COURSE NOTES

Elasticity & Inelasticity

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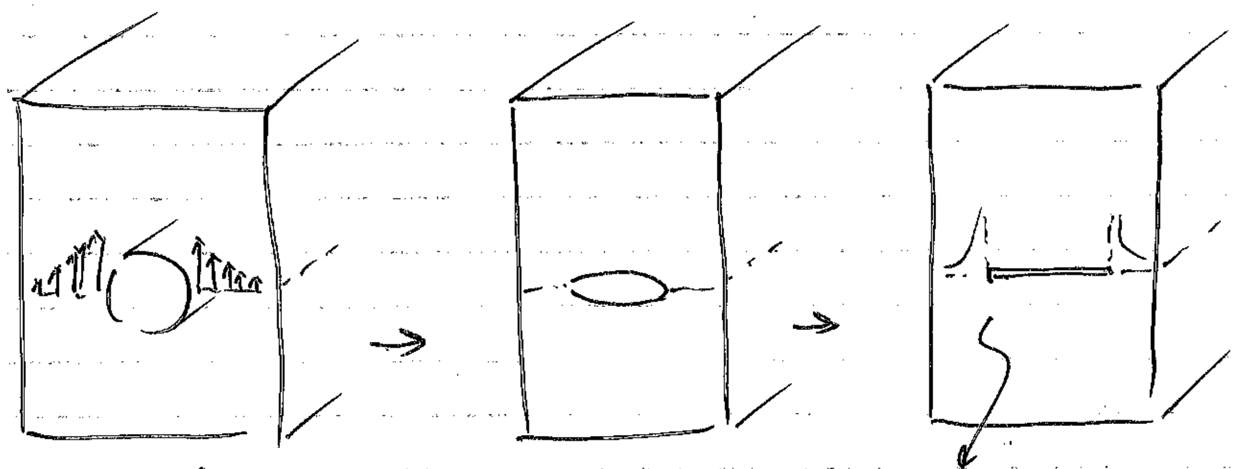
2024

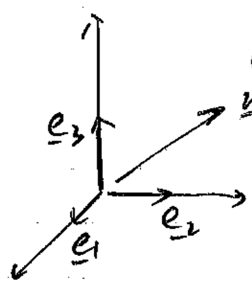
Elasticity & Inelasticity. 4/1/2024



Soln : 3

"Singularity"

- Tensor transform.
- Stress-strain relations



vector

$\underline{u} = u_1\underline{e}_1 + u_2\underline{e}_2 + u_3\underline{e}_3$

$= u_i\underline{e}_i$

Einstein notation.

"not a vector" --> $u_i = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

column vector

$u_i'\underline{e}_i'$

$u_i' = \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix}$

Define a matrix $Q_{ij} = (\underline{e}_i' \cdot \underline{e}_j)$

orthogonal matrix: $Q^{-1} = Q^T$

$$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ & & \\ & & \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$u_i' = Q_{ij}\, u_j$

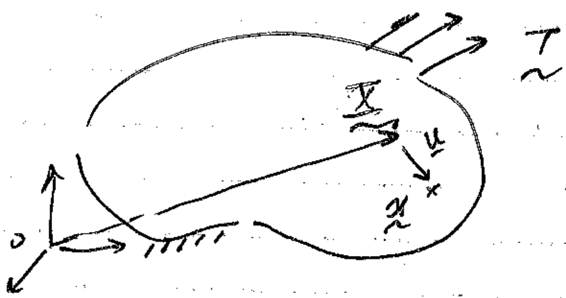
$a_i' = Q_{ij}\, a_j$

index notation.

$$u_i P_{ij} = v_j$$

$$u_k P_{kj} u_j \rightarrow \mathbb{R}.$$

unless we specify the element-wise multiplication.



$$\underline{u} = \underline{x} - \underset{\sim}{X}$$

Displacement field.

$\underline{u}(\underset{\sim}{X}) \sim \underline{u}(\underset{\sim}{x})$ ... no gradient deformation tensor here.

Strain field.

$$u_{i,j} \equiv \frac{\partial u_i}{\partial x_j}$$

"Small deformation assumption"

↓

no higher-order terms.

Strain: $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \varepsilon_{ji}.$

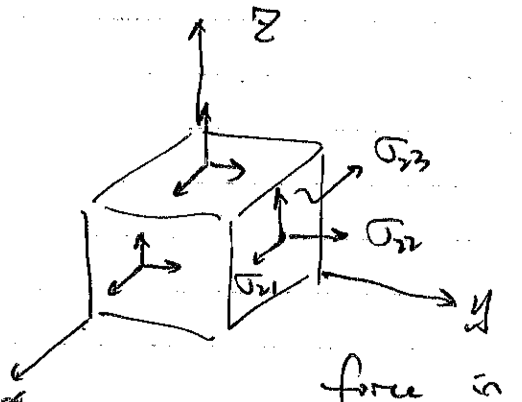
$w_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) \leftarrow$ rotation

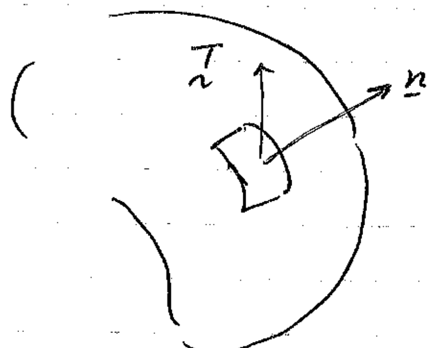
$$\varepsilon'_{ij} = Q_{im} Q_{jn} \varepsilon_{mn}.$$

epsilonp = Q * epsilon * Q'

```
epsilonp = zeros(3,3).
for i = 1:3
    for j = 1:3.
        for m = 1:3.
            for n = 1:3.
                epsilonp(i,j) = ~ + Q(i,m)*Q(j,n)
                                *epsilon(m,n);
            end
        end
    end
end.
```

equivalent.



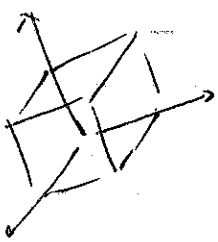


$$\sigma_{ij} = \frac{\text{force in j-th direction}}{\text{area in i-th face}}$$

$$T_j = \sigma_{ij} n_i$$

symmetric tensors.

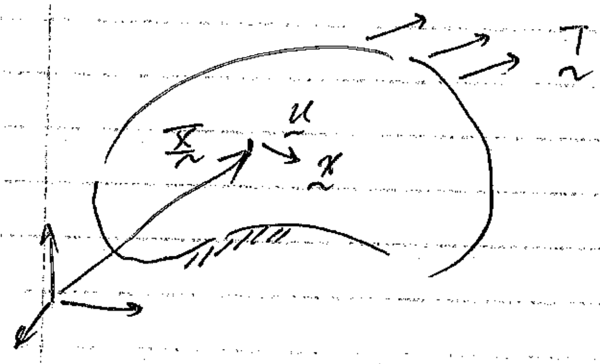
$$\sigma_{ij} = \sigma_{ji}$$



$$\sigma'_{ij} = Q_{ip} Q_{jq} \sigma_{pq}$$

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} \rightarrow C'_{ijkl} = Q_{im} Q_{jn} Q_{kp} Q_{lq} C_{mnpq}$$

11/13/2024 Lecture 2.



Displacement field. $u_i(\underset{\sim}{x})$

Strain field: $\boxed{\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})}$

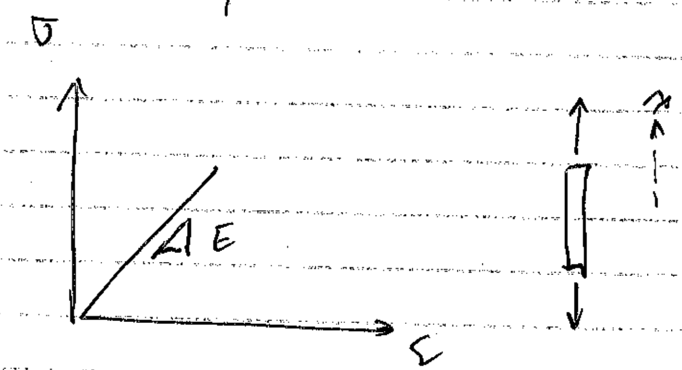
Stress field: σ_{ij}

traction field: $T_j = \sigma_{ij} n_i$

generalized Hooke's law: $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$

Today: PDE for elasticity.

Attempt to solve it. Anisotropic vs. isotropic elasticity.



... generalized Hooke's law.

Voigt notation.

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} \leftarrow \begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{31}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{16}\\ \vdots & & \ddots & \vdots\\ C_{61} & & \cdots & C_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{31}\\ 2\varepsilon_{12}\end{bmatrix} \rightarrow \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{bmatrix}$$

$$\sigma_{11} = C_{1111}\varepsilon_{11} + \begin{pmatrix} C_{1112}\varepsilon_{12} \\ C_{1121}\varepsilon_{21} \end{pmatrix} + \cdots + C_{1133}\varepsilon_{33}$$

$\hookrightarrow C_{1112}\, 2\varepsilon_{12}$

$\downarrow$

$$C_{16} = C_{1112} = C_{1121}$$

$$\sigma_I = C_{IJ}\varepsilon_J \quad , \quad I, J = 1, 2, \ldots, 6$$

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} .$$

$\nwarrow$ inverse of C_{ijkl}

$\sigma_{ij} \rightarrow$ 9 components, 6 ind. comp.

$C_{ijkl} \rightarrow$ 81 components, 21 ind. comp.

$\uparrow$ Symmetric C_{ijkl} $\Big\}$ $6+5+\cdots+1 = 21$.

$$\boxed{\begin{aligned} \sigma_I &= \frac{\partial W}{\partial \varepsilon_I} \\ W &= \frac{1}{2}\varepsilon_I C_{IJ}\varepsilon_J \end{aligned}}$$

$\rightarrow E, \nu, G.$

$\rightarrow S_{ijkl}$

Isotropic Material

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ & & \frac{1}{E} & 0 & 0 & 0 \\ & & & \frac{1}{G} & 0 & 0 \\ & \text{sym} & & & \frac{1}{G} & 0 \\ & & & & & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

Relationship between the material parameters.

$$E = 2(1+\nu) \cdot G$$

HWs:

$$\begin{array}{ccccc} C_{IJ} & \rightarrow & C_{ijkl} & \rightarrow & C'_{ijkl} \\ \downarrow & & & & \\ S_{IJ} & \rightarrow & S_{ijkl} & \rightarrow & S'_{ijkl} \end{array} \quad E' = \frac{1}{S'_{1111}}$$

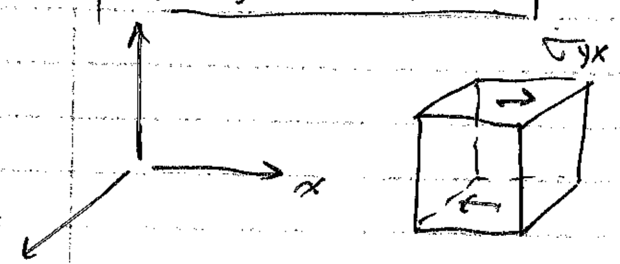
Equations for elasticity.

- compatibility condition

$$\boxed{\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - (\varepsilon_{ik,jl} + \varepsilon_{jl,ik}) = 0}$$ only for small deformation

- equilibrium condition

$$\boxed{\sigma_{ij,i} + f_j = 0}$$



$u_i(\underline{x})$ – 3 DoF

$\varepsilon_{ij}(\underline{x})$ – 6 DoF

$$\underline{\nabla} \cdot \underline{\underline{\sigma}} + \underline{F} = 0$$

- constitutive relation.
- B.C.s

For isotropic elasticity:

$$\boxed{\begin{aligned} \sigma_{ij} &= \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \\ \varepsilon_{ij} &= -\frac{\nu}{E} \sigma_{kk} \delta_{ij} + \frac{1+\nu}{E} \sigma_{ij} \end{aligned}}$$

$$\delta_{ij} = \begin{cases} 1 & , \; i=j \\ 0 & , \; i\neq j \end{cases}$$ kronecker delta

$\varepsilon_{kk} \rightarrow \Sigma$ trace. i.e., hydrostatic.

$\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$

General strategies for sol'n.

1) $\sigma_{ij} = \lambda u_{k,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i})$

$$\boxed{\mu u_{i,kk} + (\lambda + \mu) u_{k,ki} + F_i = 0}$$

3 eqs.

$$\mu \nabla^2 \underline{u} + (\lambda + \mu) \cdot \nabla (\nabla \cdot \underline{u}) + \underset{\sim}{F} = 0$$

2). write compatibility condition in terms of stress.

(2D)

↳ Equilibrium condition

$$\begin{cases} \sigma_{xx,x} + \sigma_{yx,y} + F_x = 0 \\ \sigma_{xy,x} + \sigma_{yy,y} + F_y = 0 \end{cases}$$

2D compatibility: $\varepsilon_{xx,yy} + \varepsilon_{yy,xx} + 2\varepsilon_{xy,xy} = 0$

trial sol'n / ansatz: $\phi(x,y)$

$$\left.\begin{aligned} \sigma_{xx} &= \phi_{,yy} \\ \sigma_{yy} &= \phi_{,xx} \\ \sigma_{xy} &= -\phi_{,xy} \end{aligned}\right\} \text{Equilibrium automatically satisfied}$$

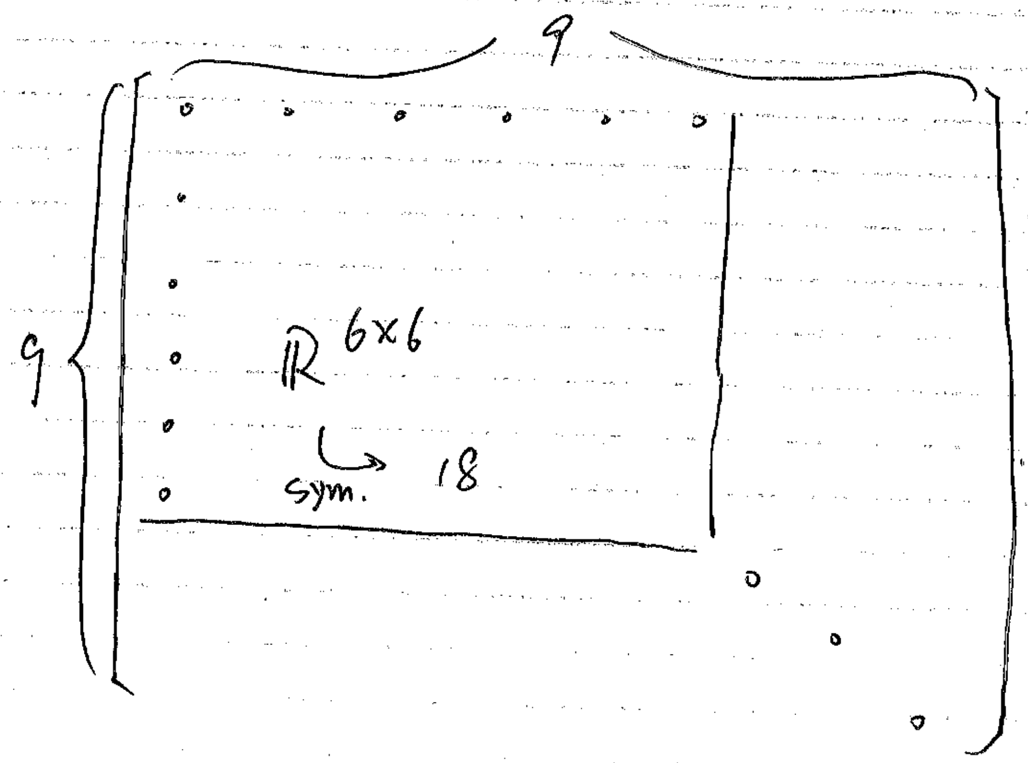
↳ compatibility condition

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(x,y) = 0$$

↳ biharmonic eqn.

$$\rightsquigarrow \nabla^2(\nabla^2\phi) = 0$$

$81 \to 21$



Review of Euler-Bernoulli Beam theory

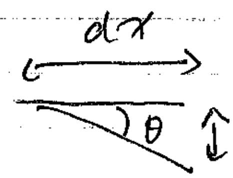
① plane strain

② N·A.

③ Small deformation

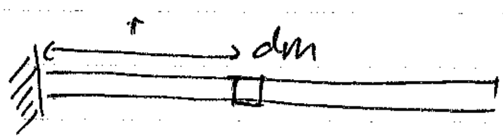
④ plane strain ⊥ N.A.

remain ⊥

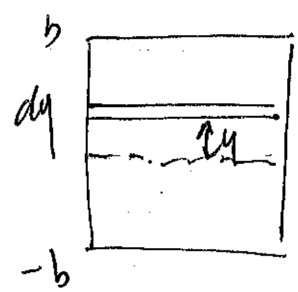


$$\tan\theta = \frac{dw}{dx} \sim \theta$$

$$K = \frac{d\theta}{dx} = \frac{\text{Moment}}{\text{Resistence to bending}}$$



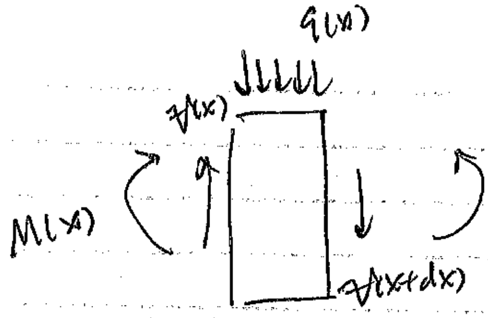
$$I = \int r^2 dm.$$



$$E\int_{-b}^{b} y^2 t\,dy$$ area moment of inertia.

$$EI_z = \frac{2Etb^3}{3}$$

$$K = \frac{M}{EI_z} = \frac{d^2w}{dx^2} \quad \cdots ①$$



$$q(x)\,dx + V(x+dx) = V(x)$$

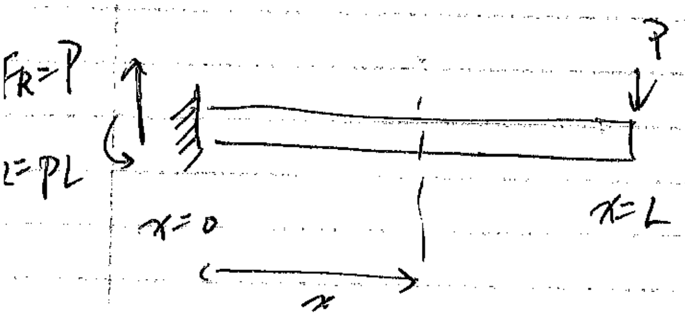
$$q(x) = \frac{dV(x)}{dx} \qquad \cdots ②$$

