-\frac{\hbar^2}{2m}\cdot\frac{d^2u}{dr^2}+\left[\mathcal{V}(r)+\frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u$$

Eigenvalue prob. ← $=Eu$.

hope to subs. the Coulomb pot. into $\mathcal{V}$.

Let $K=\frac{\sqrt{-2mE}}{\hbar}$ → $E<0$. (Energy st.)

bound st. of atoms

$$\mathcal{V}(r)=\frac{-e^2}{4\pi\varepsilon_0}\cdot\frac{1}{r}$$

for $E<0$, $K\in\mathbb{R}$.

3. Divide the both sides by E, and use the substitution of K:

$$\frac{1}{K^2}\frac{d^2u}{dr^2}=\left[1-\frac{me^2}{2\pi\varepsilon_0\hbar^2K}\frac{1}{(Kr)}\right.$$

$$\left.+\frac{\ell(\ell+1)}{(Kr)^2}\right]u.$$

Let $\rho=Kr$, &

$$\rho_0=\frac{me^2}{2\pi\varepsilon_0\hbar^2K}$$

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{\ell(\ell+1)}{\rho^2}\right]u.$$

as $\rho = kr \rightarrow \infty$

$$\frac{d^2u}{d\rho^2} = u.$$

the soln to this eq'n:

$$u(\rho) = A e^{-\rho} + B e^{\rho}.$$

$e^{\rho} \rightarrow \infty$ as $\rho \rightarrow \infty$, So we don't want this term.

$u(\rho) \sim A e^{-\rho}$; large ρ

↳ IMPORTANT: always follow the Soln ansatz.

if $\rho \rightarrow 0$. $\frac{\ell(\ell+1)}{\rho^2}$ dominates

∴ Methods of dominance balances.

$$\frac{d^2u}{d\rho^2} = \frac{\ell(\ell+1)}{\rho^2}u$$

$\rho^{\lambda} \rightarrow u(\rho) = C\rho^{\ell+1} + D\rho^{-\ell}$

as $\rho \rightarrow 0$, $\rho^{-\ell} \rightarrow \infty$ ∴ $D = 0$.

∴ $u(\rho) = C\rho^{\ell+1}$. $(\ell > 0)$

Let $v(\rho) = \rho^{\ell+1} e^{-\rho} u$

Ideally, we want to find $\frac{du}{d\xi}$ & $\frac{d^2u}{d\xi^2}$.

$$\rho\frac{d^2v}{d\rho^2} + 2(\ell+1-\rho)\frac{dv}{d\rho} + [\rho_0 - 2(\ell+1)]v = 0$$

NEW EQN. in term of v

if you chose: $\begin{cases} \nu = 2\ell+1. \\ x = 2\rho. \\ \lambda = j_{max} = n-\ell-1. \end{cases}$

$$\boxed{x\phi'' + (\nu+1-x)\phi' + \lambda\phi = 0} \qquad \phi = u.$$

↳ special eqn: Associated Laguerre eqn.

the sol'ns to this eqn, is called Associated Laguerre polynomials.

$$L^{p}_{q-p}(x) = (-1)^p\left(\frac{d}{dx}\right)^p \underbrace{L_q(x)}$$

↳ Laguerre polynomials.

"they are orthogonal" ←

$$L_q(x) = e^{x}\left(\frac{d}{dx}\right)^q\left(e^{-x}x^q\right)$$

$$a_{j+1} = \left\{ \frac{2(j+l+1) - \rho_0}{(j+1)(j+2l+2)} \right\} a_j .$$

$\hookrightarrow a_{j_{max}} = 0$

$\rightarrow 2(j_{max} + l + 1) - \rho_0 = 0$

Let $n = j_{max} + l + 1$.

$2n - \rho_0 = 0. \rightarrow \rho_0 = 2n$

$\downarrow$

Principal Quantum Number

$$E_n = - \left[\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi \varepsilon_0} \right)^2 \right] \frac{1}{n^2}, \quad \forall n = 1, 2, \ldots$$

$\hookrightarrow$ Bohr formula

$$E = \frac{\hbar^2 k^2}{2m} = \frac{-me^4}{8\pi^2 \varepsilon_0^2 \hbar^2 \rho_0^2}$$

we can then express the sol'n:

$$\psi_{nlm}(r, \theta, \phi) = R_{nl}(r) \cdot Y_l^m(\theta, \phi).$$

$$R_{nl}(r) = \frac{1}{r} \rho^{l+1} e^{-\rho} v(\rho).$$

$$v(\rho) = L_{n-l-1}^{2l+1}(2\rho)$$

Laguerre polynomials

"Schiff's QM."

We can then write the sol'n for hydrogen atoms :

Normalization

$$\psi_{nlm} = \underbrace{\sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}}}\, e^{-r/na} \left(\frac{2r}{na}\right)^l L_{n-l-1}^{2l+1} \left(\frac{2r}{na}\right) Y_l^m(\theta, \phi).$$

IMPORTANT RESULTS

where $a = \frac{4\pi \varepsilon_0 \hbar^2}{me^2}$, denoted as "Bohr radius"

$$\int \psi^*_{nlm} \psi_{n'l'm'} \underbrace{r^2 \sin\theta \, dr \, d\theta \, d\phi}_{d\Omega} = \delta_{nn'} \delta_{ll'} \delta_{mm'}$$

... Orthonormality

Inhomogeneties

till now, → linear PDE for SoV.

→ Separable < SoV ansatz. gives ODEs.