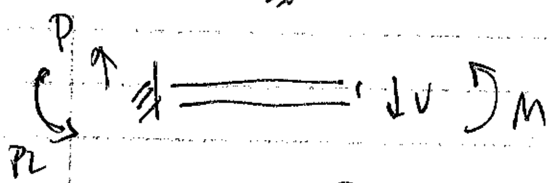
$$M(x) + V(x)\frac{dx}{2} + V(x+dx)\frac{dx}{2} = M(x+dx)$$

$$V(x) = \frac{dM(x)}{dx} \qquad \cdots ③$$

$$EI_z \frac{d^4 w}{dx^4} = -q(x).$$

Examples

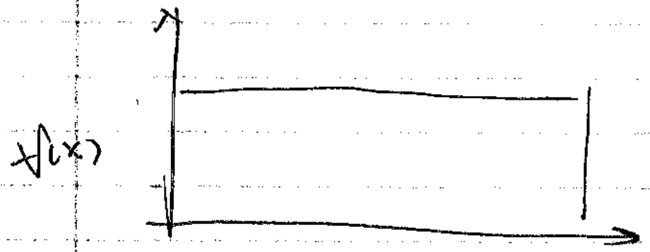


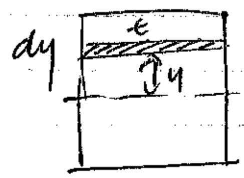


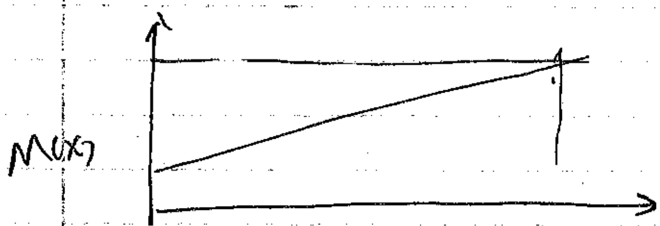
$$V(x) = P$$

$$PL + M = Px.$$

$$M = P(x-L)$$





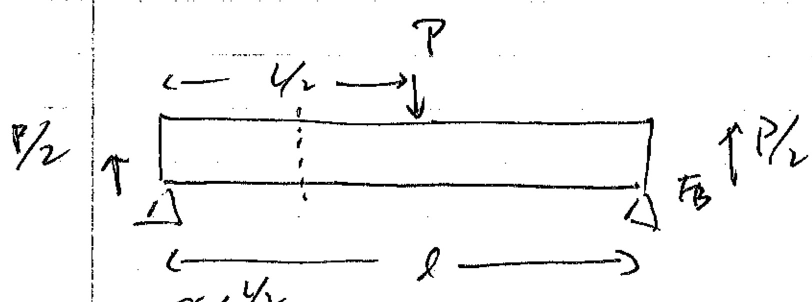


$$M(x) = \int_{-b}^{b} y\, \sigma_{xx}(x,y)\, t\, dy.$$

$$\sigma_{xx} = \frac{-M\, y}{I_z}$$

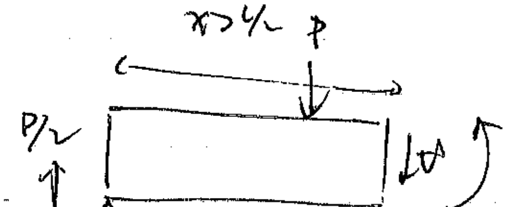
$$V(x) = \int_{-b}^{b} \sigma_{xy}(x,y)\, t\, dy.$$

$$\sigma_{xy} = \frac{3V(x)}{2A}\left[1 - \left(\frac{y}{b}\right)^2\right] \quad A = 2tb.$$



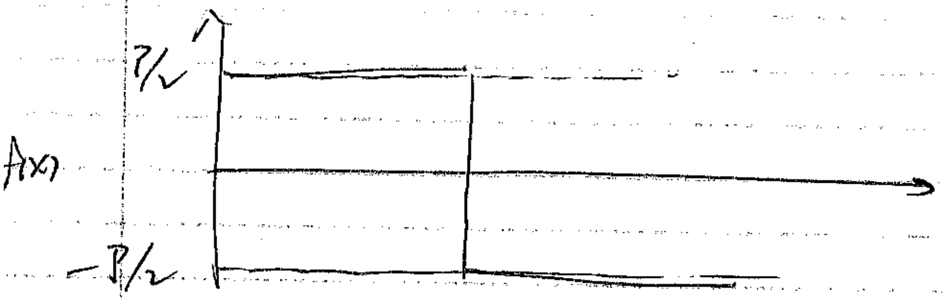
$V = P/2$

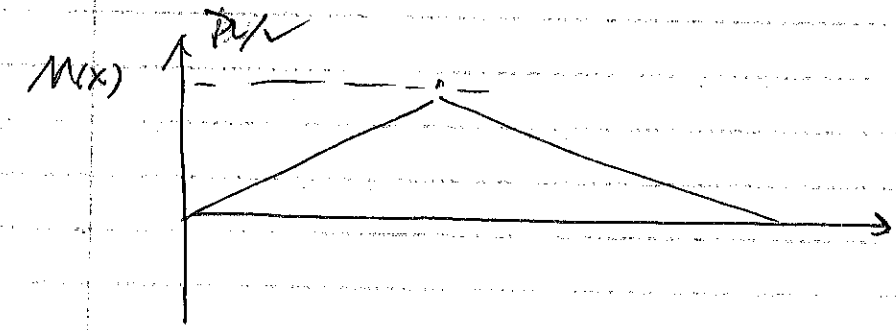
$M = Px/2$



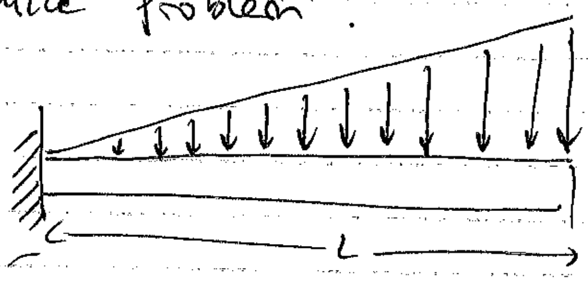
$V = -P/2.$

$PL/2 - Px/2 = M \quad \frac{P}{2}(L-x)$

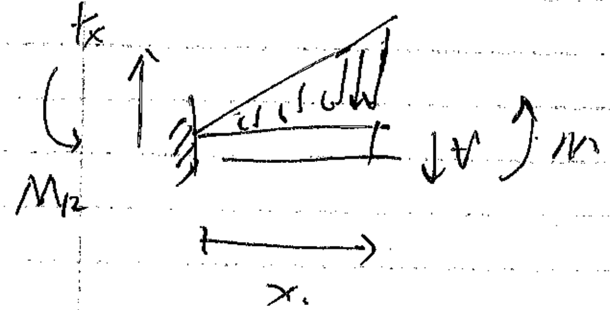




Practice Problem.



$q(x) = \frac{kx}{L}$



$\int_0^L q(x)\,dx$

$F_x = \frac{kL}{2}$

$M_R = \int_0^L \frac{kx}{L} x\,dx$

$= \frac{1}{L}\frac{1}{3}kx^3\Big|_0^L$

$= \frac{1}{3}kL^2$

(x)dx $V(x) = F_x = \frac{k(L^2 - x^2)}{2L}$

$M(x) = \frac{k}{2L}(L^2 - x^2)x + \frac{k}{3L}x^3 - \frac{kL^3}{3L}$

$= N \cdot x + \int_0^x q(x) \cdot x\,dx - M_R$

Stiffness tensor

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & \cdots & C_{16} \\ \vdots & \ddots & \vdots \\ C_{61} & \cdots & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \vdots \\ \varepsilon_{12} \end{bmatrix}$$

4/8/2024. Lecture 3.

Elasticity Equations.

- compatibility. $\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$.
- equilibrium. $\sigma_{ij,i} + F_i = 0$

Approach ①: (3D). ← solve displacements.

$$\sigma_{ij} = \lambda u_{k,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i})$$

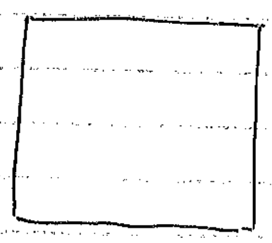
$$\mu u_{i,kk} + (\lambda+\mu) u_{k,ki} + F_i = 0$$

$i = x$ $$\mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u_x + (\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + F_x = 0$$

⋮

Approach ②: (2D).

- plane strain.



$u_x(x,y)$

$u_y(x,y)$.

$u_z = 0 \rightsquigarrow \frac{\partial}{\partial z} = 0$

$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy} \rightarrow \varepsilon_{xz} = 0, \ \varepsilon_{yz} = 0, \ \varepsilon_{zz} = 0$

$\sigma_{xx}, \sigma_{yy}, \sigma_{xy} \rightarrow \sigma_{xz} = 0, \ \sigma_{yz} = 0. \ \sigma_{zz} \neq 0$

$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$

$$\begin{cases} \varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz} \\ \varepsilon_{yy} = -\frac{\nu}{E}\sigma_{xx} + \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz} \\ \varepsilon_{zz} \end{cases}$$

(under plane strain assumption).

$$\begin{cases} \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu(1+\nu)}{E}\sigma_{yy}. \\ -\frac{\nu(1+\nu)}{E}\sigma_{xx} + \frac{1-\nu^2}{E}\sigma_{yy}. \end{cases}$$

Equilibrium. $\begin{cases} \sigma_{xx,x} + \sigma_{yx,y} + F_x = 0 \\ \sigma_{xy,x} + \sigma_{yy,y} + F_y = 0 \end{cases}$

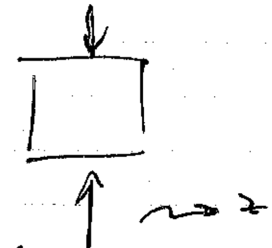
compatibility: $\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2\varepsilon_{xy,xy} = 0$

∘ plane stress.

$\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \quad \sigma_{xz} = 0, \sigma_{yz} = 0, \sigma_{zz} = 0.$

$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}, \quad \varepsilon_{xz} = 0, \varepsilon_{yz} = 0, \varepsilon_{zz} \neq 0$

$$\begin{cases} \varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy}. \\ \varepsilon_{yy} = -\frac{\nu}{E}\sigma_{xx} + \frac{1}{E}\sigma_{yy}. \\ \varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy} \end{cases}$$



free to expand in z.

Kolosov's constant $K = 3 - 4\nu$ ← plan strain.

$K = \frac{3-\nu}{1+\nu}$ → plane stress.

ansatz ⇝ Airy stress function, $\phi(x,y)$.

$$\begin{cases} \sigma_{xx} = \phi_{,yy} \\ \sigma_{yy} = \phi_{,xx} \\ \sigma_{xy} = -\phi_{,xy} \end{cases}$$ → Equilibrium condition automatically satisfied

$$\left(\frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial y^4}+2\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}\right)\phi(x,y)=0.$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi(x,y)=0$$

$$\nabla^2(\nabla^2\phi)=0$$

$$\nabla^4\phi=0 \quad \leftarrow \text{ biharmonic eqn.}$$

Examples.

$$\phi(x,y)=\alpha x+\beta y+\gamma$$

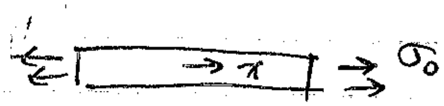
$\rightarrow$ stresses are zero

$$\phi(x,y)=\frac{1}{2}Ax^2+\frac{1}{2}By^2-Cxy$$

$\rightarrow$ $\sigma_{xx}=B$

$\sigma_{yy}=A$

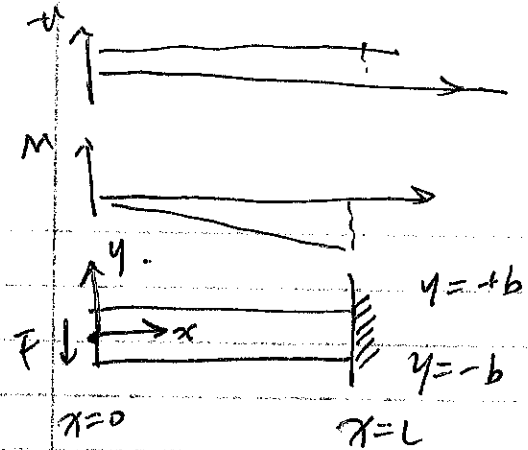
$\sigma_{xy}=C$



$$\phi=\frac{1}{2}\sigma_0 y^2$$

$$\phi=-\frac{1}{6}\frac{M}{I}y^3$$

$$\sigma_{xx}=-\frac{M}{I}y$$



B.V.P.

Top. Bottom. → traction free. $\begin{cases} \sigma_{xy} = 0 \\ \sigma_{yy} = 0 \end{cases}$ $y = \pm b$

(weak) $\sigma_{xx} \cdots ?$

"strong B.C."

Left side

$$\left.\begin{aligned} &\int_{-b}^{b} \sigma_{xy}\, dy = F \\ &\int_{-b}^{b} \sigma_{xx}\, y\, dy = 0 \\ &\int_{-b}^{b} \sigma_{xx}\, dy = 0 \end{aligned}\right\} x=0$$

... weak B.C.s

Right side

↳ automatically satisfied. $x=L$
(only for stress).

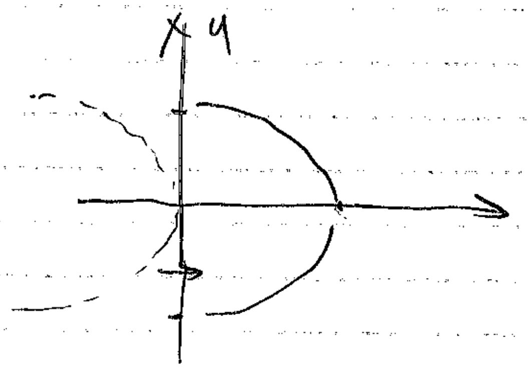
$$\underbrace{u_x = 0, \quad u_y = 0,}_{x=L} \quad \leftarrow \text{strong B.C.s.}$$

Guess. $\phi = C_1 xy^3 \quad -3C_1 b^2 xy$

all the stress field.

$$\begin{cases} \sigma_{xx} = 6C_1 xy & \longrightarrow \frac{3F}{2b^3}xy \\ \sigma_{yy} = 0 & \\ \sigma_{xy} = -3C_1 y^2 + 3C_1 b^2 & \longrightarrow \frac{3F}{4b^3}(b^2 - y^2) \end{cases}$$

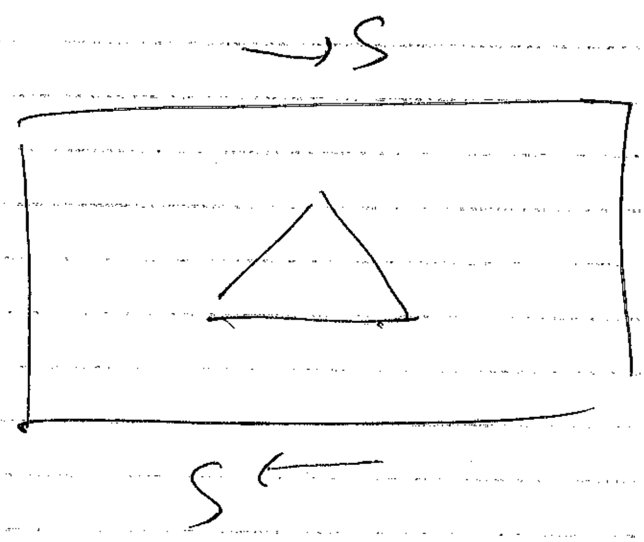
$\sigma_{xy} = -3C_1 b^2, \quad y = \pm b$



$(x=0) \quad \int_{-b}^{b} \sigma_{xy}\, dy = F$

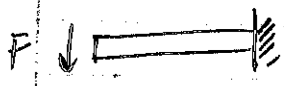
$\downarrow$

$C_1 = \frac{F}{4b^3}$



$\frac{\sqrt{3}}{2} = 0.5774 < 0.6$

Lecture 4. 4/10/2014



$\phi(x,y)$.

$\hookrightarrow \sigma_{xx}, \ \sigma_{yy}, \ \sigma_{xy}$.

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} = \frac{3F}{2Eb^3}xy$$

$$\varepsilon_{yy} = -\frac{\nu}{E}\sigma_{xx} + \frac{1}{E}\sigma_{yy} = \frac{-3F\nu}{2b^3}xy.$$

$$\varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy} = \frac{1+\nu}{E}\sigma_{xy} = \frac{3F(1+\nu)}{4Eb^3}(b^2-y^2).$$

$$u_x = \int \varepsilon_{xx}\, dx = \frac{3F}{4Eb^3}x^2y + f(y).$$

$$u_y = \int \varepsilon_{yy}\, dy = -\frac{3F\nu}{4Eb^3}xy^2 + g(x)$$

$$\varepsilon_{xy} = \frac{1}{2}(u_{x,y} + u_{y,x}).$$

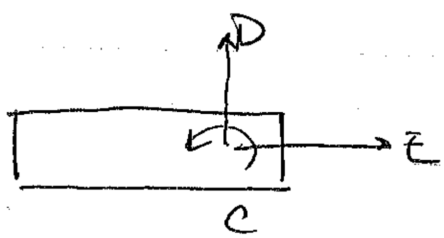
2. $$\frac{3F(1+\nu)}{4Eb^3}(b^2-y^2) = \frac{3F}{4Eb^3}x^2 + f'(y) - \frac{3F\nu}{4Eb^3}y^2 + g'(x)$$

$$\underbrace{\frac{3F}{4Eb^3}x^2 + g'(x)}_{\text{function of } x} = \underbrace{\frac{3F(1+\nu)}{2Eb^3}(b^2-y^2) + \frac{3F\nu}{4Eb^3}y^2 - f'(y)}_{\text{function of } y} = C$$

has to be const.

$$g'(x) = C - \frac{3F}{4Eb^3}x^2$$

$$g(x) = Cx - \frac{F}{4Eb^3}x^3 + D$$



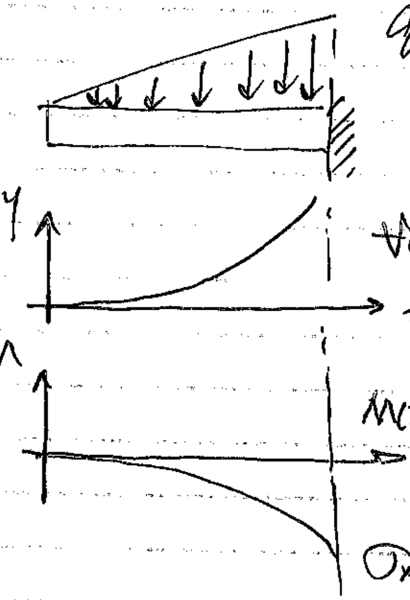
$$f'(y) = \cdots$$

$$f(y) = -Cy + \cdots + E$$

Weak B.C.s → St. Venant's principle.

discussion on exactly satisfying the imposed B.C.s.

... correction term decays exponentially.



$q(x) \sim x^n$

$V(x) \sim x^{n+1}$

$M(x) \sim x^{n+2}$

$\sigma_{xx} \sim x^{n+2} y$

B.C.s.

$\sigma_{yy}(x, y=b) = -q(x)$

$\sigma_{yy}(x, y=-b) = 0$

strong BC has to satisfy.

$\phi \sim x^{n+2} y^3$

$\phi(x,y) = C_1 x^2 + C_2 xy + C_3 y^2 + C_4 x^3 + \dots$

Max order n+5 ← biharmonic.

$\nabla^4 \phi = 0$

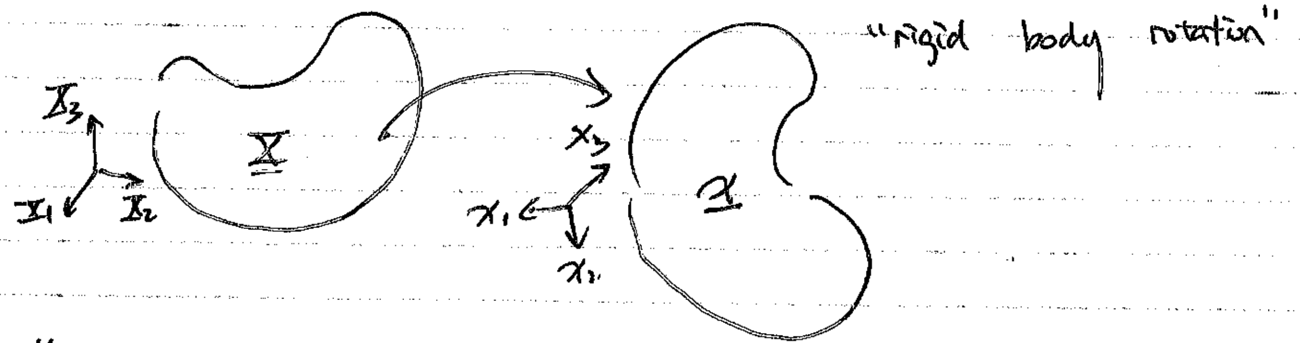
B.C.s → satisfy to determine the coeff.

	x	y		1
x^2	xy	y^2		2
x^3	x^2y	xy^2	y^3	3

⋮

Derivation of rotational tensor

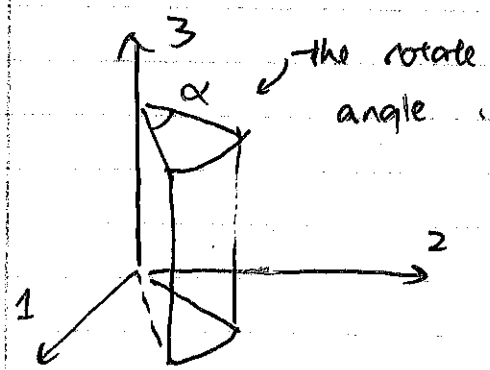
Begin with the continuum potato:



let's assume there's no deformation in the potato, i.e., pure rigid body rotation. the original coordinate writes $\underline{X} = [X_1, X_2, X_3]^T$, and the rotated coordinate is $\underline{x} = [x_1, x_2, x_3]^T$. Now let's assume if we just rotate the X_3 axis (or z-axis), the transformation writes

$$\boxed{x_1 = X_1 \cos\alpha - X_2 \sin\alpha, \quad x_2 = X_1 \sin\alpha + X_2 \cos\alpha, \quad x_3 = X_3}$$

→ it looks something like:



this is how we introduce the rotational tensor

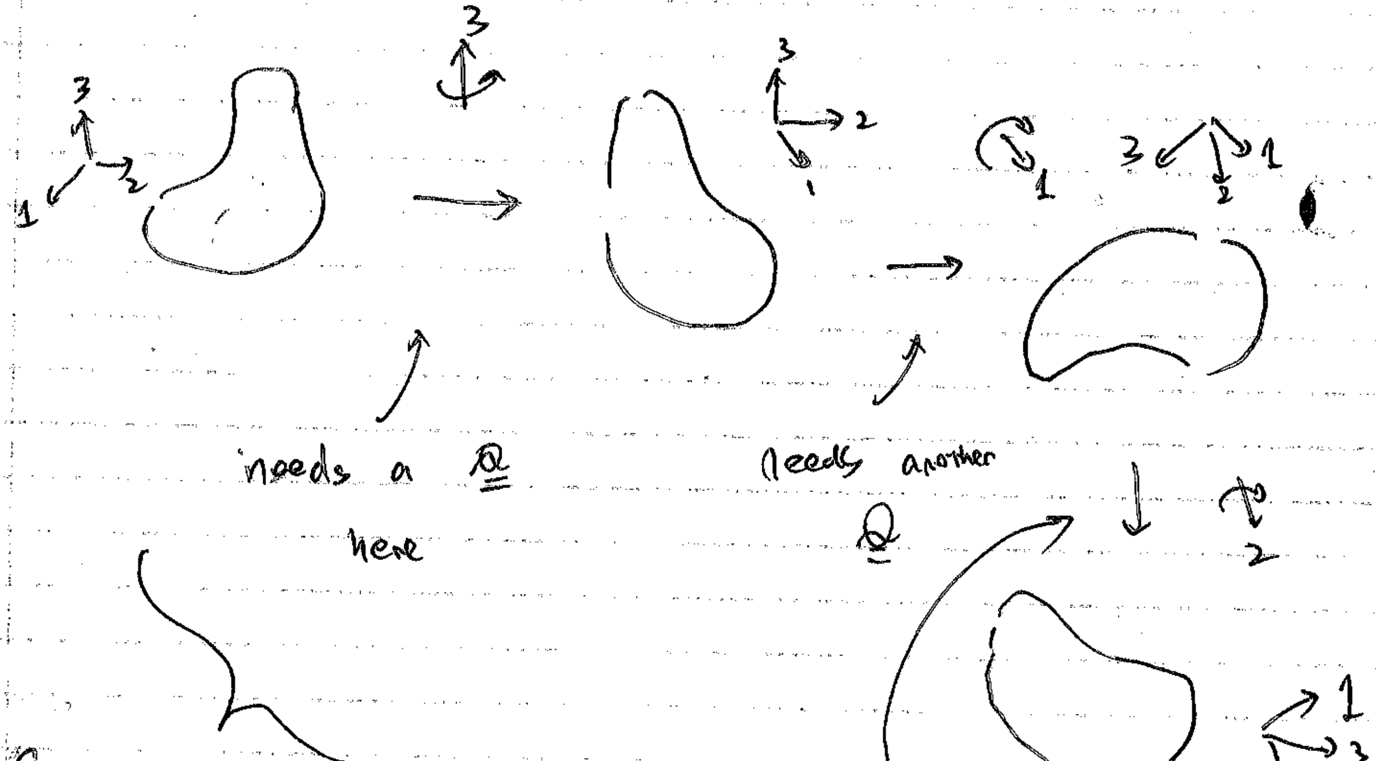
↳ Now, this process can be written as $\underline{x} = \underline{Q} \cdot \underline{X}$

this $\underline{Q}$ is $\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

LINEAR ALGEBRA

From $\underline{X}$ to $\underline{x}$ if we just rotate along the 3-axis, we need 1 $\underline{\underline{Q}}$ tensor,

if we want to rotate for both 1, 2, 3-axes, then we will need a bunch of $\underline{\underline{Q}}$ tensors to represent the transformations, something like



So this transformation process can be decomposed into a bunch of $\underline{\underline{Q}}$'s.

and... another $\underline{\underline{Q}}$!!

→ that's why you'll need to multiply by many $\underline{\underline{Q}}$s if you want to rotate around many directions.

← that example just illustrates how we rotate a vector $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. → or x_i.

if we want to rotate a tensor,

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \text{ or } A_{ij},$$ (indicial notation)

we'll need to rotate both the two axes.

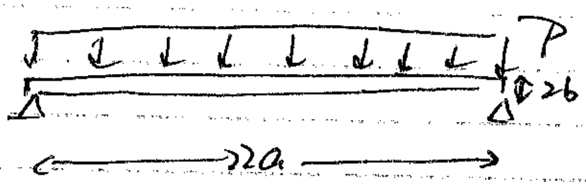
we then need two transformation tensors,

because a second-order tensor can be think of the out. product of 2 vectors (axes in our example). → So, if you want to rotate a second-order tensor, you'll need 2 $\underline{\underline{Q}}$s.

Similarly, if you want to rotate a fourth order tensor, you then need 4 $\underline{\underline{Q}}$s to transform (rotate) the 4 axes in that tensor

Problem Session #2

Problem 1 using Airy stress function



$q(x) \propto x^0$

$V(x) \propto x^1$

$\phi(x,y) = C_1 x^2 \ldots C_{18}$

$M(x) \propto x^2$

$\sigma_{xx} \propto M(x)\, y$

$t_1 = \sigma_{yy}(x, y = \pm b) = S_1 x^3 + S_2 x^2 \ldots$

$\phi \propto x^{n+5}$

t_2

t_3

t_4

$\hookrightarrow$ 4 strong B.C.s $\rightarrow$ 16 coefficients.

$F_{x_1} = \int_{-b}^{b} \sigma_{xx}\, dy = 0$

$\leftarrow$ Weak B.C.s

$F_{y_1} = \int_{-b}^{b} \sigma_{xy}\, dy = -pa$

$M = \int_{-b}^{b} \sigma_{xx}\, y\, dy = 0$ $\hookrightarrow$ 6 equations & 6 coefficients

$\nabla^4 \phi = 0$

$\hookrightarrow$ 3 eqns

Obtain the exact form of ϕ

$\downarrow$

get σ_{xx}, σ_{yy}, ...

$$u_x = \int \varepsilon_{xx}\, dx + f(y)$$

$$u_y = \int \varepsilon_{yy}\, dy + f(x)$$

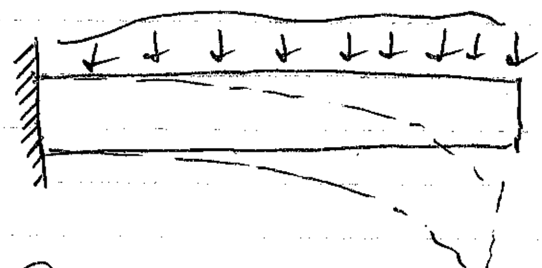
$$\varepsilon_{xy} = \frac{1}{2}\left[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right].$$

$\downarrow$

$F(x) = G(y) = C$ (separate variables).

$Cx + D$,

$Cy + E$



@ $x=0$, $\forall y$.

$\underbrace{u_x = 0,\ u_y = 0}$

"not satisfied"

Weak B.C.s

$x=0$. $y=0$

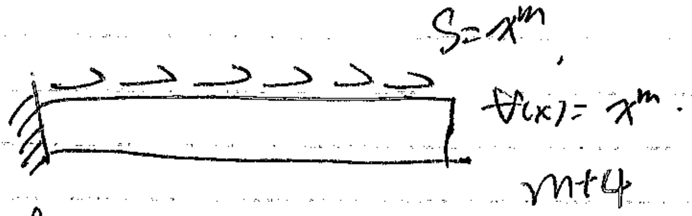
$$\int u_x\, dy = 0$$

$$\int u_y\, dx = 0$$

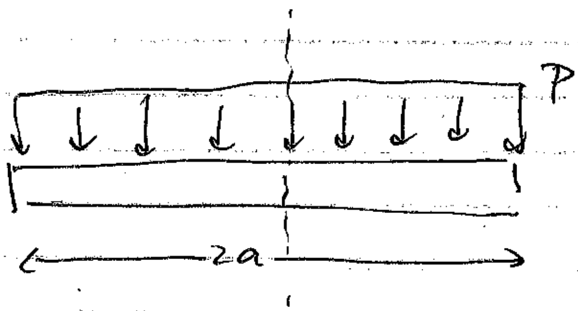
moment. $\int u_x y\, dy = 0$

$\frac{\partial u_y}{\partial x} = 0$ ← horizontal.

or ∂u_x, $y = 0$



den 2 Using Fourier Series.



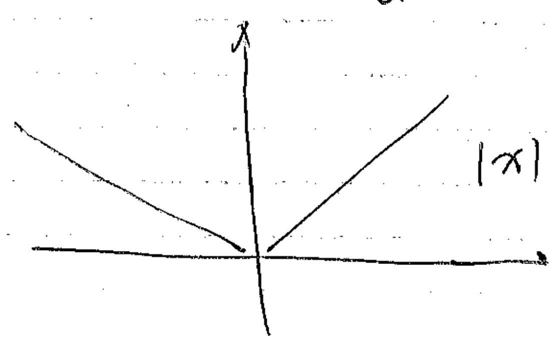
$$f(x) = a_0 + \underbrace{\sum_{n=1}^{N} a_n \cos(\lambda_n x)}_{\text{even func.}} + \underbrace{\sum_{n=1}^{N} b_n \sin(\lambda_n x)}_{\text{odd func.}}$$

$$\lambda_n = \frac{2n\pi}{2a} = \frac{n\pi}{a}$$

$$P = \sum_{n=1}^{N} a_n \cos(\lambda_n x)$$

$$\lambda_n = \frac{(2n-1)\pi}{2a} \qquad a_n = \frac{2}{2a}\int_{-a}^{a} P \cdot \cos(\lambda_n x)\, dx$$

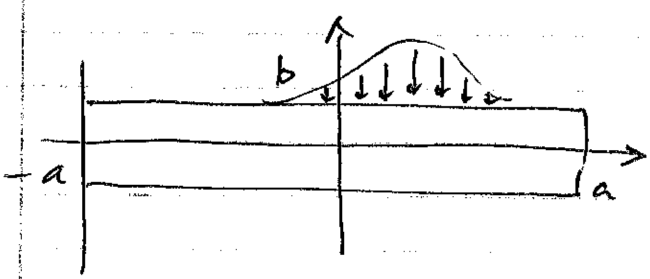
integration to determine the coefficients



4/15/2024. Lecture 5

Stress function. $\phi(x,y)$. (Recap).

$$\begin{cases} \sigma_{xx} = \phi_{,yy} \\ \sigma_{yy} = \phi_{,xx} \\ \sigma_{xy} = -\phi_{,xy} \end{cases} \rightarrow \nabla^4 \phi = 0$$



trial $\phi(x,y)$ polynomials.

$\sigma_{yy}(x, y=\pm b) = \pm t_{y\pm}(x)$.

$\sigma_{xy}(x, y\pm b) = \pm t_{x\pm}(x)$

trial sol'n: $\phi(x,y) = e^{\alpha x} e^{\beta y}$

... separation of var.

$e^{ikx} = \cos kx + i \sin kx$

$$\nabla^2 \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(e^{\alpha x} e^{\beta y}\right) = (\alpha^2 + \beta^2)\left(e^{\alpha x} e^{\beta y}\right).$$

$\nabla^2 \phi = 0$ ... harmonic eqn $\rightarrow \alpha^2 + \beta^2 = 0$

$\hookrightarrow \alpha^2 = -\beta^2$

General form of the sol'n:

$\phi(x,y) = e^{\pm i\lambda x} e^{\pm \lambda y} \longleftarrow \alpha = \pm i\beta$.

if not harmonic.

$$\nabla^2(\nabla^2\phi) = 0$$

$\phi(x,y)$

$$e^{i\lambda x}e^{\lambda y},\quad e^{i\lambda x}e^{-\lambda y},\quad e^{i\lambda x}ye^{\lambda y},\quad e^{i\lambda x}ye^{-\lambda y}$$

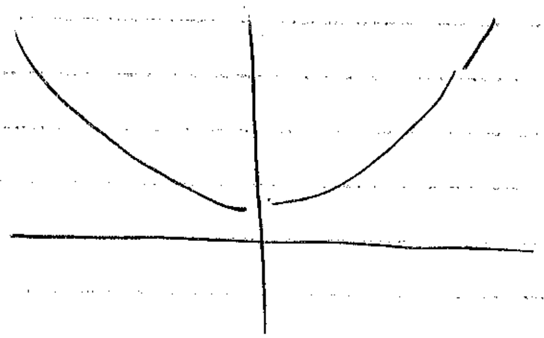
general expression for ϕ.

$$\phi(x,y) = e^{i\lambda x}\left[(C_1 + C_2 y)e^{\lambda y} + (C_3 + C_4 y)e^{-\lambda y}\right]$$

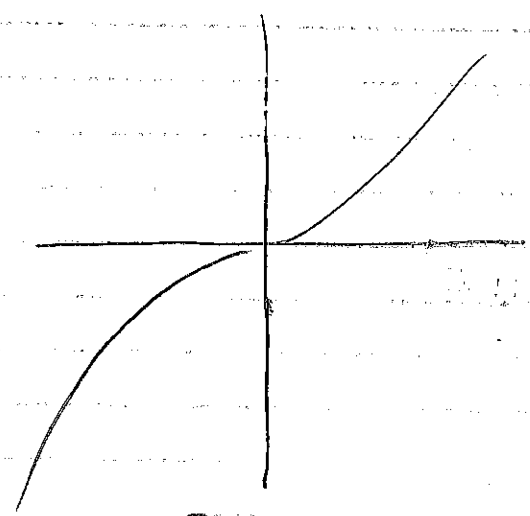
$$\cosh y = \frac{e^{\lambda y} + e^{-\lambda y}}{2}$$

$$\sinh y = \frac{e^{\lambda y} - e^{-\lambda y}}{2}$$

$$\cos kx = \frac{e^{ikx} + e^{-ikx}}{2},\quad \sin kx = \frac{e^{ikx} - e^{-ikx}}{2i}$$



cosh y



sinh y

even func. of x.

(prb. even w.r.t. x).

$$\phi(x,y) = \cos\lambda x \left[\underbrace{Ae^{\lambda y}}\ \ By e^{\lambda y}\ \ \underline{Ce^{-\lambda y}}\ \ Dy e^{-\lambda y} \right]$$

$\sin\lambda x$

$+ B'y\cosh\lambda y$
$+ D'y\sinh\lambda y.$

$$A'\cosh\lambda y. \quad + \quad C'\sinh\lambda y.$$

$$= \frac{A'}{2}(e^{\lambda y} + e^{-\lambda y}) \quad + \quad \frac{C'}{2}(e^{\lambda y} - e^{-\lambda y})$$

$$= \frac{A'+C'}{2} e^{\lambda y} \quad + \quad \frac{A'-C'}{2} e^{-\lambda y}.$$

(try to group them ...)

$$\phi(x,y) = \underbrace{\cos\lambda x \quad \left[A'\cosh\lambda y + D'y\sinh\lambda y \right]}_{\text{even in both}}$$

even fn. in y.

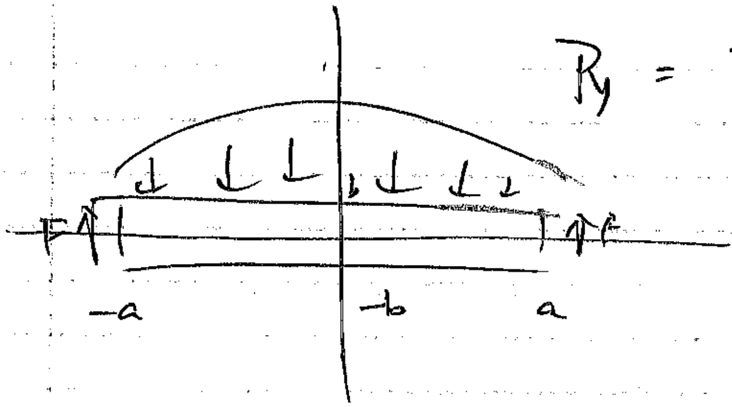
$$\cos\lambda x \quad [B'y\cosh\lambda y + C'\sinh\lambda y]$$

even in x.
odd in y.

$$\sin\lambda x \quad [\ \ldots\ ]$$

$$\sin\lambda x \quad [\ \ldots\ ].$$

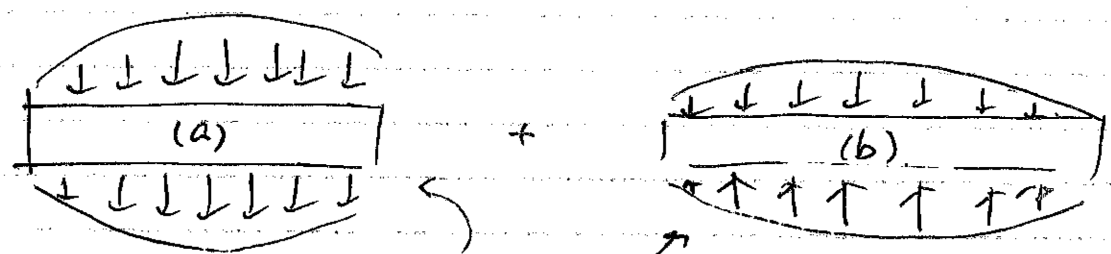
$$P_y = P_0 \cos\frac{\pi x}{2a}$$



B.C.
$$\begin{cases} \sigma_{yy}(x, y=\pm b) = -P_0 \cos\frac{\pi x}{2a} \\ \sigma_{yy}(x, y=-b) = 0. \\ \sigma_{xy}(x, y=+b) = 0 \\ \sigma_{xy}(x, y=-b) = 0 \end{cases}$$

find c_1, c_2, c_3, c_4 (with $\lambda = \frac{\pi}{2a}$).