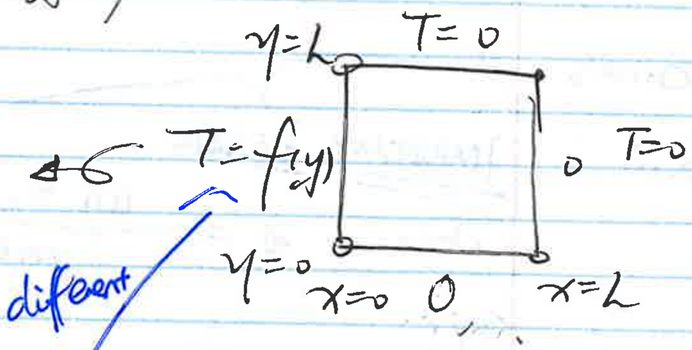
↳ derivatives are w.r.t. only one ind. var.

→ All but "1" condition have to be homogeneous.

→ Simple Inhomogeneities.

eq. Poisson's eqn.

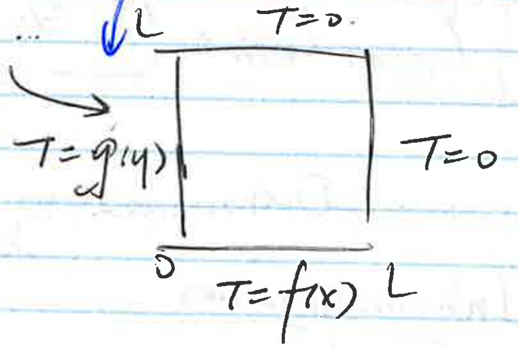
$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right) = 0$$



One can solve with ⇐ SoV.

≠ different

What happens if ...



we are solving ⇐ linear eqns. ...

we may construct.

$$T(x,y) = T_1(x,y) + T_2(x,y).$$

↳ superposition

$$\begin{cases} T_1(x,0) = f(x). \\ T_1(x,L) = 0, \\ T_1(0,y) = 0 \\ T_1(L,y) = 0 \end{cases} \qquad \begin{cases} T_2(x,0) = 0 \\ T_2(x,L) = 0. \\ T_2(0,y) = g(y). \\ T_2(L,y) = 0 \end{cases}$$

2 SoV.

superimpose two fields?

Lecture #16 2/29/2024.

Recap Schrödinger eqn.

→ Associated Legendre eqn.

→ radial, polar, azimuthal.

→ eigenfunc, Quantum number were eigenvals.

→ Inhomogeneous B.C. (more than 1).

Inhomogeneities

$$T_{xx} + T_{yy} = 0$$

$$T(x,0) = 0 \quad ; \quad T(x,L) = 0$$

$$T(0,y) = f(y) \; ; \; T(L,y) = 0.$$

Soln:

$$T(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{L}\right) \sinh\left(\frac{n\pi(L-x)}{L}\right)$$

Recall the separated ODEs:

$$X'' - \lambda^2 X = 0. \rightsquigarrow \text{SL (nonsql.}$$

$$Y'' + \lambda^2 Y = 0.$$

— Coefficients (using homogeneity).

$$T(0,y) = \sum_{n=1}^{\infty} K_n \sin\left(\frac{n\pi y}{L}\right) \sinh(\pi x) = f($$

$$K_n = \frac{\left(f(y),\ \sin\left(\frac{n\pi y}{2}\right)\right)}{\left(\sin\left(\frac{n\pi y}{L}\right),\ \sin\left(\frac{n\pi y}{L}\right)\right)} \rightarrow (f(y), \phi_n)$$

(ϕ_n, ϕ_n)

if $T(x,0) = f(x), \quad T(x,L) = 0,$

$T(0,y) = g(y). \quad T(L,y) = 0.$

By linearity:

$$T(x,y) = T_1(x,y) + T_2(x,y).$$

T_2 will satisfy

$$\begin{cases} \boxed{T(x,0) = f(x)}, & T(x,L) = 0. \\ T(0,y) = 0, & T(L,y) = 0 \end{cases}$$

T_1 will satisfy

B.C.s to solve for x & y seperately

$T(x,0) = 0, \quad T(x,L) = 0, \quad \boxed{T(0,y) = g(y)}$

$T(L,y) = 0$

Super-position.

Inhomogeneities

-- two stretegies.

1). Use superposition to reduce the problem to multiple standard S.o.V problems.

2). Iff. there is a bulk of inhomogeneity.

... Can one $\boxed{\text{guess}}$ a substitution s.t. PDE becomes homogeneous.

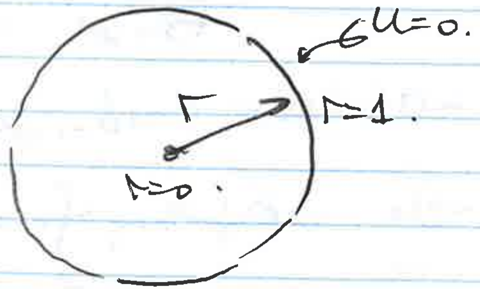
--New variable: $v = u + u_p$.

\# General Inhomogeneties.

Recall $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial t} - f(r,t).$

i.e, 2D heat conduction in polar coordinates.

(infinitely long cylinder).



→ there is an inhomogeneity in the eqn. recall eigenfunction exp.:

B.C.s: $\begin{cases} u(r=0, t) = \text{finite}. \\ u(r, t=0) = 0 \end{cases}$

Reasonably assuming: $f(r,t) = \sum_{n=1}^{\infty} c_n \phi_n$

S.o.V: $r^2R'' + rR' + \lambda^2 r^2 R = 0$

$T' = -\lambda^2 T$

$\rightarrow T(t) = a e^{-\lambda^2 t}$

Contains all the temporal terms

$R(r) = C_1 J_0(z_n r)$

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0(z_n r) \cdot e^{-z_n^2 t}$$

Sol'n to the homogeneous PDE.

why [illegible] as B.f.

$$f(r,t) = \sum_{n=1}^{\infty} C_n(t) J_0(z_n(r))$$

$$C_m(t) = \frac{1}{N_m} \int_0^1 f(r,t) J_0(z_n r)\, r\, dr$$

$$N_m = \frac{J_1^2(z_m)}{2}.$$

Bessel func.,
the inner prod. is weighted,
the weighting func is r.