... principles of superposition.



$\frac{1}{2}P_0 \cos\frac{\pi x}{2a}$

σ_{yy} even in x. odd in y.	σ_{yy} even in x even in y.
ϕ even in x, odd in y. σ_{xy} odd x even y.	ϕ even in x. even in y

σ_{xx} even in x

(a). $\phi = \cos\lambda x \, [B y \cosh\lambda y + C \sinh y]$

B.C. $\sigma_{yy}(x, y=b) = -\frac{1}{2}P_0 \cos\frac{\pi x}{2a}$.

$\lambda \equiv \frac{\pi}{2a}$

$\sigma_{xy}(x, y=b) = 0$

$\cos\lambda x$

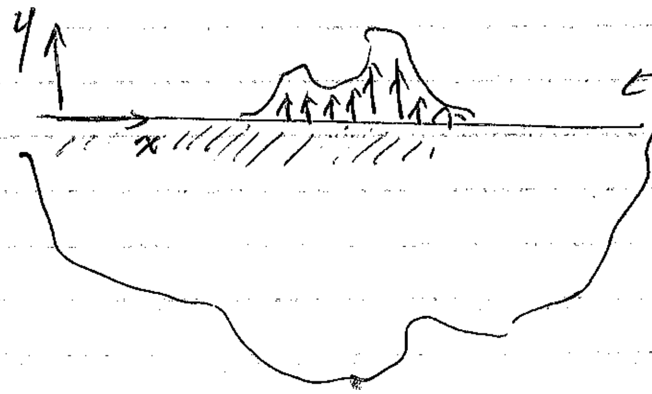
$\sigma_{yy} = -\lambda^2 \cos\lambda x \, [\cdots]$

$\sigma_{xy} = \cdots$

plug in $y=b$.

Lecture 6 6/17/2024.

Elastic half Space



Property of interest: displacement at top surface.

① Euler-Bernoulli beam theory.

$w(x) \propto \frac{1}{I}$.

$V(x) \propto \sigma_{xy}(x) \propto \frac{1}{I}$.

$M(x) \propto \sigma_{xx}(x) \propto \frac{1}{I}$.

② Airy stress function approach ... ϕ,

$\nabla^4 \phi = 0$

$\sigma_{xx} = \phi_{,yy}$

$\sigma_{yy} = \phi_{,xx}$

$\sigma_{xy} = -\phi_{,xy}$

$\sigma \to \varepsilon \to u$

Find my displacement at (on surface)
due to a force at x'.

$$u(x, x') = \underbrace{\Im(x, x')}\, T_y(x')$$

$$u(x) = \int_{\Omega x} u(x, x')\,dx' = \int_{\Omega x} G_s(x - x')\, T_y(x')\, dx'$$

$i \in \{x, y\}$.

$$u_i(x) = \int_{\Omega x} G_{s_{ij}}(x - x')\, T_j(x')\, dx'$$

if we just replace by e^{-ikx}

we then get into Fourier trans.

$$T_y(x) = e^{ikx}$$

$$u_y(x) = \int_{-\infty}^{\infty} G_s(x - x')\, e^{ikx'}\, dx'$$

$$x'' = x - x' \rightarrow dx' = -dx''$$

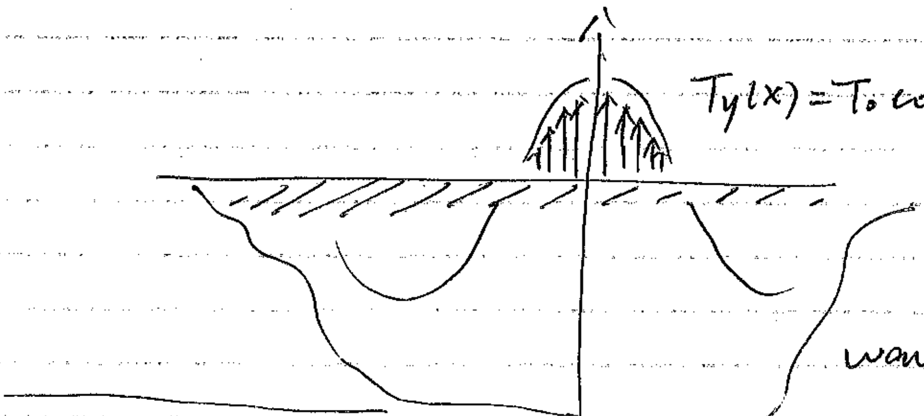
$$u_y(x) = \int_{-\infty}^{\infty} G_s(x'')\, e^{ikx} \cdot e^{-ikx''}\, (-dx'')$$

$$= e^{ikx} \underbrace{\int_{-\infty}^{\infty} G_s(x'')\, e^{-ikx''}\, dx''}$$ Fourier trans. of $G_s(x)$

$u_y(x) = \hat{G}_s(k)\, T_y(x)$.

or $T_0 \cos(kx)$

G_s set: G_{sxx}, G_{sxy}, G_{syx}, G_{syy}



B.C.s: $\sigma_{yy}(y=0, x) = T_y(x)$.

$\sigma_{xy}(y=0, x) = 0$

Stress function:

$\phi(x,y) = \cos\lambda x\,[A + By]\,e^{\lambda y}$

reject $e^{-\lambda y}$.

$\lim_{y \to -\infty} e^{-\lambda y} \to \infty$

$\sigma_{yy} = \phi_{,xx} = -\lambda^2 \cos\lambda x\,[A+By]\,e^{\lambda y}$

$\sigma_{xy} = -\phi_{,xy}$

$= \lambda \sin\lambda x\,[A\lambda + B + B\lambda y]\,e^{\lambda y}$.

$B = -A\lambda$

$A = -\frac{T_0}{\lambda^2}$

$B = T_0/\lambda$

$\sigma_{xx} = \phi_{,yy} = \cos\lambda x\,[A\lambda^2 + 2B\lambda + B\lambda^2 y]\,e^{\lambda y}$

$\sigma_{xx} = T_0 \cos\lambda x\,(1+\lambda y)\,e^{\lambda y}$

$\sigma_{yy} = T_0 \cos\lambda x\,(1-\lambda y)\,e^{\lambda y}$

$\sigma_{xy} = T_0\,\lambda \sin\lambda x\; y\,e^{\lambda y}$

Under plane-strain assumption.

$$\varepsilon_{xx} = \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu(1+\nu)}{E}\sigma_{yy}$$

$$\varepsilon_{yy} = \frac{-\nu(1+\nu)}{E}\sigma_{xx} + \frac{1-\nu^2}{E}\sigma_{yy}.$$

$$\varepsilon_{xy} = \frac{\sigma_{xy}}{2\mu} = \frac{1+\nu}{E}\sigma_{xy}.$$

$$u_x = \int \varepsilon_{xx}\,dx = \sim\sim + C(y)$$

$$u_y = \int \varepsilon_{yy}\,dy = \_\_\_\_ + D(x)$$

$$\frac{1}{2}(u_{x,y} + u_{y,x}) = \varepsilon_{xy}. \rightarrow \begin{array}{l} C(y) = C \\ D(x) = D \end{array}$$

We can then obtain forms for u_x & u_y:

$$u_x(x,y) = \frac{T_0}{\lambda E}\sin\lambda x\left[(1-\nu-2\nu^2) + (1+\nu)\lambda y\right]e^{\lambda y} + C$$

$$u_y(x,y) = \frac{T_0}{\lambda E}\cos\lambda x\left[(2-2\nu^2) - (1+\nu)\lambda y\right]e^{\lambda y} + D$$

$$u_x(x, y=0) = \tilde{u}_x(x) = \frac{T_0}{\lambda E}\sin\lambda x\,(1-\nu-2\nu^2).$$

$$u_y(x, y=0) = \tilde{u}_y(x) = \frac{T_0}{\lambda E}\cos\lambda x\,(2-2\nu^2).$$

$$\hat{u}_y(x) = T_y(x)\left(\frac{(2-2\nu^2)}{kE}\right)$$

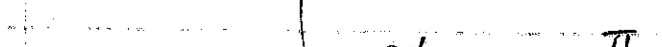
$$\hat{G}_{syy}(k) = \frac{2(1-\nu^2)}{kE}$$

↳ Can be replaced by $2\mu(1+\nu)$.

$$= \frac{1-\nu}{k\mu}$$

↳ $\dfrac{1-\nu}{|k|\mu}$

↳ the postiveness does not really matter here



$T_y \propto \sin kx$

$\propto \cos(kx')$.

$$x' = x - \frac{\pi}{2k}.$$

$$T_y(x) = e^{ikx} = \cos kx + i\sin kx$$

$$\hat{u}_y(x) = T_y(x)\,\frac{1-\nu}{|k|\mu}$$

where:

$$\mathcal{F}^{-1}\left[\frac{1}{|k|}\right] = \frac{-\log(x)}{\pi}$$

$$G_{syy}(x) = \mathcal{F}^{-1}\left[\frac{1}{|k|}\right]\frac{1-\nu}{\mu}$$

$$G_{syy}(x) = \frac{(1-\nu)}{\pi\mu} - \log(x)$$

$$K = 3 - 4\nu$$

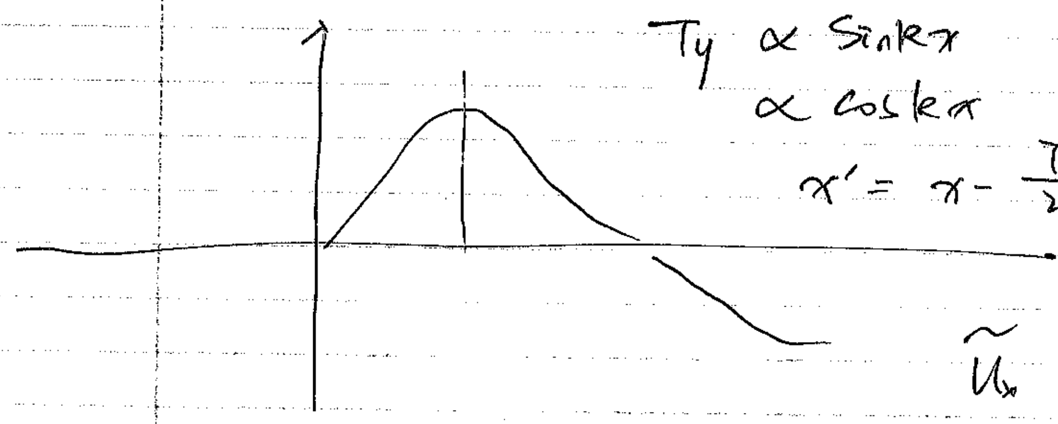
$$G_{syy}(x) = -\frac{K+1}{4\pi\mu}\log(x)$$

$$\tilde{u}_x = \frac{T_0 \sin\lambda x}{\lambda E}\left(1-\nu-2\nu^2\right)$$

$$= \frac{T_0 \sin\lambda x\,(1+\nu)(1-2\nu)}{\lambda\, 2\mu(1+\nu)}$$ ← loading of $\cos\lambda x$

$$= \frac{T_0 \sin\lambda x\,(1-2\nu)}{2\mu\lambda}$$

k & λ's are interchangable



$T_y \propto \sin kx$

$\propto \cos kx$

$x' = x - \frac{\pi}{2k}$

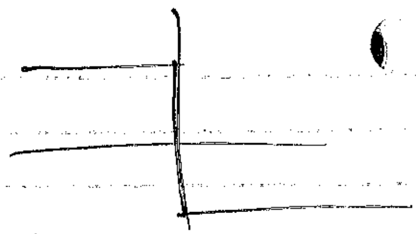
$\tilde{u}_x \propto \sin\left(kx - \frac{\pi}{2}\right)$

$\Longleftarrow \propto -\cos kx$

$$T_y = e^{ikx} = \cos kx + i\sin kx$$

$$\tilde{u}_x = A\left[\sin kx - i\cos kx\right]$$

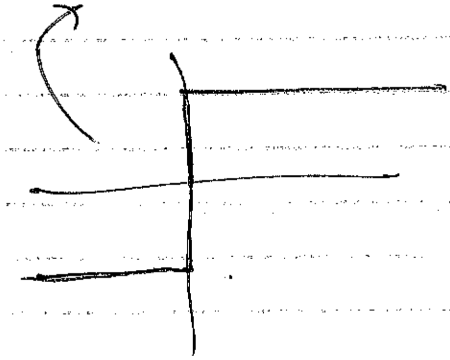
$$\hat{u}_x = -iA\left[\cos kx - \frac{1}{i}\sin kx\right]$$



$-\frac{1}{i}\,\mathrm{sgn}(x)$

$$-i^{-1} = i$$

$$\tilde{u}_x = -iA\, e^{ikx}$$



$$= \left[-i\,(1-2\nu)/2\mu k\right] e^{ikx}$$

$$\hat{u}_{sxy}(k)$$

$\frac{1}{i}\,\mathrm{sgn}(x)$

$$\mathcal{F}^{-1}\left[\hat{G}_{sxy}(k)\right] = \frac{-(1-2\nu)}{2\mu}\,\mathcal{F}^{-1}\left[\frac{i}{k}\right]$$

ane Strain.

$$\kappa = 3-4\nu \;\rightarrow\; \frac{\kappa-1}{2} = 1-2\nu$$

$$= \frac{-(\kappa-1)}{4\mu}\cdot\frac{\mathrm{sgn}(x)}{2}$$

		Fourier	Real space
y-loading	G_{syy}	$\frac{\kappa+1}{4\mu}\frac{1}{\lvert k\rvert}$	$-\frac{\kappa+1}{4\pi\mu}\log(x)$
	G_{sxy}	$\frac{-(\kappa-1)}{4\mu}\left(\frac{i}{k}\right)$	$-\frac{(\kappa-1)}{8\mu}\mathrm{sgn}(x)$
x-loading	G_{sxx}	$\frac{\kappa+1}{4\mu}\frac{1}{\lvert k\rvert}$	$-\frac{\kappa+1}{4\pi\mu}\log(x)$
	G_{syx}	$\frac{\kappa+1}{4\mu}\left(\frac{i}{k}\right)$	$-\frac{(\kappa-1)}{8\mu}\mathrm{sgn}(x)$

$$T(x) = T_y(x)\,\hat{e}_y + T_x(x)\cdot\hat{e}_x$$

$$\tilde{u}_x = \int_{-\infty}^{\infty} -\frac{\kappa+1}{4\pi\mu}\log(x-x')\,T_x(x')\,dx'$$

$$+\int_{-\infty}^{\infty} \frac{\kappa-1}{8\mu}\,\mathrm{sgn}(x-x')\,T_y(x')\,dx'$$

Problem Session #3.

Generalized Airy Stress function.

$$\phi(x,y) = \sum_i C_i \phi_i(x,y) \begin{cases} \text{not universal.} \\ \text{unstable.} \\ \nabla^4\phi = 0 \text{ not guaranteed} \end{cases}$$

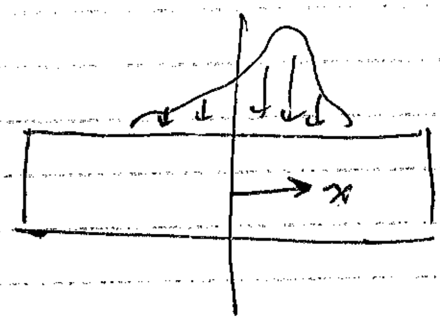
$\phi(x,y) = e^{\alpha x} e^{\beta y}$ $\quad \nabla^2 \phi = 0$

$= e^{i\lambda x} e^{\lambda y}$ ① $\quad \alpha = \pm i\beta \rightarrow \lambda.$

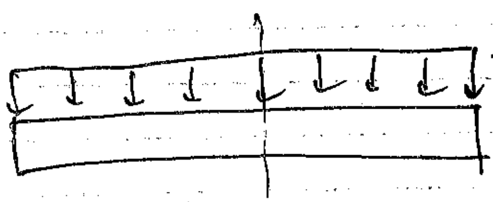
$e^{i\lambda x} e^{-\lambda y}$ ②

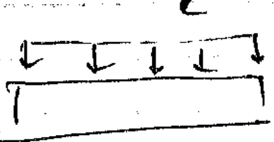
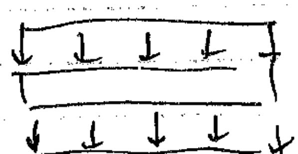
$e^{i\lambda x} y e^{\lambda y}$ ③

$e^{i\lambda x} y e^{-\lambda y}$ ④



$$\phi(x,y) = e^{i\lambda x}\left[(C_1 + C_2 y)e^{\lambda y} + (C_3 + C_4 y)e^{-\lambda y}\right]$$



①  +  $P_0/2$ ②

		x	y
Problem ①	σ_{yy}	even	even.
	ϕ		

		x	y
Problem ②	σ_{yy}	even	odd.

② Expand the loading as Fourier series.

$$p(x) = \sum_{n=1}^{\infty} a_n \cos(\lambda_n x)$$

$$\hookrightarrow \lambda_n = \frac{(2n-1)\pi}{2a}$$

③ B.C.s: $\sigma_{yy}(x, y=b) = -P_0/2 = -\frac{1}{2}\sum_{n=1}^{\infty} a_n \cos(\lambda_n x)$

$\sigma_{yx}(x, y=0) = 0$

$$\sigma_{yy}\Big|_{x, y=b} = \sum_{n=1}^{\infty} -\lambda_n^2 \cos(\lambda_n x)\left[B_n \sinh(\lambda b) \quad \ldots \quad \right]$$

$$-\frac{a_n}{2} = -\lambda_n^2 \left[B_n \sinh(\lambda_n b) + C_n b \cosh(\lambda_n b) \right]$$

$$\sigma_{xx}, \; \sigma_{yy}, \; \sigma_{yx}$$

$$\downarrow$$

$$\varepsilon_{xx}, \; \varepsilon_{yy}, \; \varepsilon_{yx}$$

$$\downarrow$$

$$\varepsilon_{xx} = u_{x,x}$$

even in x, odd in y.