General sol'n writes:

$u = u^h + u^p.$

Sol'n to homogeneous PDE

Sol'n to inhomogeneous PDE with zero B.C.s.

$$u^p(r,t) = \sum_{n=1}^{\infty} A_n(t) J_0(z_n r).$$

Substitute in the PDE:

$$\sum_{n=1}^{\infty} A_n(t) \left[J_0(z_n r) \right]'' = \overbrace{\sum_{n=1}^{\infty} \frac{dA_n}{dt} J_0(z_n r)}^{\frac{\partial u}{\partial t}}$$

Spatial derivatives.

$$- \underbrace{\sum_{n=1}^{\infty} C_n(t) J_0(z_n r)}_{f(r,t).}$$

#Notation on the 1st Bessel func (w.r.t. deriv. of r).

"Should be"

$$\left[J_0(r) \right]'' \rightarrow \frac{1}{r} \cdot \frac{\partial}{\partial r} \left[r \frac{\partial}{\partial r} \left(J_0(z_n r) \right) \right]$$

$$\text{LHS}: \sum_{n=1}^{\infty} A_n(t) \left[-z_n^2 J_0(z_n r) \right]$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{dA_n(t)}{dt} J_0(z_n r)$$

(properties of the Bessel functions).

$$- \sum_{n=1}^{\infty} C_n(t) J_0(z_n r)$$

$$- A_n(t) z_n^2 = -C_n(t) + \frac{dA_n(t)}{dt}.$$

$$\frac{dA_n(t)}{dt} = C_n(t) - A_n(t) z_n^2.$$

Solving 1st-order ODE w/ inhomog.

coeff. from inhomog.

ξ → dummy var.

$$A_n(t) = \int_0^t e^{\lambda_n^2 t'} C_n(t')\, dt'.$$

↓

Sol'n in terms of func. of t.

$$C_n(t) = \frac{(f(x,t), \phi_n)_w}{(\phi_n, \phi_n)}$$

... End of S.F

Fourier Transform

$\mathcal{F}(f)$. → transform PDE into ODEs.

↓

Inverse Fourier map

By def. : $$\hat{f}(\underline{k}) = \frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{\infty} f(\underline{\xi})\, e^{ik\underline{\xi}}\, d\underline{\xi}.$$

wave vector

N → dimension, i.e., $\mathbb{R}^N$

$$f(\underline{x}) = \frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{\infty} \hat{f}(\underline{k})\, e^{-\underline{k}x}\, d\underline{k}.$$

Cartesian vec.

General form.

$$\mathcal{F}[f(\underline{x})] = \hat{f}(\underline{k}) = \frac{r}{2\pi} \int_{-\infty}^{+\infty} f(\underline{\xi})\, e^{ai\, \underline{k}\cdot\underline{\xi}}\, d\underline{\xi}$$

... Fourier transform

$$f(\underline{x}) = \frac{1}{\gamma} \int_{-\infty}^{+\infty} \hat{f}(\underline{k})\, e^{-ai\underline{k}\cdot\underline{x}}\, d\underline{k}$$

$$a = \pm 1, \quad \gamma = \sqrt{2\pi}.$$

$f(\underline{x})$ & $\hat{f}(\underline{k})$ are integrable:

$$L^1 = \int_{\mathbb{R}^n} |f|\, dx < \infty$$

"bounded".

Derivatives.

$$\mathcal{F}(f') = -ik\,\mathcal{F}(f).$$

IBP to check

* Boundary terms go away cus' of fn decay at infinity.

$$\mathcal{F}(f'') = -k^2\,\mathcal{F}(f).$$

Convolution thm

$$\mathcal{F}(f) = \hat{f}(k).$$

$$\mathcal{F}(g) = \hat{g}(k).$$

$$\mathcal{F}^{-1}(\hat{f}\hat{g}).$$

hoping to find.

1D case:

$$\mathcal{F}^{-1}\{\hat{f}\hat{g}\} = H(x)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\xi)\, g(x-\xi)\, d\xi.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} g(\xi)\, f(x-\xi)\, d\xi.$$

Fundamental soln

$$u_t - u_{xx} = p(x,t), \quad -\infty < x < \infty$$

$$u_t - u_{xx} = \delta(x-\xi)\,\delta(t-t')$$

$$\delta(x-a) = \begin{cases} \infty & x = a. \\ 0 & x \neq a \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(x-a) = 1, \quad \int_{-\infty}^{+\infty} \delta(x-a) f(x)\, dx = f(a)$$

fundamental soln: $G(x-\xi, t-t')$.

$$u(x,t) = \int_{\xi=-\infty}^{\infty}\int_{t=0}^{t} G(x-\xi', t-t')\, p(\xi', t')\, dt\, d\xi'$$

$$u_t - u_{xx} = \delta(x-\xi)\,\delta(t-t')$$

B.C.s

$u \to 0$ as $|x| \to \infty$

$$\Omega: \begin{cases} -\infty < x < \infty \\ 0 < t < \infty \\ |\xi| < \infty \end{cases}$$

$\hat{u}_t$ ~ "Fourier transform of u"

$+ k^2 \hat{u}$

... turning PDE in ODE and F.T. only in space.

$$= \frac{\delta(t-t')}{\sqrt{2\pi}} \underbrace{\int \delta(x-\xi) e^{ikx}\, dx}_{e^{ik\xi}}.$$

Recall the convolution thm.

Solve for ξ and $t' = 0$

$$\hat{u}_t + k^2 \hat{u} = \frac{\delta(t)}{\sqrt{2\pi}}$$

Using Green's func. to solve the ODE.

$$\hat{u}(k,t) = \frac{1}{\sqrt{2\pi}} e^{-k^2 t}.$$

complex conj.

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-k^2 t} e^{-ikx}\, dk.$$

$$m = \xi + i\frac{x}{2t^{1/2}} \Big\} \; e^{-m^2}$$ is analytic

↓

$$G(x,t) = \sqrt{\pi}\, e^{x/2t^{1/2}}$$ [illegible] deriv. can be calculated

Problem Session 8

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \underset{\text{source term}}{Q(x,t)}. \quad 0 \le l \le x. \quad t \ge 0$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = G(x)$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

$$u = X(x)\,T(t).$$

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

$$X_n(x) = B_n \sin(n\pi x). \quad n = 0, 1, 2, \dots$$

↖ series expansion

$$u(x,t) = \sum_{n=0}^{\infty} B_n \sin(n\pi x)\,T(t).$$

incorporate the const. into T(t)

$$= \sum_{n=0}^{\infty} \sin(n\pi x)\,T(t)$$

... Eigenfunction expansion

$$\sum_{n=0}^{\infty} \sin(n\pi x)\,T' = \sum_{n=0}^{\infty} -(n\pi)^2 \sin(n\pi x)\,T + Q$$

$$\sum_{n=0}^{\infty} \left[T' + (n\pi)^2 T\right] \sin(n\pi x) = Q$$

$$\left(\sum_{n=0}^{\infty} \left[T' + (n\pi)^2\right] \sin(n\pi x),\ \sin(m\pi x)\right) = \left(Q, \sin(m\pi x)\right)$$

#why?

$$\left[T' + (\underset{n}{\underset{\updownarrow}{m}}\pi)^2\right]\left(\sin(\underset{n}{\underset{\updownarrow}{m}}\pi x),\ \sin(\underset{n}{\underset{\updownarrow}{m}}\pi x)\right) = \left[Q, \sin(n\pi x)\right]$$

$$T' + (n\pi)^2 T = \frac{\left(Q, \sin(n\pi x)\right) \rightsquigarrow q_n}{\left(\sin(n\pi x), \sin(n\pi x)\right) \rightsquigarrow 1/2}$$

$$\int \sin^2(m\pi x)\,dx = \frac{1}{2}$$

$$\Rightarrow 2q_n$$

$$\mu(t) = e^{\int p(t)dt} = e^{\int (n\pi)^2 dt} = e^{(m^2)t}.$$

$$\frac{d}{dt}\left(e^{(m)^2 t} T\right) = 2e^{(m)^2 t} q_n.$$

$$e^{(n\pi)^2 t} T - T(0) = \int_0^{\pi} 2e^{(n\pi)^2 t} q_n(t')\, dt'$$

modify a little bit:

$$T(t) = e^{-(n\pi)^2 t}\left[\int_0^t 2e^{(n\pi)^2 t'} q_n(t')dt' + T(0)\right].$$

$$\rightarrow T(t) = e^{-(n\pi)^2 t}\left[\int_0^1 2e^{(n\pi)^2 t'} q_n(t')\, dt' + 2q_n\right].$$

$$u(x,t) = \sum_{n=0}^{\infty} \sin(n\pi x) T(t)$$

$$= \sum_{n=0}^{\infty} \sin(n\pi x)\, e^{-(n\pi)^2 t}\left[\int_0^1 2e^{(n\pi)t'} q_n(t')dt' + 2q_n\right]$$

Fourier transform.

$$\frac{\partial \hat{\phi}}{\partial x} = ik\hat{\phi}$$

$$\frac{\partial^2 \hat{u}}{\partial x^2} = \frac{\partial}{\partial x}\cdot\widehat{\left(\frac{\partial u}{\partial x}\right)} = ik\widehat{\frac{\partial u}{\partial x}}$$

$$= ik(ik)\hat{u} = -k^2\hat{u}(k,t)$$

$$\frac{\widehat{\partial^2 u}}{\partial t^2} = \frac{\partial^2 \hat{u}}{\partial t^2}.$$

$$\frac{\partial^2 \hat{u}(k,t)}{\partial t^2} + c^2k^2\hat{u}(k,t) = 0$$

$$\hat{u}(k,0) = -(k)$$

$$\frac{\partial \hat{u}(k,0)}{\partial t} = 0$$

$$\frac{\partial^2 \hat{u}(k,t)}{\partial t} = -ikc\hat{A}e^{-ikct} + ikc\hat{B}e^{ikct}.$$

$$\rightarrow \hat{A} + \hat{B} = \hat{f} \rightarrow \hat{B} = \hat{f} - \hat{A} \rightarrow 2\hat{A} = \hat{f}$$

$$\hat{A} = \frac{1}{2}\hat{f} = \hat{B}$$

$$\rightarrow -ikc\hat{A} + ikc\hat{B} = 0 \rightarrow \hat{B} = \hat{A}$$

$$\hat{u} = \hat{f}(k).\left(\frac{e^{ikct} + e^{ikct}}{2}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{f}\, e^{-ikct}}{2} e^{-ikx} dk$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{f}\, e^{ikct}}{2} e^{-ikx} dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{f}\, e^{-ik(x+ct)}}{2} dk$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}\, e^{-ik(x-ct)} dk$$

$$f(x) = IFT(\hat{f}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k).\, e^{-ikx} dk$$

$$u(x,t) = \frac{f(x+ct)}{2} + \frac{f(x-ct)}{2}$$

only 2 vars. govern the PDE

we can find a similarity soln.

Q1). $\rho C_p. \frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial t^2}$

Q2). Prove $\int_0^{\infty} e^{-s^2} ds = \frac{\sqrt{\pi}}{2}$.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy = \int_0^{\infty}\int_0^{\infty} e^{-r^2} r\, dr d\theta$$

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \pi$$

Q3). $q = -K\frac{\partial T}{\partial x}$.