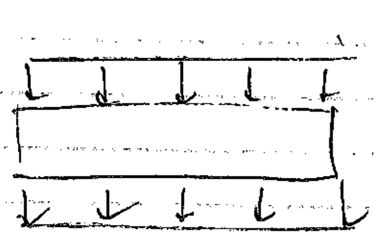
$$\phi_{,yy}$$

$$\uparrow$$

$$\varepsilon_{xx} = \_\,\sigma_{xx} + \_\,\sigma_{yy}$$

$$\downarrow$$

even in x,
odd in y.



$$u_x = \int \varepsilon_{xx}\, dx$$

$$\downarrow$$

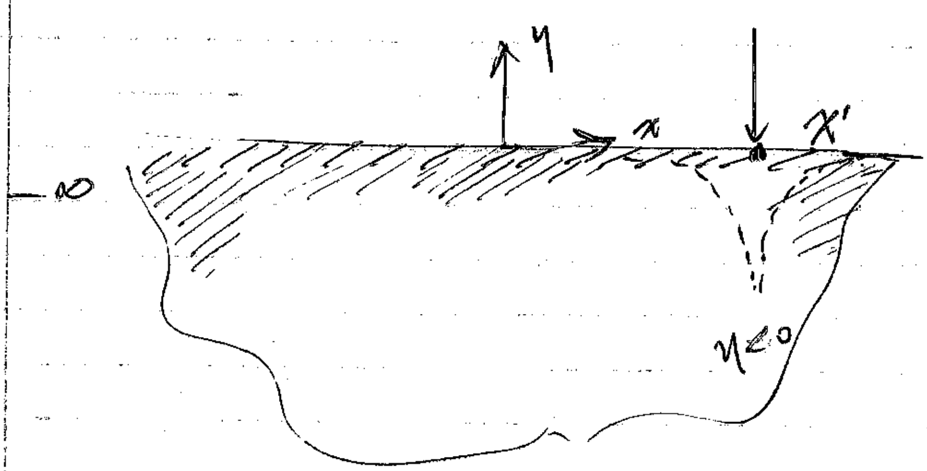
odd in x,
odd in y

$$u_x = \sum_{n=1}^{\infty} \sin(\lambda_n x) \left[D_n \sinh(\lambda_n y) + E_n y \cosh(\lambda_n y) \right]$$

$$u_y = \ldots$$

$$u_{x,x} = \varepsilon_{xx} = \frac{\sigma_{xx}}{E'} - \frac{\nu' \sigma_{yy}}{E'}$$

2> Surface Green's function

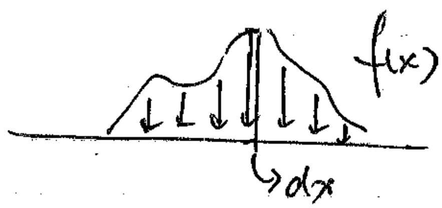


$G^{s}_{yy}(x-x')$

↳ direction of force

↳ displacement direction

Distributed loading



$$\int_{-\infty} G^{s}_{yy}(x-x') f(x') dx'$$

f(x)dx

$$\tilde{u}_y = \int_{-\infty}^{\infty} G^{s}_{yy}(x-x') T_y(x') dx'$$

$$+ \int_{-\infty}^{\infty} G^{s}_{yy}(x-x') T_x(x') dx'$$

↳ general form of surface displacement.

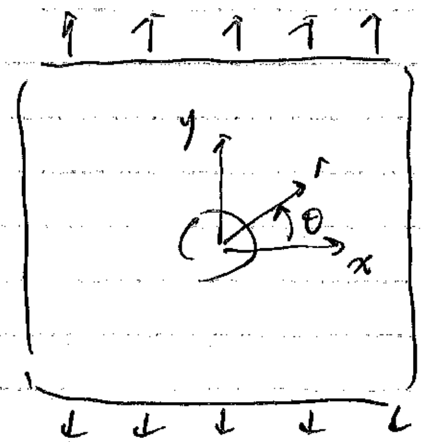
Recap. apply loading e^{ikx}.

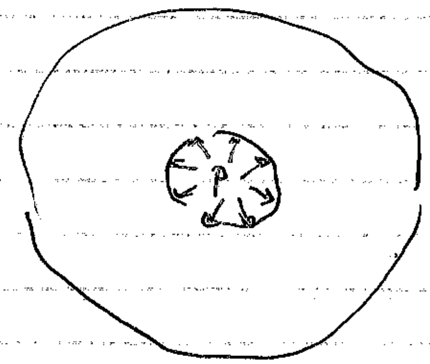
$$\tilde{u}_y = G^{s}_{yy}[k] \cdot e^{ikx}.$$

$$\mathcal{F}(G^{s}_{yy}(x))$$

Lecture 7 4/22/2024.

→ Polar Coordinates





Stress Concentration. ~ 3.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \rightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \operatorname{atan}\frac{y}{x} \end{cases}$$

… Cartesian coordinate.

$$\sigma_{xx} = \phi_{,yy}(x,y).$$

$$\sigma_{yy} = \phi_{,xx}(x,y).$$

$$\sigma_{xy} = -\phi_{,xy}(x,y).$$

… Polar Coordinate.

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \qquad \sigma_{\theta\theta} = \phi_{,rr}, \quad \sigma_{r\theta} = -\frac{1}{r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

r, θ

$\downarrow$

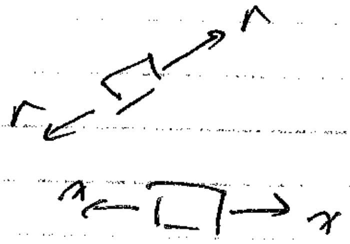
$\phi(r,\theta) = \phi(x,y).$

$\downarrow$

$\sigma_{xx}, \sigma_{yy}, \sigma_{xy}.$

$\downarrow$ coord. trans.

$\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta}.$



$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \longrightarrow \frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}.$$

chain rule.

Vector calculus.

$$\underset{\sim}{\nabla} f = \underline{e}_x f_{,x} + \underline{e}_y f_{,y}.$$

gradient operator

$$\underset{\sim}{\nabla} = \underline{e}_x \frac{\partial}{\partial x} + \underline{e}_y \frac{\partial}{\partial y}.$$

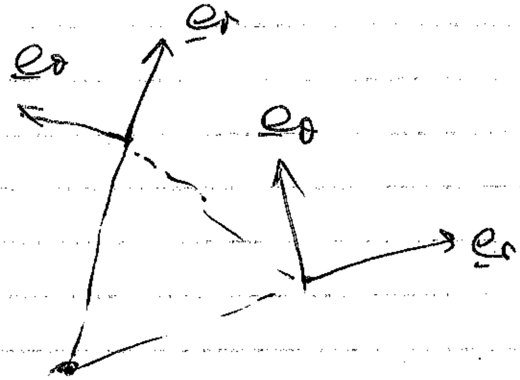
$$= \underline{e}_{x'} \frac{\partial}{\partial x'} + \underline{e}_{y'} \frac{\partial}{\partial y'}$$

$$\nabla^2 = \underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$= \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}$$

... Laplacian in polar coordinates?

in polar coordinates.



$$\underset{\sim}{\nabla} = \underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\theta \frac{\partial}{\partial \theta} \frac{1}{r}$$

(grad).

Laplacian.

$$\nabla^2 = \underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla} = \left(\underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right) \cdot \left(\underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right)$$

your Laplacian:

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial \theta^2}$$

One can show that:

$$\begin{cases} \dfrac{\partial \underline{e}_r}{\partial \theta} = \underline{e}_\theta \\[2ex] \dfrac{\partial \underline{e}_\theta}{\partial \theta} = -\underline{e}_r \end{cases}$$

$$\underline{\underline{\sigma}} \neq \underset{\sim}{\nabla} \otimes \underset{\sim}{\nabla} \phi$$

not true !!!

$$\begin{cases} \dfrac{\partial \underline{e}_r}{\partial r} = 0 \\[2ex] \dfrac{\partial \underline{e}_\theta}{\partial r} = 0 \end{cases}$$

$$\underset{\sim}{\nabla}^a = \underset{\sim}{\nabla} \times \underset{\sim}{e}_z$$

$$= \left(\underline{e}_x \frac{\partial}{\partial x} + \underline{e}_y \frac{\partial}{\partial y}\right) \times \underline{e}_z$$

$$= \left(-\underline{e}_y \frac{\partial}{\partial x} + \underline{e}_x \frac{\partial}{\partial y}\right)$$

$$\underline{\underline{\sigma}} = \nabla^a \otimes \nabla^a \phi$$

for Cartesian

in polar coordinates

$$\underset{\sim}{\nabla}^a = \left(\underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\theta \frac{1}{r} \cdot \frac{\partial}{\partial \theta} \right) \times \underset{\sim}{e}_z$$

$$= \left(- \underline{e}_\theta \frac{\partial}{\partial r} + \underline{e}_r \frac{1}{r} \cdot \frac{\partial}{\partial \theta} \right)$$

$$\underset{\sim}{\nabla}^a \otimes \underset{\sim}{\nabla}^a = \underset{\sim}{e}_\theta \otimes \underset{\sim}{e}_\theta \frac{\partial^2}{\partial r^2} + \cdots$$

Stress function satisfies biharmonic ...

$$\nabla^4 \phi = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = 0$$

→ in polar coordinates

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \phi = 0$$

displacement vs. strain

$$\varepsilon_{ij} = \frac{1}{2} \left(u_{i,j} + u_{j,i} \right) \rightarrow \text{Cartesian}$$

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left[\underline{\nabla} \otimes \underline{u} + (\underline{\nabla} \otimes \underline{u})^T\right]$$

$$\underset{\sim}{u} = u_r \underline{e}_r + u_\theta \underline{e}_\theta$$

$$\begin{cases} \varepsilon_{rr} = \dfrac{\partial u_r}{\partial r} \\ \varepsilon_{\theta\theta} = \dfrac{1}{r} \cdot \dfrac{\partial u_\theta}{\partial \theta} + \dfrac{u_r}{r} \\ \varepsilon_{r\theta} = \dfrac{1}{2}\left(\dfrac{1}{r} \cdot \dfrac{\partial u_r}{\partial \theta} - \dfrac{u_\theta}{r} + \dfrac{\partial u_\theta}{\partial r}\right) \end{cases}$$

Generalized Hooke's law.

$$\underline{\underline{\sigma}} = \lambda \, tr\left[\underline{\underline{\varepsilon}}\right]\underline{\underline{I}} + 2\mu \underline{\underline{\varepsilon}}$$

Traction force vs. Stress.

$$T_j = \sigma_{ij} n_i$$

$$\underset{\sim}{T} = \underline{n} \cdot \underline{\underline{\sigma}}$$

$$\begin{cases} T_r = \sigma_{rr} n_r + \sigma_{\theta r} n_\theta \\ T_\theta = \sigma_{r\theta} n_r + \sigma_{\theta\theta} n_\theta \end{cases}$$

Equilibrium Condition.

$$\sigma_{ij,i} + F_j = 0$$

$$\underline{\nabla} \cdot \underline{\underline{\sigma}} + \underline{F} = 0$$

↳ "automatically satisfied by stress func. approach"

PDE to be solved:

$$\nabla^4 \phi(r,\theta) = 0$$

Expect $\phi(r,\theta) = \phi(r, \theta + 2\pi)$

$$\phi(r,\theta) = f(r)\, e^{in\theta} \qquad (n = 0, 1, 2, \dots)$$

$$\frac{\partial}{\partial\theta}\phi = in\phi, \qquad \frac{\partial^2}{\partial\theta^2}\phi = -n^2\phi.$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right) f(r)\, \cancel{e^{in\theta}} = 0$$

deriv. replaced by $-n^2$

Try. $f(r) = r^m$

$$\frac{1}{r}\cdot\frac{\partial}{\partial r} f(r) = m r^{m-2}.$$

$$\frac{\partial^2}{\partial r^2} f(r) = m(m-1)\, r^{m-2}.$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right) r^m = (m^2 - n^2)\, r^{m-2}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right) r^{m-2} = \left((m-2)^2 - n^2\right) r^{m-4}.$$

$$\left[(m-2)^2 - n^2\right](m^2-n^2)\, r^{m-4} = 0$$

... four possibilities. $\begin{cases} m = \pm n \\ m = 2 \pm n \end{cases}$

$$\phi(r,\theta) = \left(A_{n1} r^{n+2} + A_{n2} r^{-n+2} + A_{n3} r^{n} + A_{n4} r^{-n}\right) e^{in\theta}$$

$\left.\begin{array}{l} n=0 \\ n=1 \end{array}\right\}$ not four sol'ns.

		n	$-n$	$2+n$	$2-n$
$n=0$	m	0	0	2	2

$\underbrace{0 \quad 0 \quad 2}_{\text{only } 2 \text{ sol'ns for } \phi.}$

$n=1$	m	1	-1	3	1

Mitchell Sol'ns ......

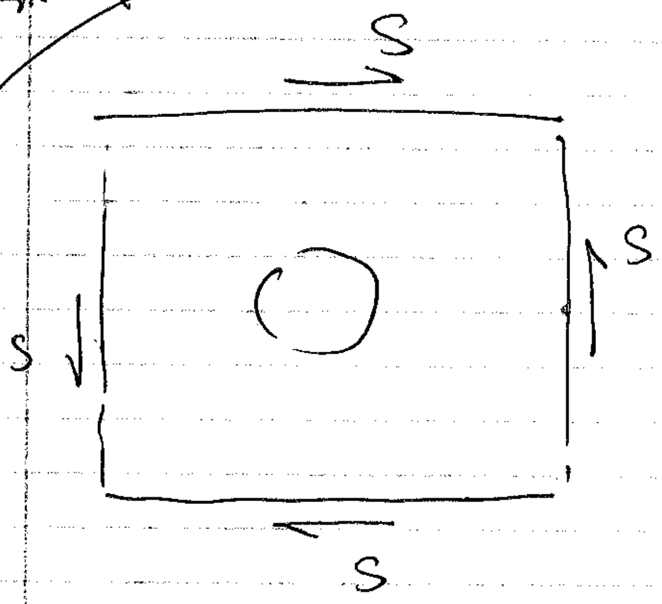
$$f_0(r) = A_{01} r^2 + A_{02} r^2 + A_{03} + A_{04}$$

$\cdots \downarrow$ some math tricks

$$A_{01} r^2 + A_{02} r^2 \ln r + A_{03} \ln r + A_{04}$$

take deriv. w.r.t. parameters

Example 1



$$\begin{cases} \sigma_{xx} = \sigma_{yy} = 0 & r \to \infty \\ \sigma_{xy} = S & r \to \infty \end{cases}$$

on the hole boundary:

$$\begin{cases} \sigma_{r\theta} = 0 & r = a \\ \sigma_{rr} = 0 & r = a \end{cases}$$

$$\phi = \phi^{(0)} + \phi^{(1)}$$

$$\sigma_{xy}^{(0)} = S$$

$$\phi^{(0)} = -S\,xy$$

in polar coordinate.

$$\phi^{(0)} = -S\,r^2 \sin\theta \cos\theta$$

$$= -\frac{1}{2} S r^2 \sin 2\theta$$

$$\begin{cases} \sigma_{rr}^{(0)} = \frac{1}{r} \cdot \frac{\partial \phi^{(0)}}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 \phi^{(0)}}{\partial \theta^2} = S \sin 2\theta \\ \sigma_{\theta\theta}^{(0)} = -S \sin\theta \\ \sigma_{r\theta}^{(0)} = S \cos 2\theta \end{cases}$$

→ satisfies infinite far B.C.s

but does not s. hole BCs

We need to come up $\phi^{(1)}$, s.t.

ϕ satisfies hole B.C.s but do not

mess up the infinite far B.C.s.

We should pick $n=2$ to cancel the

$\sin(2\theta)$ term.

$$\phi^{(1)} = \left(A_{21} r^4 + A_{22} + A_{23} r^2 + A_{24} r^{-2}\right) \sin 2\theta$$

want to cancel these terms

terms survived. (match BCs)

Rewrite the stress function:

$$\phi^{(1)} = \left(A + B r^{-2}\right) \sin 2\theta$$

$r=a$: $\sigma_{rr}=0$

... same $\sigma_{r\theta}$...

$$\boxed{A = Sa^2,\ B = -\tfrac{1}{2}Sa^4}$$

$$\begin{cases} \sigma_{rr} = \left(S - \dfrac{4A}{r^2} - \dfrac{6B}{r^4}\right) \sin 2\theta \\ \sigma_{r\theta} = \left(S + \dfrac{2A}{r^2} + \dfrac{6B}{r^4}\right) \cos 2\theta \end{cases}$$

$$\sigma_{\theta\theta} = \left(-S + \frac{6B}{r^4}\right) \sin 2\theta$$

effect of hole

Problem Session 4

Polar Coordinates. & Michell Sol'ns.

$\phi(r,\theta) \rightarrow$ compatibility $\nabla^4 \phi(r,\theta) = 0$

$\phi(r,\theta) = f(r) \cdot e^{in\theta} \rightarrow \phi(r, \theta + 2\pi) = \phi(r,\theta)$

$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2}{\partial \theta^2}$ $\qquad f(r) = r^m$.

$\nabla^2 \phi = \left(\ldots - \frac{n^2}{r^2} \right) \phi \hookrightarrow r^m e^{in\theta}$

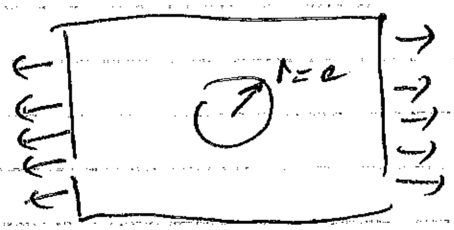
$\nabla^4 = \nabla^2 \cdot \underbrace{\nabla^2 r^m e^{in\theta}}_{\text{not zero everywhere}} = 0$

$m = (2+n,\ 2-n,\ +n,\ -n)$.

... 4 unique sol'ns

$$\phi(r,\theta) = \underbrace{\left(A_{n1} r^{2+n} + A_{n2} r^{2-n} + A_{n3} r^{n} + A_{n4} r^{-n} \right) e^{in\theta}}_{\text{Michell sol'ns}}$$

Example



$\ldots \infty$

(α) No hole uniform axial stress

(β) hole w/o loading

problem (α):

$$\sigma_{xx} = S$$

$$\sigma_{yy} = \sigma_{xy} = 0$$

IMPORTANT

$$\phi^{(0)}(x,y) = \frac{Sy^2}{2} = \frac{Sr^2 \sin^2\theta}{2} \qquad \left(\frac{1-\cos 2\theta}{2}\right)$$

$$\phi^{(0)}(x,y) = \underbrace{\frac{Sr^2}{4}}_{n=0} - \underbrace{\frac{Sr^2 \cos 2\theta}{4}}_{n=2}$$

"base solution".

$$\phi^{(1)}(r,\theta) = \left(\cancel{A_{01} r^2} + \cancel{A_{02} r^2 \ln(r)} + A_{03} \ln r + A_{04}\,\theta\right)$$

$$+\left(\cancel{A_{21} r^4} + A_{22} + \cancel{A_{23} r^2} + \frac{A_{24}}{r^2}\right)\cos 2\theta \;+ \ldots$$

at $r=a$.

<u>B.C.s</u> traction free. $\sigma_{rr} = \sigma_{r\theta} = 0, \quad \forall\,\theta.$

$$\sigma_{rr} = C_1 \cos 2\theta - \frac{S r^2}{4} \cos 2\theta = 0$$

$$\phi^{(1)}(r,\theta) = A \ln r + B\theta + C \cos 2\theta + \frac{D}{r^2} \cos 2\theta.$$

not periodic.

$$\phi^{tot} = \frac{S r^2}{4} + A \ln r + B\theta + \left(C + \frac{D}{r^2} - \frac{S r^2}{4}\right)\cos 2\theta$$

$$\sigma_{rr} = \frac{1}{r}\cdot\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$= \frac{S}{2} + \frac{A}{r^2} + 0 + \frac{1}{r}\cdot\left(\frac{-2D}{r^3} - \frac{2Sr}{4}\right)\cos 2\theta$$

$$+ \frac{(-4)}{r^2}\left(C + \frac{D}{r^2} - \frac{S r^2}{4}\right)\cos 2\theta$$

or go to Michell's table

From Michell's sol'n table,

$$\sigma_{rr} = \frac{S}{4}(2) + A\left(\frac{1}{r^2}\right) + 0\left(\frac{-4C}{r^2} - \frac{6D}{r^4} + \frac{S}{2}\right)\cos 2\theta \overset{0}{//}$$

Get A, B, C, D just as in class.

Given disp. B.C.s.