$q(x=0). \sim t^{-1/2}$

---

1 a).

	$\tilde{T}$	x	t	β	α.
deg.	1	0	0	1	0
length.	0	1	0	-1	2
time	0	0	1	-1	-1.

rank $m = 3$, $n = 5$

$\# \pi$ grps $= 2$

$\pi_1 = fcn(\pi_2)$

apply similarity.

2nd-order in space
(→ 1st order)

b). $n = 1/2$
$m = 3/2$

$A =$
$B =$

ODE for $F(y)$.

chose A, B to make ODE for F. B.C.s simple

$\hat{T} = T - T_0 = Bt^m F(\eta)$

~ ODE for $F(\eta)$.
BCs for $F(\eta)$

$\hat{T} + T_0 = T$

$T = Bt^m F(\eta) + T_0$

match the order of t

$$\frac{\partial (Bt^m F(\eta) + T_0)}{\partial t} = \alpha \frac{\partial^2 [Bt^m F(\eta) + T_0]}{\partial x^2}$$

$$\frac{\partial [Bt^m F(\eta)]}{\partial t} = \alpha \cdot \frac{\partial^2 [Bt^m F(\eta)]}{\partial x^2}$$

$$\frac{\partial}{\partial t}\left[Bt^m F\left(\frac{x}{At^n}\right)\right] = \alpha \cdot \frac{\partial^2}{\partial x^2}\left[Bt^m F\left(\frac{x}{At^n}\right)\right]$$

$$Bt^{m-1} \cdot F\left(\frac{x}{At^n}\right) + Bt^m \cdot F\left(\frac{x}{At^n}\right) \cdot \frac{x}{A} \cdot t^{-n-1} \cdot (-n).$$

LHS: $F(\eta) \cdot Bt^{m-1} + F(\eta) \cdot \frac{x}{A} \cdot B \cdot t^{m-n-1} (-n).$

RHS: $\frac{\partial}{\partial x}\left[\alpha B \cdot t^m \cdot F(\eta) \cdot \frac{1}{At^n}\right]$

$$\alpha \cdot B \cdot t^m \cdot F(\eta) \cdot \frac{1}{(At^n)^2}$$

$$\frac{\alpha B F(\eta)}{A^2} \cdot t^{m-2n}.$$

$$m-1 = m-2n = m-n-1.$$

$$2n = 1. \rightarrow n = \frac{1}{2}$$

$m = \frac{3}{2}$

$$F(\eta) \cdot Bt^{m-1} + F(\eta) \cdot \frac{x}{A} \cdot B \cdot t^{m-\frac{3}{2}} \left(-\frac{1}{2}\right)$$

$$= \frac{\alpha B F(\eta)}{A^2} \cdot t^{m-1}$$

1(c). ~ superposition

ODE for F(η)

$2F'' + \eta F' - 3F = 0.$ $F(0)=1.$ $F(\infty)=0$

$A = \sqrt{\alpha^2}$, $B = -\beta\sqrt{\alpha^2}$

... these ODEs are all valid options depending on A & B.

$F'' + \eta F' - 3F = 0.$

$A = \sqrt{2\alpha^2}$, $B = -\beta A = -\beta\sqrt{2\alpha^2}$

(P2). Domain should be $0 \le x < \infty$

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D_0 c \cdot \frac{\partial C}{\partial x}\right).$$

$C \quad x \cdot t \cdot D_0$

$n = 1/2$

$m = 0$

$$\frac{\partial\left(Bt^m F(\eta)\right)}{\partial t} = \frac{\partial}{\partial x}\left\{D_0\, c \cdot \frac{\partial^2\left[Bt^m F(\eta)\right]}{\partial x^2}\right\}.$$

$$Bt^{m-1} F(\eta) + Bt^m F(\eta) \cdot (-n) \cdot t^{-n-1} \cdot \frac{x}{A}.$$

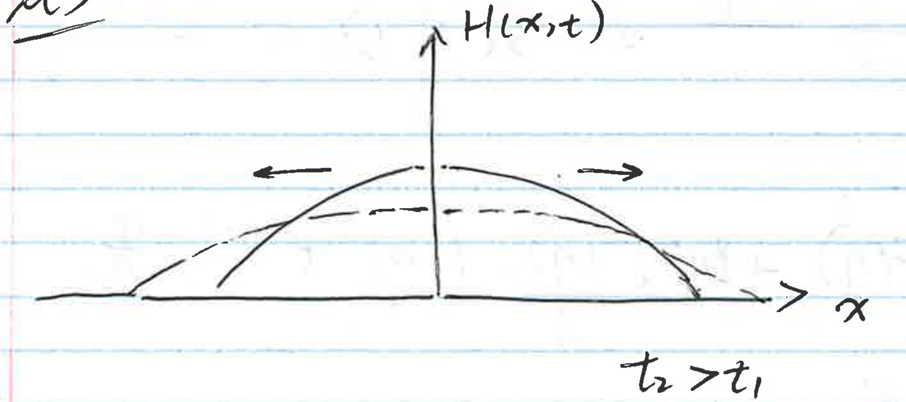
$$= \frac{\partial}{\partial x}\left\{D_0 \cdot c \cdot Bt^m\right.$$

b). $0 = F''F + (F')^2 + 2\eta F'.$

$F(0) = 1,$ $F(\infty)$

HW#8 -

Q3



$$\frac{\partial H}{\partial t} = -\frac{\gamma}{3\mu}\cdot\frac{\partial}{\partial x}\left[H^3\cdot\frac{\partial^3 H}{\partial x^3}\right]$$

	$\tilde{H}$	x ,	t .	α
depth	1	0	0	0
length	0	1	0	1
time	0	0	1	-1

$\longrightarrow -\frac{\gamma}{3\mu}.$

$n=4$

$m=3$ > # π group : 1.

$\Downarrow$

there exists a similarity soln.

$$\frac{\partial H}{\partial t} = Bt^{m-1}\cdot[mF - m\eta F']$$

$$H(x,t) = Bt^m F(\eta). \quad \eta = \frac{x}{At^n}.$$

$$\frac{\partial \eta}{\partial t} = (-n)t^{-n-1}\cdot\frac{x}{A}.$$

$$\frac{\partial \eta}{\partial x} = \frac{1}{At^n}.$$

$$\frac{\partial^3 H}{\partial x^3} = \frac{\partial^2}{\partial x^2}\cdot\left(Bt^m F'(\eta)\cdot\frac{1}{At^n}\right).$$

$$= (Bt^m)^3 F'''(\eta)\cdot\frac{1}{(At^n)^3}.$$

$$= \left(\frac{B}{A}\right)^3 t^{3m-3n}\cdot F'''(\eta).$$

$$\frac{\partial}{\partial x}\left[H^3\cdot H'''\right] = (H^3)'H''' + H^3\cdot H''''$$

$$= 3H^2\cdot H'\cdot\frac{1}{At^n}H''' + H^3\cdot\left[\left(\frac{B}{A}\right)^4 t^{4m-4n} F''''(\eta)\right]$$

$$H' = \frac{Bt^m}{At^n}\cdot F'(\eta)$$

$$R.H.S. = -\frac{\gamma}{3\mu} \cdot \frac{\partial}{\partial x}\left(H^3 \cdot \frac{\partial^3 H}{\partial x^3} \right).$$

$$= -\frac{\gamma}{3\mu}\left[3H^2 \frac{\partial H}{\partial x} \cdot \frac{\partial^3 H}{\partial x^3} + H^3 \frac{\partial^4 H}{\partial x^4} \right]$$

$$ds = At^m \, d\eta$$

$$\int_0^{\delta(t)} Bt^m \; F(\eta) \cdot At^m \, d\eta = M.$$

$$\rightarrow \; AB t^{m-n} \int_{\eta=0}^{\frac{b(t)}{At^m}} F(\eta)\, d\eta = M.$$

$$\frac{\partial H}{\partial x} = \frac{B}{A} \cdot t^{m-n} \cdot F'(\eta).$$

$$\frac{\partial^2 H}{\partial x^2} = \frac{B}{A^2} \cdot t^{m-2n} \cdot F''(\eta).$$

$$\frac{\partial^3 H}{\partial x^3} = \frac{B}{A^3} \cdot t^{m-3n} \; F'''(\eta).$$

$$\frac{\partial^4 H}{\partial x^4} = \frac{B}{A^4} \cdot t^{m-4n} \; F''''(\eta).$$

Plugging in the derivatives to RHS.

$$RHS: = -\frac{\gamma}{3\mu} \cdot \left[3H^2 \cdot \frac{B}{A} \cdot t^{m-n} \cdot F'(\eta) \cdot \frac{B}{A^3} t^{m-3n} F'''(\eta). \right.$$

$$\left. + H^3 \cdot \frac{B}{A^4} \cdot t^{m-4n} \cdot F''''(\eta) \right]$$

$$LHS: \; \frac{\partial H}{\partial t} = Bmt^{m-1} F + Bt^m \cdot F' \frac{\partial \eta}{\partial t}.$$

$$= Bt^{m-1}\left[mF - m\eta F' \right]$$

Condition: t exponential coefficients are the same ... ?

$$RHS = -\frac{\gamma}{3\mu}\left[3H^2 \cdot \frac{B^2}{A^4} \cdot t^{2m-4n} \; F'(\eta) \cdot F''(\eta) \right.$$

$$\left. + H^3 \cdot \frac{B}{A^4} \cdot t^{m-4n} \cdot F''''(\eta) \right]$$

$$4m - 4n = m - 1 \; ✓$$

$$RHS = -\frac{\gamma}{3\mu}\left[3 \cdot F^2 F' F''' + F^3 F'''' \right] \frac{B^4}{A^4} t^{4m-4n}.$$

$$4m = 3n + 1$$

$m+n=0.$

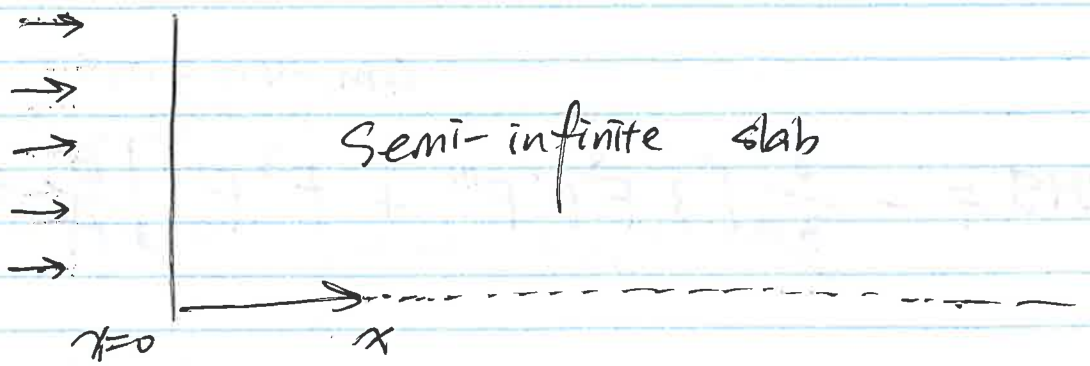
$m=\frac{1}{7}\ ,\quad n=-\frac{1}{7}\ .$

Q4 $\frac{\partial}{\partial x}\left(\sqrt{T}\cdot\frac{\partial T}{\partial x}\right)=\gamma T^2\frac{\partial T}{\partial t}.$

② $\frac{\partial T}{\partial x}(0,t)=0.\quad T(\infty,t)=0$

① $T(x,0)=0.$ ③ $\gamma\int_0^\infty T^3 dx=\beta.$

	$\tilde{T}$	$x,$	t	γ	β
deg	1	0	0	$-3/2$	$3/2$
length	0	1	0	-2	$-1.$
time	0	0	1.	1	1



Dimensional analysis.

$$\frac{[k]^{-3/2}}{[L]^2}=\gamma\frac{[k]^3}{T}.$$

$[\gamma][k]^3[L]=[\beta].$

rank $=3$, $(m=3)$

\# of dimensionless groups $n=5$

\# π groups: $n-m=2<3.$

↳ num. ind var. + num. dep. var.