Say $u_r = 0$ at $r = a$ $\forall \theta$

From Michell tab., $2\mu u_r = \frac{S}{4}(k-1)r$

$+A(-\frac{1}{r}) \ldots$

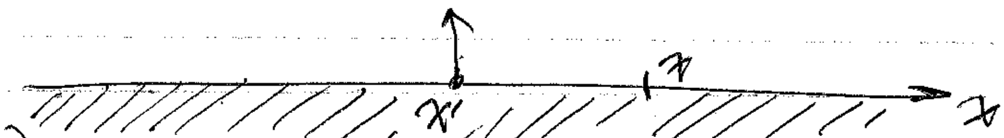
$u_r\Big|_{r=a} = 0$

Stress concentration factor

$\sigma_{\theta\theta} = \frac{S}{2}\left(1 + \frac{a^2}{r^2}\right) - S\cos 2\theta\left(\frac{3a^4}{r^4} + 1\right)$

Lecture 9 4/29/2024

Contact. → Surface Green function

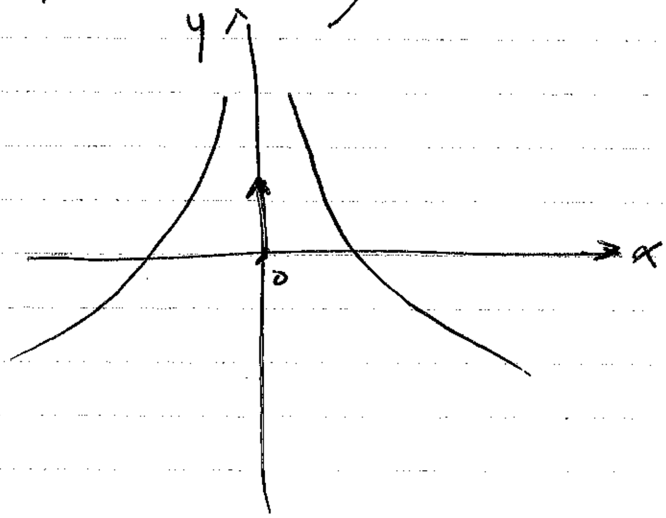


$G_{ij}(x, -x')$

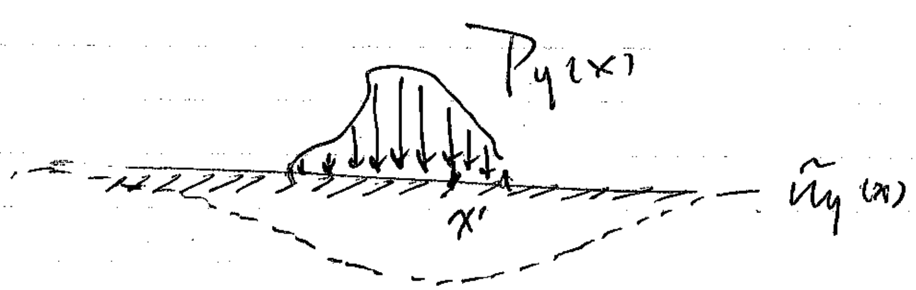
direction of the displacement

direction of the force

$$G^{s}_{yy}(x) = -\frac{\kappa+1}{4\pi\mu}\log|x|$$



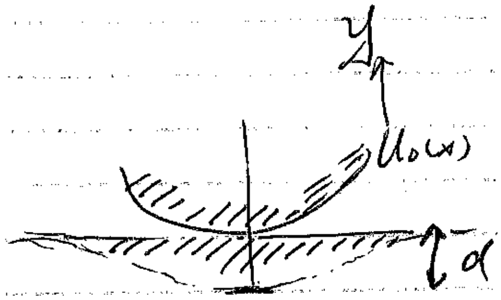
$$\tilde{u}_y(x) = \int_{-\infty}^{+\infty} - P_y(x')\, G^{s}_{yy}(x-x')\, dx'$$

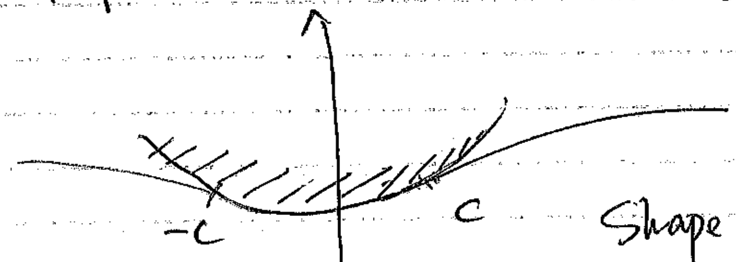


$$\tilde{u}_y(x) = \int_{-\infty}^{+\infty} \frac{\kappa+1}{4\pi\mu} P_y(x') \log|x-x'|\, dx'$$

Set up frictionless contact problem

- Simplification: rigid indenter





Shape: $u_0(x)$

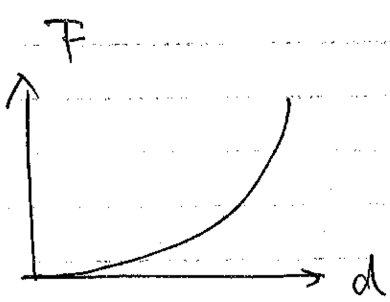
"symmetric"

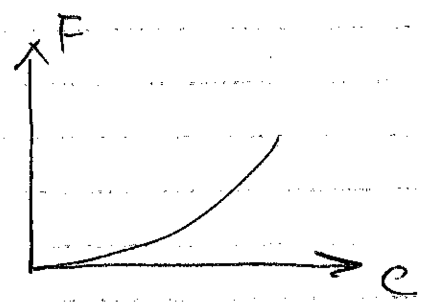
$$u_0(x) - d = \int_{-\infty}^{+\infty} P_y(x') \frac{\kappa+1}{4\pi\mu} \log|x-x'|\,dx' \quad \left[-c < x < c \ \text{(contact area)}\right]$$

or $\int_{-c}^{+c}$

"compact support"

$$F = \int_{-c}^{c} P_y(x')\,dx'$$





B.C.s ① contact area.

$$-c < x < c, \; (y=0) \quad \begin{cases} \tilde{u}_y(x) = u_0(x) - d \\ \sigma_{yy}(x) = -P_y(x) < 0 \\ \sigma_{xy}(x) = 0 \end{cases}$$

② gap area.

$$|x| > c \quad y=0 \quad \begin{cases} \tilde{u}_y(x) < u_0(x) - d \\ \sigma_{yy}(x) = 0 \\ \sigma_{xy}(x) = 0 \end{cases}$$

try to invert $u_0(x) - d$

Johnson & Barber provided some approaches

$$\frac{du_0(x)}{dx} = \int_{-c}^{c} P_y(x') \frac{\kappa+1}{4\pi\mu} \frac{1}{x-x'} \, dx'$$

does not have abs. val.

$$= \frac{\kappa+1}{4\pi\mu} \int_{-c}^{c} \frac{P_y(x')}{x-x'} \, dx'$$

$$\cdots (-c < x < c) \qquad (*)$$

Eqn. (*) is a general relationship between 2 functions.

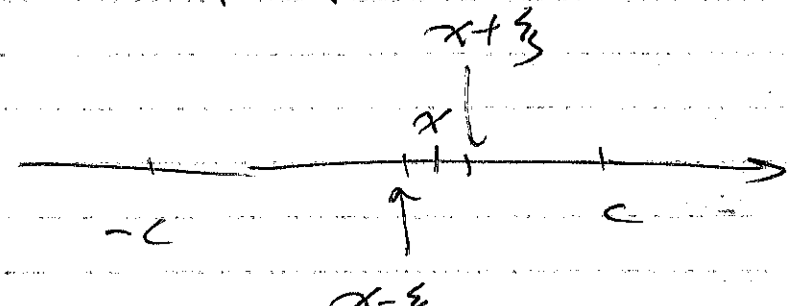
$$\boxed{g(x) = \int_{-c}^{c} \frac{f(x')}{x-x'}\,dx'}$$

integral eqn.

"the kernel is singular" ⟵ (implicit).

$\Downarrow$

the principle value



$$\int_{-c}^{c} \rightarrow \int_{-c}^{x-\xi} + \int_{x+\xi}^{c}$$

Introducing: ⟵ Hilbert transform

$$g(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f(x')}{x-x'}\,dx'$$

"Hilbert transform is its own inverse".

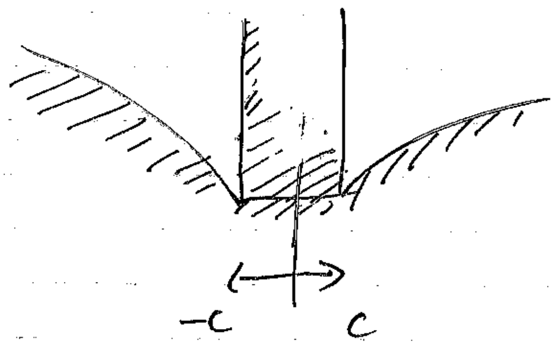
— the answer is

$$f(x) = -\frac{1}{\pi^2\sqrt{c^2-x^2}}\int_{-c}^{c}\frac{\sqrt{c^2-x'^2}\,g(x')}{x-x'}\,dx' + \frac{F}{\pi\sqrt{c^2-x^2}}$$

indenter force.

Example 1

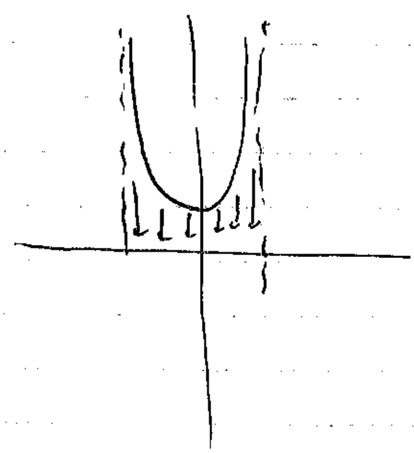
Flat punch



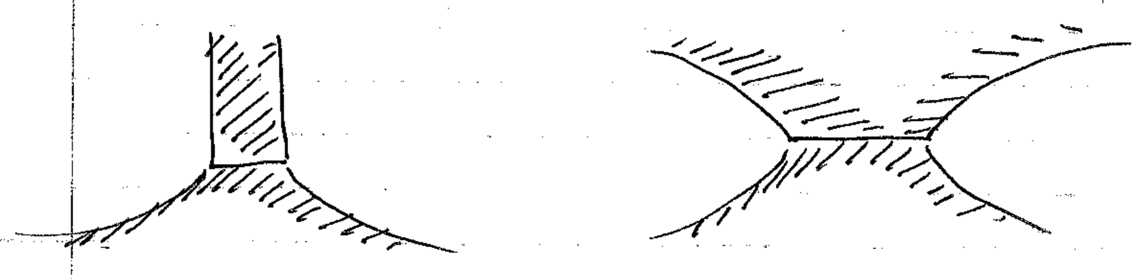
$x = c - r$, $r \to 0$

$\sigma \sim \frac{1}{\sqrt{r}}$

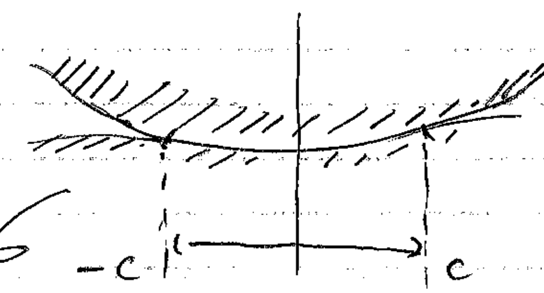
$$P_y(x) = \frac{F}{\pi\sqrt{c^2-x^2}}$$ plot it



other similar examples



Example 2 Cylindrical Punch



$u_0(x) = \frac{x^2}{2R}$

to sharp corner ?? no singularities (theoretically)

$$\frac{du_0(x)}{dx} = \frac{x}{R} \qquad p_y(x) = \frac{4\mu}{(\kappa+1)R}\left(\sqrt{c^2 - x^2} - \frac{c^2}{2\sqrt{c^2-x^2}} \right) + \frac{F}{\pi\sqrt{c^2-x^2}}$$

has to cancel to avoid singularities

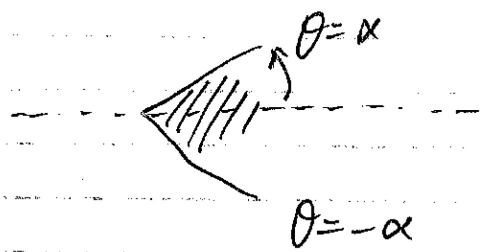
Q: what is C? How does c depend on F?

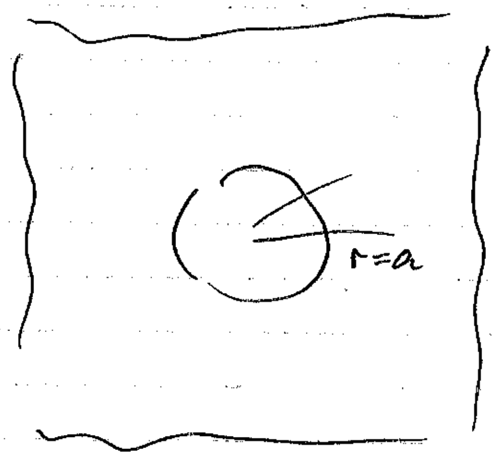
Stress singularities?

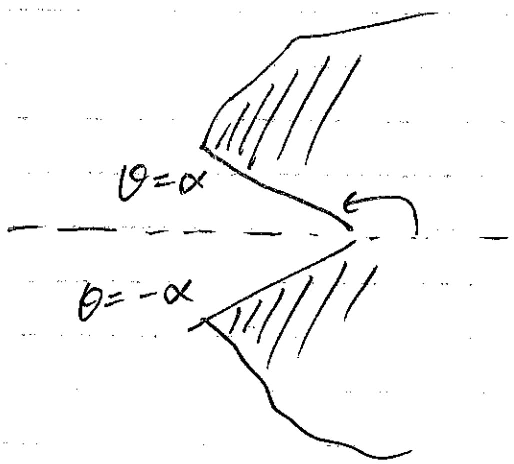
$$\frac{F}{\pi} = \frac{2\mu c^2}{(\kappa+1)R}$$

Lecture 10. 5/1/2024.

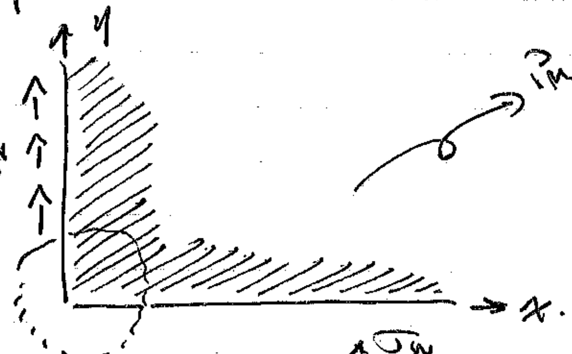
Wedge and Notch.







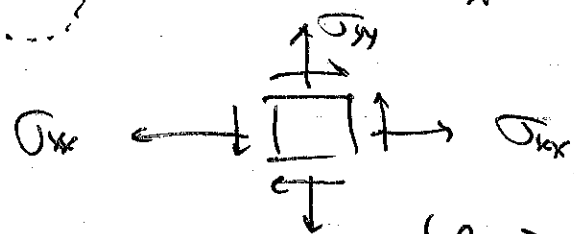
Example 1



Imagine an infinite large block.

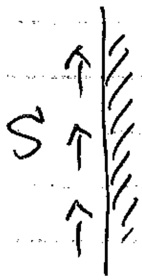
↓

look at area of interest, i.e., corner



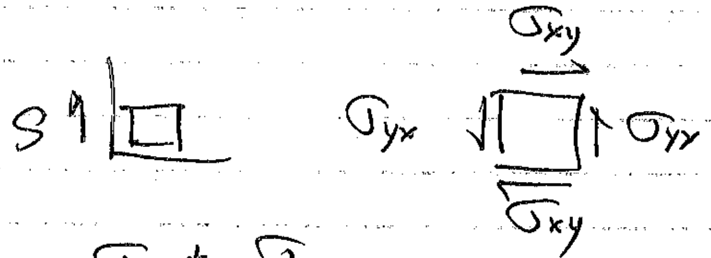
↳ traction free B.C.s ...

$\sigma_{xy} = \sigma_{yy} = 0 , \quad y=0$

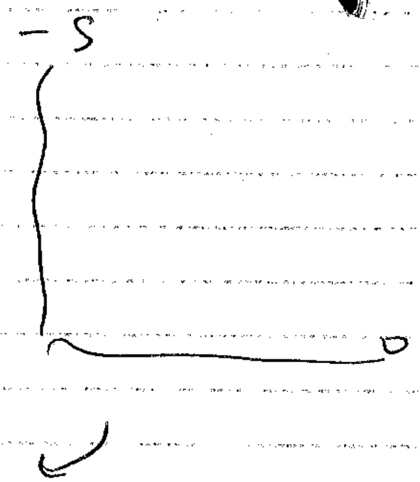


B.C.s.

$$\left.\begin{aligned}\sigma_{xx} &= 0\\ \sigma_{xy} &= -S.\end{aligned}\right\} x=0$$

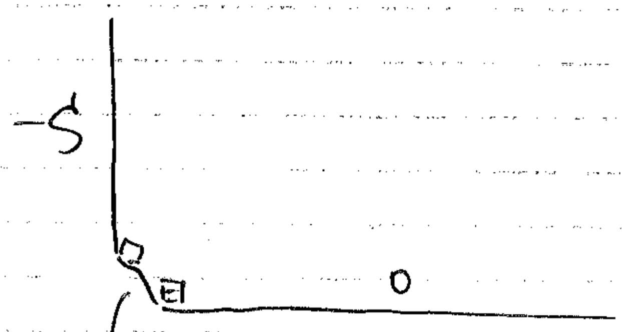


$\sigma_{xy} \neq \sigma_{yx}$



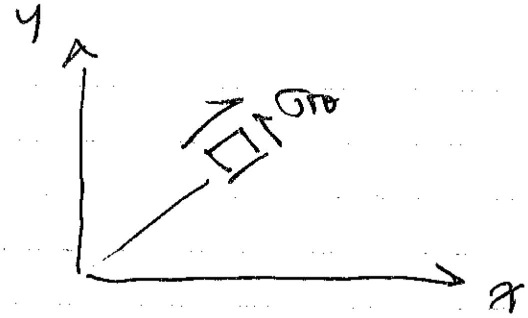
$$\sigma_{xy} = \sigma_{yx} = -\phi_{,xy}.$$

we will have.



dig a corner to avoid singularity.

$$\nabla^4 \phi(r,\theta) = 0.$$



$$\phi(r,\theta) = r^m e^{in\theta}$$

$$m = n,\ -n,\ 2+n,\ 2-n.$$

$$\sigma_{r\theta} \sim r^0. \quad \phi \sim r^2.$$

$$m = 2, \qquad n = 2, 0$$

Barber. Tab. 8.1.

we create it manually.

$$\phi = r^2,\ r^2\cos 2\theta,\ r^2\sin 2\theta.\ r^2\theta$$

σ_{rr}, $\sigma_{r\theta}$, $\sigma_{\theta\theta}$

satisfies biharmonic eqn.

find corresponding terms in table.