$\underbrace{2+1}$

$\pi_1,\pi_2=f^n(\pi_1)$

let. $\left.\begin{array}{l}\pi_1=x^a t^b\gamma^c\beta^d=\eta.\\ \pi_2=T^a t^b\gamma^c\cdot\beta^d=\textcircled{H}.\end{array}\right\}.$

$\nwarrow$ dimensional analysis

$\eta=\frac{x}{At^m}.\quad T=Bt^m F(\eta).$

$a-2c-d=0\qquad \frac{3}{2}(d-c)=0$

$b+c+d=0\qquad \longrightarrow$

Lecture 20 3/15/2024.

Review of the Course.

→ Start :

PDEs → Classifications.

→ Properties.

↳ Order : $A_1 \frac{\partial^2\phi}{\partial x^2} + B_1 \frac{\partial^2\phi}{\partial x \partial y} + C_1 \frac{\partial^2\phi}{\partial y^2}$

$+ a\frac{\partial\phi}{\partial x} + b\frac{\partial\phi}{\partial y} + c(\phi)$

$+ d(x,y) = 0$

Independent var. x, y

$\Delta = B_1^2 - 4A_1C_1$

$\Delta = 0$, parabolic. ← eigenval. non-distinct.

$\Delta > 0$, hyperbolic

$\Delta < 0$, elliptic ← eigenval. complex.

Characteristics

SoV - Eigenfunction expansions

Integral Transforms

Similarity

} • Linear • Nonlinear

Definition on Linearity.

$L\phi = 0.$

$L(c_1\phi_1 + c_2\phi_2) = c_1 L(\phi_1) + c_2 L(\phi_2)$

	Linear	Non-linear.
Char.	✓	✓
Sol.	✓	X
Integral.	✓	X*
Similarity.	✓	✓

→ requires the eqn. to be hyperbolic.

- for hyperbolic. $\frac{dx}{dt} = \pm C_0$ ⇝ wave eqn.

  ↳ to be finite.

- for parabolic. $\frac{dx}{dt} = 0$

  $t \to$ const.

- for elliptic. $\frac{dx}{dt} \to$ complex.

... characteristics exists, just we can't really use it to solve efficiently.

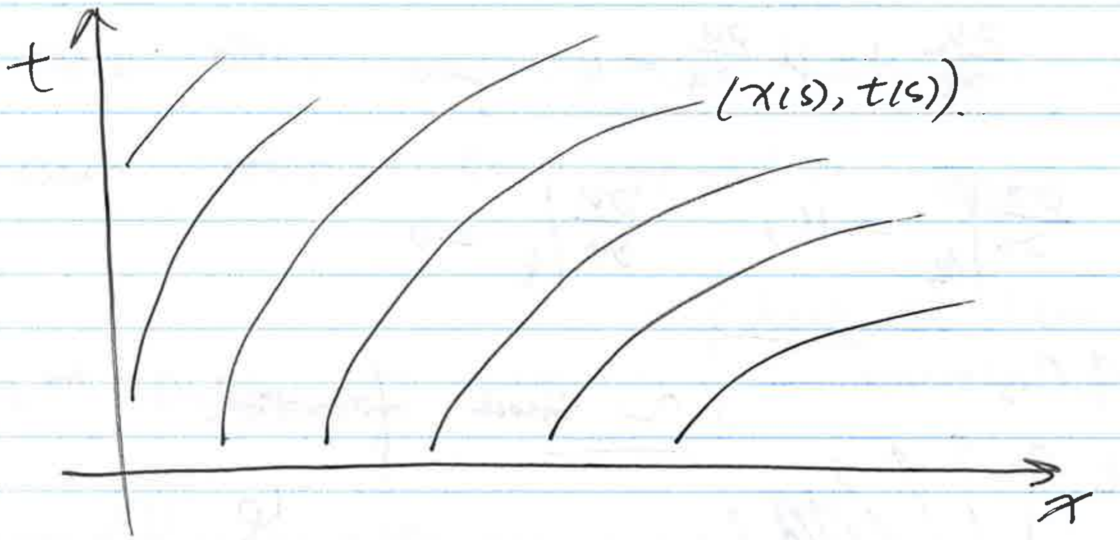
* Characteristics

$$A\frac{\partial\phi}{\partial t} + B\frac{\partial\phi}{\partial x} + C(\phi) + D = 0$$

$$\phi(x,t) \xrightarrow{\text{Parametrized}} \phi(x(s), t(s)).$$

... coordinate transformation.

$$\frac{\partial\phi}{\partial s} = \frac{\partial\phi}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial\phi}{\partial t}\cdot\frac{\partial t}{\partial s}.$$



rewrite the PDE as:

$$\frac{\partial\phi}{\partial t} + \frac{B}{A}\cdot\frac{\partial\phi}{\partial x} + \frac{C(\phi)}{A} + \frac{D}{A} = 0$$

... assuming $A \neq 0$.

$$\frac{dx}{ds} - \frac{B}{A}\cdot\frac{dt}{ds} = 0. \rightarrow \frac{d\phi}{ds} = -\frac{(C+D)}{A}\cdot\frac{dt}{ds}$$

Chose $t=s$.

$$\frac{dt}{ds}=1 \Rightarrow \frac{dx}{ds}=\frac{B}{A}, \quad \frac{d\phi}{ds}=-\frac{(C+D)}{A}.$$

It can also be written as:

$$\frac{dt}{A}=\frac{dx}{B}=-\frac{d\phi}{(C+D)}$$

Burgers Eqn.