for $r^2\theta \rightarrow$

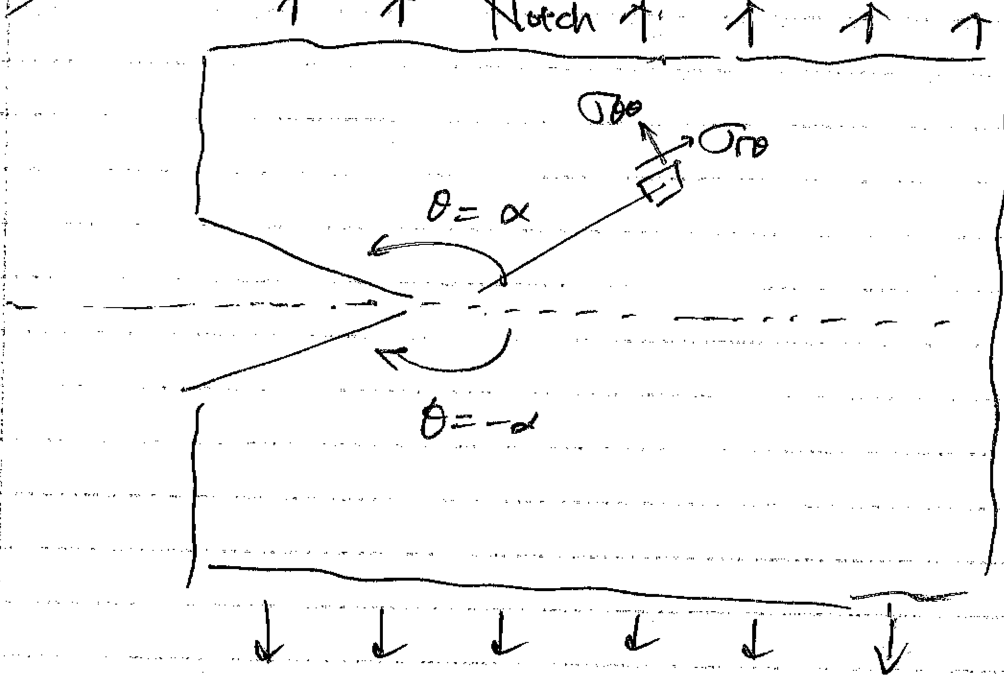
2θ	-1	2θ
(σ_{rr})	$(\sigma_{r\theta})$	$(\sigma_{\theta\theta})$

$$\phi = S\left(-\frac{\pi r^2 \cos 2\theta}{8} + \frac{\pi r^2}{8} + \frac{r^2 \sin 2\theta}{4} - \frac{r^2\theta}{2}\right)$$

$$\rightarrow \quad \phi(x,y) \rightarrow \sigma_{xy} = \sigma_{yx} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

$$= -\frac{Sy^2}{x^2+y^2}$$

Notch problem



B.C.s

$\sigma_{r\theta} = \sigma_{\theta\theta}$

$\theta = \pm\alpha$

→ Williams Solution

$$\phi = r^{n+2}\{A_1 \cos(n+2)\theta + A_2 \cos n\theta + A_3 \sin(n+2)\theta + A_4 \sin n\theta\}$$

$n = \lambda - 1, \quad n+2 = \lambda + 1.$

$$\phi = r^{\lambda+1}\{ \quad \cdots \quad \}$$

$\sigma_{rr} = r^{\lambda-1} \{ \cdots$

$\sigma_{r\theta} = r^{\lambda-1} \{ \cdots$

$\sigma_{\theta\theta} = r^{\lambda-1} \{ \cdots$

if $\lambda < 1$, Stress field is Singular

$\sigma \sim A \cdot r^{\lambda-1}$

substitute $\begin{cases} \theta = \alpha \\ \theta = -\alpha \end{cases}$

$\sigma_{r\theta} = 0$. $\theta = \alpha$ $\theta = -\alpha$.

$\sigma_{\theta\theta} = 0$ $\theta = \alpha$, $\theta = -\alpha$

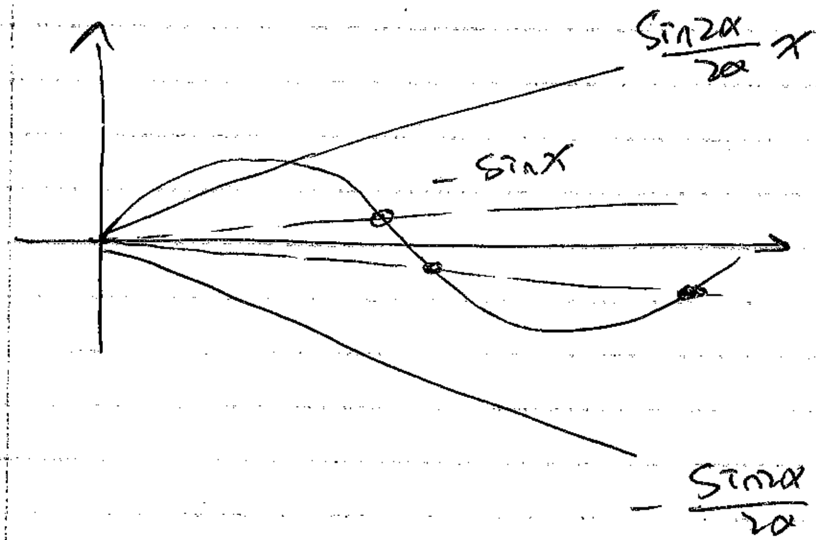
$$\begin{bmatrix} \cdot & M_1 & 0 & 0 \\ \cdot & \cdot & 0 & 0 \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & M_2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[M_1] \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad [M_2] \begin{bmatrix} A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In order to have non-trivial soln.

then, $\det(M_1) = 0 \Rightarrow \lambda \sin 2\alpha + \sin 2\lambda\alpha = 0$

or $\det(M_2) = 0 \Rightarrow \lambda \sin 2\alpha + \sin 2\lambda\alpha = 0$



$x = 2\lambda\alpha \rightarrow \lambda = \frac{x}{2\alpha}$

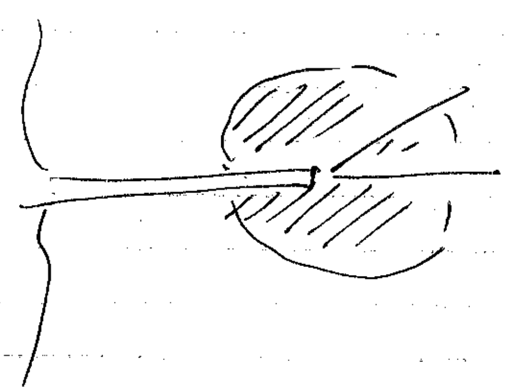
$$\boxed{\frac{\sin 2\alpha}{2\alpha} x \pm \sin x = 0}$$

$\alpha \rightarrow \pi$: $\lambda = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

$\sigma \sim \frac{1}{r}, \frac{1}{\sqrt{r}}, \ldots$ non-singular

$\checkmark$

reject this sol'n term.

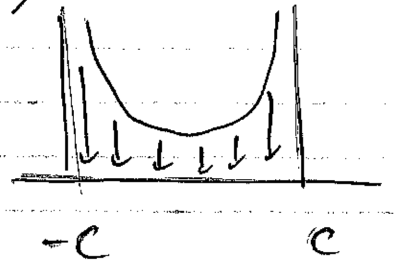


$\sigma \sim \frac{1}{\sqrt{r}}$, $\varepsilon \sim \frac{1}{\sqrt{r}}$

$w^e = \frac{1}{2}\sigma\varepsilon \sim \frac{1}{r}$

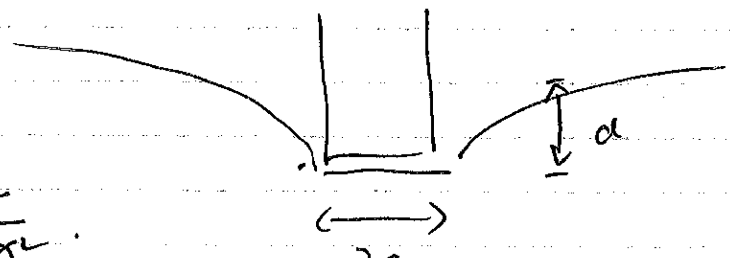
$\Sigma = \int w r dr = \int \frac{1}{r} \cdot r dr$ ↳ finite

Problem Session #5



$u_0(x) = 0$

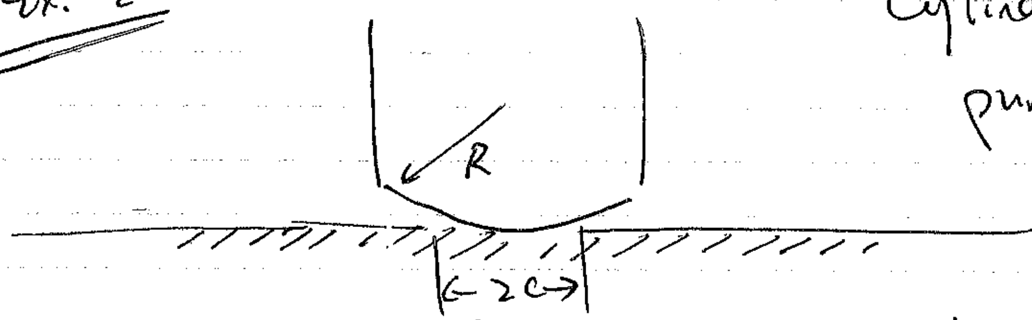
$g(x) = 0$



$$P_y(x) = \frac{F}{\pi\sqrt{c^2 - x^2}}.$$

Ex. 2

Cylindrical punch



$d << R.$ $\quad u_0(x) = \frac{x^2}{2R}.$ (parabolic)

$$g(x) = \frac{4\pi\mu}{k+1}\left(\frac{x}{R}\right) \hookrightarrow \frac{du_0(x)}{dx}$$

$$P_y(x) = \frac{-1}{\pi\sqrt{c^2 - x^2}} \, \text{P.V.} \int_{-c}^{c} \frac{\sqrt{c^2 - x'^2}}{x - x'} \frac{4\pi\mu}{k+1}\left(\frac{x'}{R}\right) dx' + \frac{F}{\pi\sqrt{c^2 - x^2}}$$

$$\text{P.V.} \int_{-1}^{1} \frac{\sqrt{1-t^2}}{x-t}\, t^n \, dt = \pi \left[x^{n+1} - \frac{x^{n-1}}{2} \cdots \right]$$

$n=1$. $\quad I_1 = \pi\left(x^2 - \frac{1}{2}\right)$. $\quad I_0 = \pi(x)$

$t \to x'/c$ $\qquad$ transform

Applying change of variables.

$$\int_{-1}^{1} \frac{\sqrt{-1-t^2}}{\left(\frac{x}{c} - t\right)}\, ctc\, dt = c^2 \pi \left(\left(\frac{x}{c}\right)^2 - \frac{1}{2} \right).$$

$$\pi\left(x^2 - \frac{c^2}{2}\right) = \pi(x^2 - c^2) + \frac{\pi c^2}{2}$$

Final Expression:

$$P_y(x) = \frac{4\mu}{(K+1)R} \sqrt{c^2-x^2} + \left(\frac{F}{\pi} \; \frac{-2\mu c^2}{(K+1)R} \right) \frac{1}{\sqrt{c^2-x^2}}$$

$$f(x) = \frac{-1}{\pi^2 \sqrt{c^2-x^2}} \text{P.V.} \int_{-c}^{c} \frac{\sqrt{c^2-x^2}}{x-x'} g(x)\, dx' + \frac{F}{\pi \sqrt{c^2-x^2}}$$

the second term has a singularity.

... we want to vanish the $\left(\frac{F}{\pi} - \frac{2\mu c^2}{(K+1)R}\right)\frac{1}{\sqrt{c^2-x^2}}$

we set it to zero.

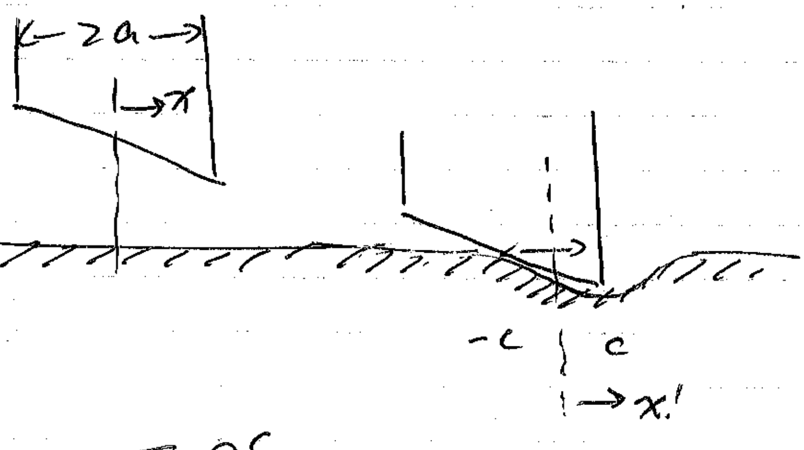
$$F = \frac{2\pi \mu c^2}{(K+1)R}$$

total indentation force

$$F = \int_{-c}^{c} P_y(x)\, dx.$$

$$\hookrightarrow P_y(x) = \frac{4\mu}{(K+1)R}\sqrt{c^2-x^2} \qquad \frac{F}{\pi} - \frac{2\mu c^2}{(K+1)R} = 0$$

<u>Ex. 3</u>



$u_0(x) = \beta(a-x)$

$$\frac{du_0(x)}{dx} = -\beta.$$

$$P_y(x) = \frac{-1}{\pi^2\sqrt{c^2-x^2}}\, \text{P.V.}\left[\int_{-c}^{c} \cdots \frac{4\pi\mu}{K+1}(-\beta)\, dx' + \cdots\right.$$

transformed coordinate.

$\tilde{x} = x - (a-c)$

Subs. $\tilde{x}$ into the eqn.

$$P_y(x) = \frac{-1}{\pi^2\sqrt{c^2-\tilde{x}^2}} \, P.V. \left[\int_{-c}^{c} \cdots \frac{4\pi\mu}{k+1}(-\beta)\, d\tilde{x}' + \cdots \right]$$

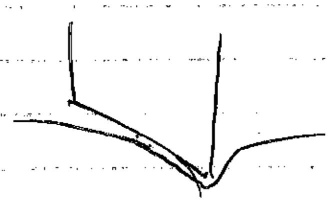
$\tilde{x}' = t^0$

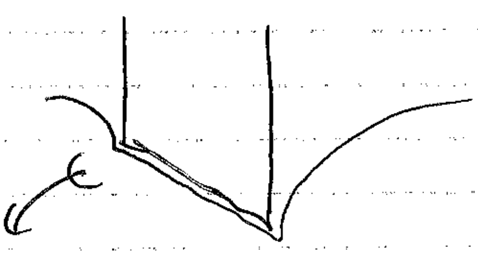
$$I_0 = c\,\frac{\pi x}{c} \qquad \pi(x) \leftarrow t^0$$

$$= \frac{4\pi\mu\beta}{\pi^2\sqrt{c^2-\tilde{x}^2}}\,\frac{c\pi\, x_n}{c} + \frac{F}{\pi\sqrt{c^2-x_n^2}} \qquad t = \frac{x_n'}{c}$$

We know $P_y(-c) = 0$

$$\frac{F}{\pi} - \frac{4\mu\beta c}{(k+1)} = 0 \rightarrow F = \frac{4\pi\mu\beta c}{(k+1)}$$

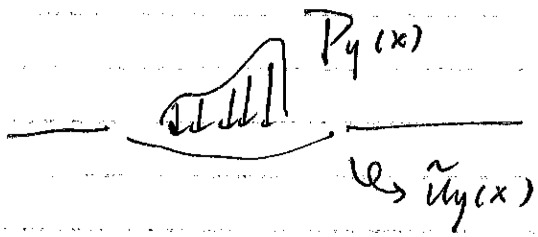




$c \le a$ critical point.

$$F_{\text{বোঁধল}} = \frac{4\pi\mu\beta a}{(k+1)}.$$

Review notes for contact problems.



surface displacement.

$$\tilde{u}_x(x) = \int_{-\infty}^{+\infty} -P_y(x) \cdot G^{s}_{xy}(x-x')\, dx'$$

$$\tilde{u}_y(x) = \int_{-\infty}^{+\infty} -P_y(x) \cdot G^{s}_{yy}(x-x')\, dx' \qquad (*)$$

compressive

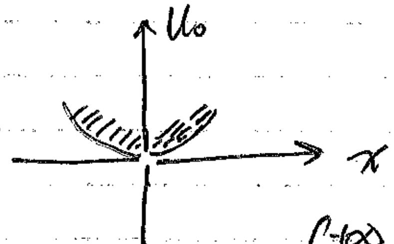
Green's function for 2D plane strain.

$$G^{s}_{xy}(x) = \frac{\kappa-1}{8\mu}\, \mathrm{sgn}(x).$$

$$G^{s}_{yy}(x) = -\frac{\kappa+1}{4\pi\mu} \log|x|.$$

integrating Green's function ...

frictionless contact. $T_x(x) = 0$



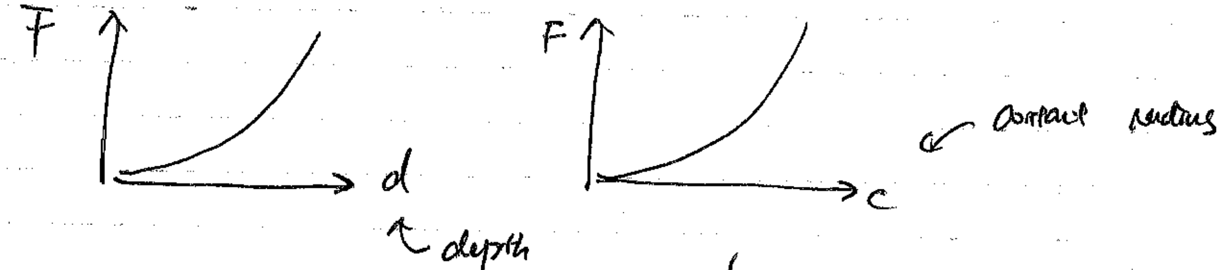
Eqn. $(*)$ gives us:

$$u_0(x) - d = \int_{-\infty}^{+\infty} P_y(x') \cdot \frac{\kappa+1}{4\pi\mu} \log|x-x'|\, dx'$$

$(**)$

total indenting fore is the integral of the load:

$$F = \int_{-c}^{c} P_y(x)\, dx$$



we want to know this.

#IMPORTANT: in $-c \sim c$ area:

$$\begin{cases} u_y(x) = u_0(x) - d & \text{indenter shape} \\ \sigma_{yy}(x) \leq 0 & \text{compressive} \\ \sigma_{xy}(x) = 0 & \text{frictionless.} \end{cases}$$

outside contact region.

$$\begin{cases} u_y(x) < u_0(x) - d & \text{no overlap.} \\ \sigma_{yy}(x) = 0 & \\ \sigma_{xy}(x) = 0 & \end{cases} \Big\} \text{ traction free.}$$

Direct inversion of integral eqn. (differentiating (**)).

$$\frac{du_0(x)}{dx} = \frac{\kappa+1}{4\pi\mu} \cdot \int_{-c}^{c} \frac{P_y(x')}{x - x'}\, dx'$$

of the form: $g(x) = \int_{-c}^{c} \frac{f(x')}{x - x'}\, dx'$

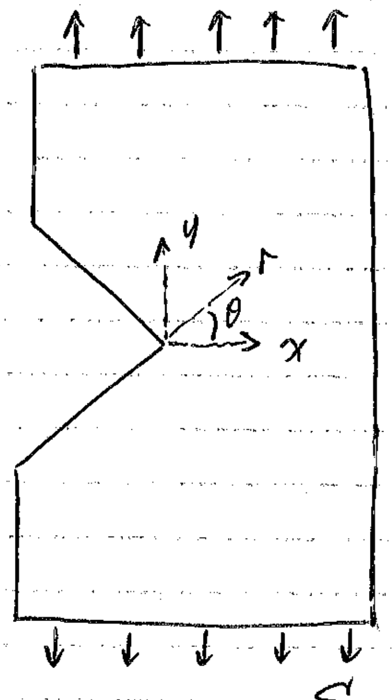
General sol'n to the integral equation

indenting force

$$f(x) = -\frac{1}{\pi^2\sqrt{c^2-x^2}}\int_{-c}^{c}\frac{\sqrt{c^2-x'^2}\cdot g(x')}{x-x'}\,dx' + \frac{F}{\pi\sqrt{c^2-x^2}}$$

Since $\int_{-c}^{c}\frac{f(x')}{x-x'}dx'$ is singular, we need to interpret its singular values, i.e., P.V.