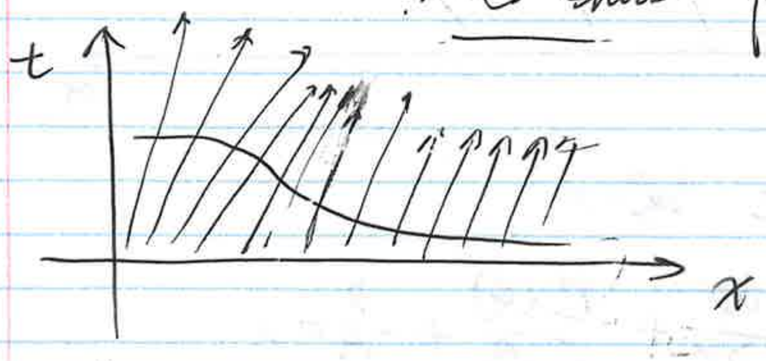
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0$$

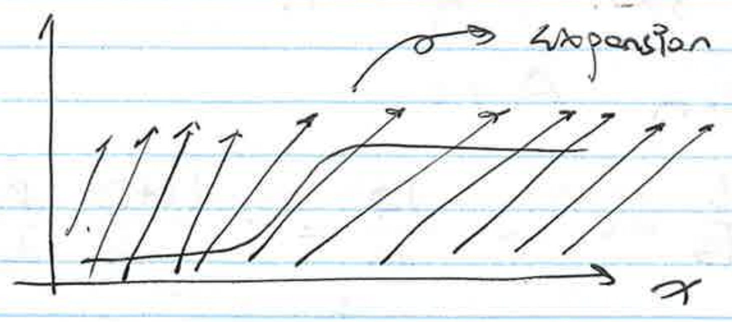
$$\left.\frac{\partial x}{\partial t}\right|_{\xi}=u, \quad \left.\frac{\partial u}{\partial t}\right|_{\xi}=0$$

I.C.s

* ~> shock formation

↳ compression exp.





Generalize to conservation law.

$$\frac{\partial f}{\partial t}+\frac{\partial F}{\partial x}=0$$

If a shock forms, I can use Rankine-Hugoniot to find shock speed:

$$U_s=\frac{F_R-F_L}{\rho_R-\rho_L}$$

these ODEs may not be independent of each other in M.C.

↓

Check the HW question for ref.

2nd Order Eqn.

$$\frac{\partial \underline{U}}{\partial t}+\underline{\underline{\beta}}\frac{\partial \underline{U}}{\partial x}=\underline{0}.$$

$$\hookrightarrow \underline{\underline{\beta}}=\begin{bmatrix}0 & -1\\ c^2 & 0\end{bmatrix} \quad \underline{U}=\begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial t}\end{bmatrix}.$$

for wave eqn

↓

positive eigenvals

↓

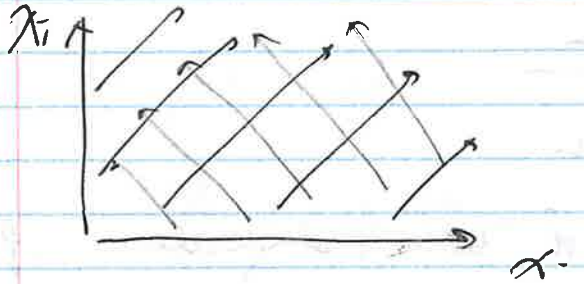
diagonalize β

$$\left\{f(x - c_0t) + \underbrace{f(x + c_0t)}\right\}$$

$$\uparrow \qquad\qquad \uparrow$$

$$\frac{\partial u}{\partial t} + c_0\frac{\partial u}{\partial x} = 0 \qquad \frac{\partial u}{\partial t} - c_0\frac{\partial u}{\partial x} = 0$$

two sets of solutions for waves



If $\underline{\beta}$ has complex eigenvals, then what?

→ ... S.o.V.s. → All types

⇓

has to be linear.

dependent variables: u.

independent variables: x, y.

Ansatz: $u = X(x)\,Y(y)$.

Subs. back to PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \leftarrow \text{Laplace}$$

$$X''Y + Y''X = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda \leftarrow \text{Const.}$$

$$\begin{cases} X'' - \lambda X = 0 \\ Y'' + \lambda Y = 0 \end{cases}$$

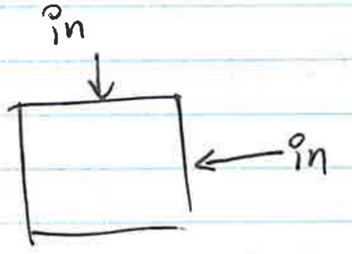
All B.C.s except 1 to be homogeneous.

this "1" is the "inhomogeneous".

Solutions to ODEs eigenfunctions.

$$f = \sum_i C_n \Phi_n.$$

$$u = \sum_i \underset{\phi(x)}{f(x)}\, \underset{\phi(y)}{f(y)}.$$



Similarity

→ Buckingham-Pi to find non-dim. group.

→ $\eta = f(t, x)$ ← inherent scaling.

→ write the solution in terms of η.

↳ appropriate scaling.

PDE → ODE (η)

$\eta = \frac{x}{At^n}$. $\hat{T} = Bt^m F(\eta)$.

↳ function.

↳ Subs. scaling in PDE.

get an ODE in $F(\eta)$.

use B.C.s to determine n & m.

$$\eta = \frac{x}{2\sqrt{\alpha t}}$$