Practice midterm.



σ_{yy}

Without notch.

$$\underline{\underline{\sigma}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow 2D$$

from $\sigma_{yy} = S$, $\rightarrow \phi = \frac{1}{2}Sx^2$

$\phi_{,xx}$

$\phi(r,\theta) = \frac{1}{2}Sr^2\cos^2\theta$

$$\phi = \frac{1}{2}Sr^2\left(\frac{1}{2}+\cos 2\theta\right)$$

$n=0$ $\quad$ $n=2$

$$= \frac{1}{4}Sr^2 + \frac{1}{2}Sr^2\cos 2\theta$$

IMPORTANT RELATIONSHIPS.

$$\cos^2\theta = \frac{1+2\cos 2\theta}{2}$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin^2\theta = \frac{1-2\cos 2\theta}{2}$$

$\sigma_{rr} = S'$ $\quad \sigma_{\theta\theta} = S'$

uniform stress field.

$\left[-\frac{1}{2}Sr^2\right]\cos 2\theta$

$\frac{1}{4}Sr^2 + \frac{1}{2}Sr^2\cos 2\theta + \ldots$

uniform.

resultant stress. $-2\cos 2\theta \cdot (\sigma_{rr})$
$+2\cdot\cos 2\theta \quad (\sigma_{\theta\theta})$
$+2\sin 2\theta \quad (\sigma_{r\theta})$

we need to create $+2\cos 2\theta\,(\sigma_{rr})$
$-2\cos 2\theta\,(\sigma_{\theta\theta})$
$-2\cos 2\theta\,(\sigma_{r\theta})$

can be applied $\Leftarrow$

to cancel the angle variation & terms in stress.

@ $\theta = \pm 135°$: apply shear $\sigma_{r\theta} = -2\cos 2\theta$

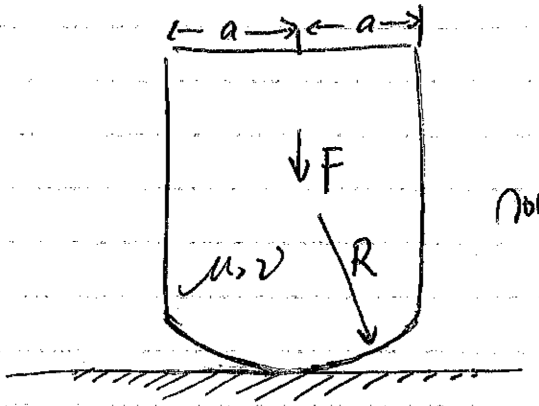
$\phi = \frac{1}{2}Sx^2 + Ay + B$

$\phi' = \frac{1}{2}Sx^2 + \frac{1}{2}Sy^2 - Sxy$

$\rightarrow$ goal: $\sigma_{xy} = S \quad \sigma_{yy} = S$

$\frac{1}{2}Sr^2\cos^2\theta + \frac{1}{2}Sr^2\sin^2\theta - \frac{1}{2}S\sin 2\theta$

5 #2



normal load dist. $P_y(x)$.

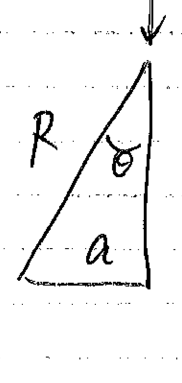
elastic half space

$$P_y(x) = \frac{2F}{\pi c^2}\sqrt{c^2 - x^2} \quad \text{where } c = \sqrt{\frac{2F(1-\nu)R}{\pi\mu}}$$

non-truncated cylindrical punch

$$f(x) = -\frac{1}{\pi^2\sqrt{c^2 - x^2}}\int_{-c}^{c}\frac{\sqrt{c^2-\xi^2}\, g(\xi)}{x-\xi}\, d\xi$$

$$+ \frac{F}{\pi\sqrt{c^2 - x^2}}$$

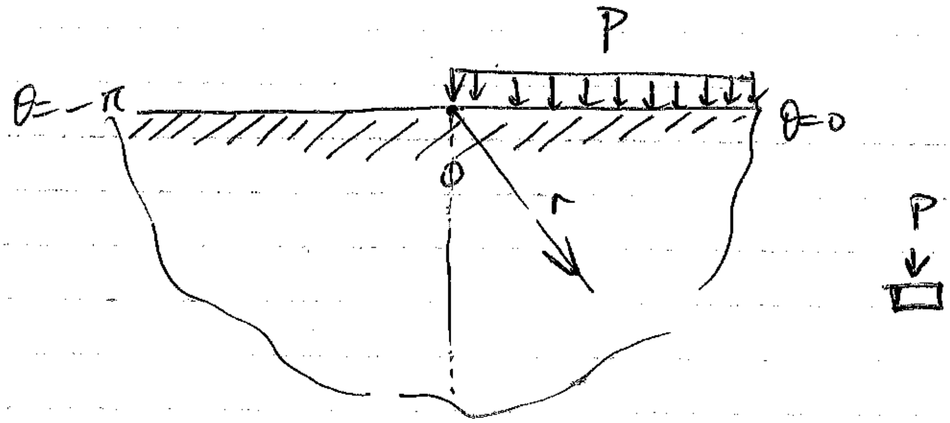


$$\sin\theta = \frac{a}{R}$$

$$\theta = \arcsin\left(\frac{a}{R}\right)$$

$$P_y(x) = \begin{cases} \frac{2F}{\pi c^2}\sqrt{c^2 - x^2} & \theta < \arcsin\left(\frac{a}{R}\right) \\ \frac{2F}{\pi c^2}\sqrt{c^2-x^2} + \frac{F}{\pi\sqrt{c^2 - x^2}} & \theta > \arcsin\left(\frac{a}{R}\right) \end{cases}$$

Pb #3



(a) $\begin{cases} \sigma_{\theta\theta} = -P \\ \sigma_{r\theta} = 0 \end{cases}$ @ $\theta = 0$

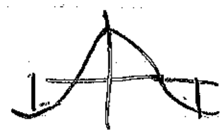
$\begin{cases} \sigma_{\theta\theta} = 0 \\ \sigma_{r\theta} = 0 \end{cases}$ @ $\theta = -\pi$

(b) From the problem, we know that $\sigma_{\theta\theta}$ has θ dependence.

@ $\theta = 0 \rightarrow \sigma_{r\theta} = 0$

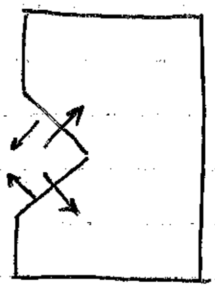
@ $\theta = -\pi \rightarrow \begin{matrix} \sigma_{\theta\theta} = 0 \\ \sigma_{r\theta} = 0 \end{matrix}$ $\Rightarrow$ 2θ term gone.

$\phi = A r^2 + B r^2 \cos 2\theta$



$\sigma_{\theta\theta} = 2A + B\, 2\cos 2\theta$ $\nearrow$ $\begin{matrix} 2A + 2B = P \\ 2A - 2B = 0 \end{matrix}$

$\sigma_{r\theta} = B\, 2\sin 2\theta$



$$\begin{cases} \underline{n}_1 = \left[-\frac{\sqrt{2}}{2} \quad -\frac{\sqrt{2}}{2}\right]. \\ \underline{n}_2 = \left[-\frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2}\right] \end{cases}$$

$$\underline{\underline{\sigma}} = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}$$

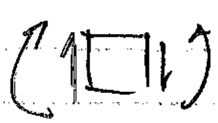
rotation matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

transform from Cartesian to polar coord.

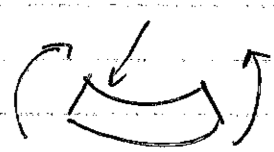
Notes on beam theory.

Euler-Bernoulli.

$$\frac{d}{dx} V(x) = -q(x)$$



$$\frac{d}{dx} M(x) = V(x).$$



$$K(x) = \frac{M(x)}{EI}$$

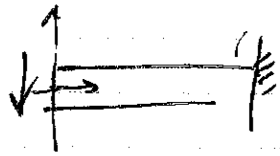
Solving beam problem using Airy stress function.

Strong B.Cs on top & bottom surfaces.

$$\begin{cases} \sigma_{xy} = 0. \\ \sigma_{yy} = 0 \end{cases} \quad y = \pm b.$$

On two edges, apply the weak B.C.s

$$\int_{-b}^{b} \sigma_{xy}\, dy = F$$



→ collection of weak B.C.s.

$$\left.\begin{aligned} &\int_{-b}^{b} \sigma_{xy}\, dy = F, \\ &\int_{-b}^{b} \sigma_{xx}\, y\, dy = 0, \\ &\int_{-b}^{b} \sigma_{xx}\, dy = 0 \end{aligned}\right\} x=0 \qquad \left.\begin{aligned} &\int_{-b}^{b} \sigma_{xy}\, dy = F \\ &\int_{-b}^{b} \sigma_{xx}\, y\, dy = Fa. \\ &\int_{-b}^{b} \sigma_{xx}\, dy = 0 \end{aligned}\right\} x=a$$

left

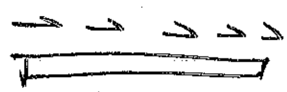
Polynomial stress function (analytic).

$$\phi = C_1 x y^3 \rightarrow \begin{cases} \sigma_{xx} = 6C_1 xy. \\ \sigma_{xy} = -3C_1 y^2 \\ \sigma_{yy} = 0 \end{cases}$$

General Solution Strategies for beam problem.

1. Determine the maximum order of polynomials.

normal loading $\sim x^n$.
shear $\sim x^{n+1}$ → moment $\sim x^{n+2}$
→ $\phi \sim x^{n+2} y^3$



loading $\sim x^m$.
shear x^m → moment x^{m+1}
→ $\phi \sim x^{m+1} y^3$

2. polynomial trial function.

$$\phi(x,y) = C_1 x^2 + C_2 xy + C_3 y^3 + \cdots$$

3. Impose compatibility condition $\nabla^4 \phi = 0$

4. Apply strong & weak B.C.s.

two constraints

5. Determine the constants.

Fourier expansion.

$$f(x) = \sum_{i=0}^{\infty} C_i \varphi_i(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \lambda_n x$$

$$+ b_n \sin \lambda_n x$$

Expansion coefficients.

$$\begin{cases} a_0 = \frac{1}{a} \int_{-a}^{a} f(x)\, dx. \\ a_n = \frac{1}{a} \int_{-a}^{a} f(x) \cos \lambda_n x\, dx. \\ b_n = \frac{1}{a} \int_{-a}^{a} f(x) \sin \lambda_n x\, dx \end{cases}$$

If even function:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{a} x.$$

Or $f(x) = \sum_{n=1}^{\infty} a_n \cos \lambda_n x$, $\lambda_n = \frac{(2n-1)\pi}{2a}$

Fourier transform.

$e^{ikx} \equiv \cos kx + i\sin kx$.

Expansion of $f(x)$ in terms of e^{ikx}.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k) e^{ikx}\, dk.$$

(Inverse Fourier transform)

↳ little confusing ?

Source: Image sampling & reconstruction.
funk @ princeton.edu, [CS426.].

- Fourier transform:

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi x}\, dx.$$

- Inverse Fourier transform:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{i2\pi k}\, dk$$

Fourier Solution (beam problem).

- how to construct stress function based on sym.

$$\phi(x,y) = \begin{cases} \cos\lambda x\,[A\cosh\lambda y + Dy\sinh\lambda y] & \text{even } x, y \\ \cos\lambda x\,[By\cosh\lambda y + C\sinh\lambda y] & \text{even } x \text{ odd } y. \\ \sin\lambda x\,[A\cosh\lambda y + Dy\sinh\lambda y] & \text{odd } x \text{ even } y \\ \sin\lambda x\,[By\cosh\lambda y + C\sinh\lambda y] & \text{odd } x \text{ odd } y \end{cases}$$

IMPORIANT.

▷ Generalized Hookes law for 2D.

$$\varepsilon_{xx} = \frac{K+1}{8\mu}\sigma_{xx} - \frac{3-K}{8\mu}\sigma_{yy}.$$

$$\varepsilon_{yy} = -\frac{3-K}{8\mu}\sigma_{xx} + \frac{K+1}{8\mu}\sigma_{yy}$$

$$\varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy}.$$

plane strain: $K = 3 - 4\nu$

plane stress: $K = \dfrac{3-\nu}{1+\nu}$.

▷ Weak B.C.s applied at the beam end.

$$\begin{cases} u_x = 0 \\ u_y = 0 \\ \dfrac{\partial u_y}{\partial x} = 0 \end{cases} \longleftarrow \text{@ beam end (Subs. pos.)}$$

$$\begin{cases} u_x = 0 \\ u_y = 0 \\ \dfrac{\partial u_x}{\partial y} = 0 \end{cases} \quad \text{or} \quad \begin{cases} \displaystyle\int_{-b}^{b} u_x \, dy = 0 \\ \displaystyle\int_{-b}^{b} u_y \, dy = 0 \\ \displaystyle\int_{-b}^{b} u_x y \, dy = 0 \end{cases}$$

~ Additional Notes for Contact.

$$g(x) = \frac{4\pi\mu}{(K+1)} \frac{du_0(x)}{dx}. \qquad f(x) = P_y(x)$$

$$f(x) = -\frac{1}{\pi^2\sqrt{c^2-x^2}} \int_{-c}^{c} \frac{\sqrt{c^2-x'^2}\, g(x')}{x-x'}\, dx'$$

$$+ \frac{F}{\pi\sqrt{c^2-x^2}}$$

$$f(x') = -\frac{4\mu}{(K+1)\sqrt{c^2-x'^2}} \sum_{n=1}^{\infty} n a_n \cos n\theta$$

$$+ \frac{P_0 c}{2\sqrt{c^2-x'^2}}$$

Flat punch. $P_y = \dfrac{F}{\pi\sqrt{c^2-x^2}}$

$$\rightarrow P_y \sim \frac{F}{2\sqrt{2c}} \frac{1}{\sqrt{r}} \rightarrow \sigma_{yy} \propto \frac{1}{\sqrt{r}}$$

Cylindrical punch.

$$u_0(x) = \frac{x^2}{2R}, \quad \frac{du_0(x)}{dx} = \frac{x}{R}.$$

$$c = \sqrt{\frac{(K+1)R}{2\pi\mu} F} \rightarrow P_y(x) = \frac{2F}{\pi c^2}\sqrt{c^2-x^2}$$

~Polar coordinates.

$$\sigma_{rr} = \frac{1}{r} \cdot \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}$$

$$\sigma_{r\theta} = \frac{1}{r^2} \frac{\partial \phi}{\partial \theta} - \frac{1}{r} \cdot \frac{\partial^2 \phi}{\partial r \partial \theta}$$

$$= -\frac{\partial}{\partial r}\left(\frac{1}{r} \cdot \frac{\partial \phi}{\partial \theta}\right)$$

$$\sigma_{rr} = (\lambda + 2\mu) \varepsilon_{rr} + \lambda \varepsilon_{\theta\theta} + \lambda \varepsilon_{zz}.$$

$$\sigma_{\theta\theta} = \lambda \varepsilon_{rr} + (\lambda + 2\mu) \varepsilon_{\theta\theta} + \lambda \varepsilon_{zz}.$$

$$\sigma_{zz} = \lambda \varepsilon_{rr} + \lambda \varepsilon_{\theta\theta} + (\lambda + 2\mu) \varepsilon_{zz}$$

$$\sigma_{r\theta} = 2\mu \varepsilon_{r\theta}$$

$$\sigma_{\theta z} = 2\mu \varepsilon_{\theta z}.$$

$$\sigma_{rz} = 2\mu \varepsilon_{rz}$$

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right)$$

▷ $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ $\quad$ $\sin 2\theta = 2\sin\theta\cos\theta$

$\downarrow$

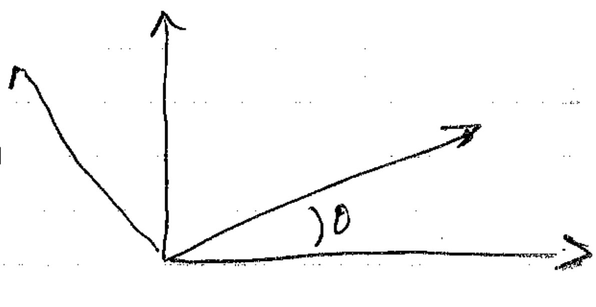
$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin^3\theta = \dfrac{3\sin\theta - \sin 3\theta}{4}$ $\quad$ $= 1 - 2\sin^2\theta$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ $\quad$ $= 2\cos^2\theta - 1$

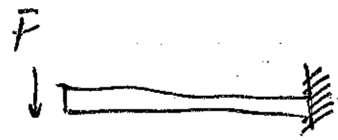
▷ rotation tensor

$$\underline{\underline{Q}} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\underline{\underline{\sigma}}' = \underline{\underline{Q}}\,\underline{\underline{\sigma}}\,\underline{\underline{Q}}^T$$

▷ Beam free end weak B.C.s.



$$\int_{-b}^{b} \sigma_{xy}\, dy = F \qquad \int_{-b}^{b} \sigma_{xx}\, dy = 0$$

$$\int_{-b}^{b} \sigma_{xx}\, y\, dy = 0$$

One integral / differentiation will change the odd / even-ess of the function in that specific direction.

HW example: $\varepsilon_{xx} \rightarrow$ even in x. even in y

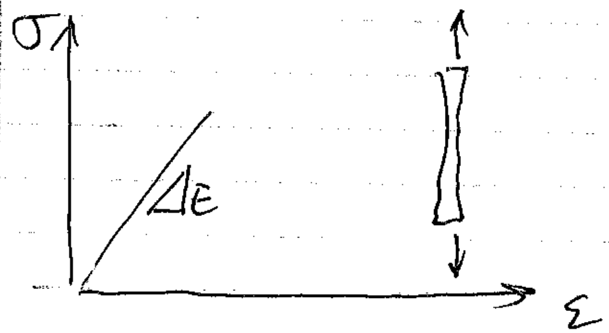
$$u_x = \int \varepsilon_{xx}\, dx$$ odd in x, even in y

Lecture 12 5/8/2014

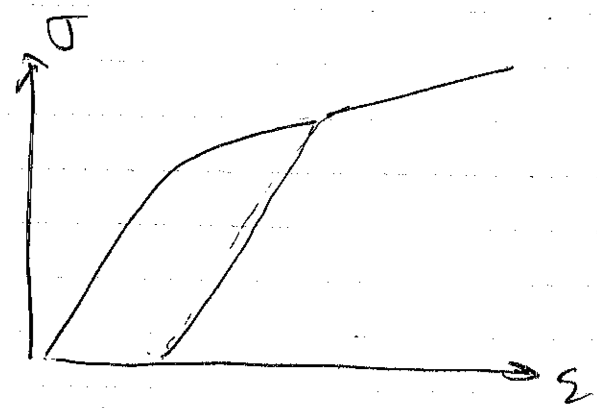
Inelasticity

Plasticity.

tensile test.

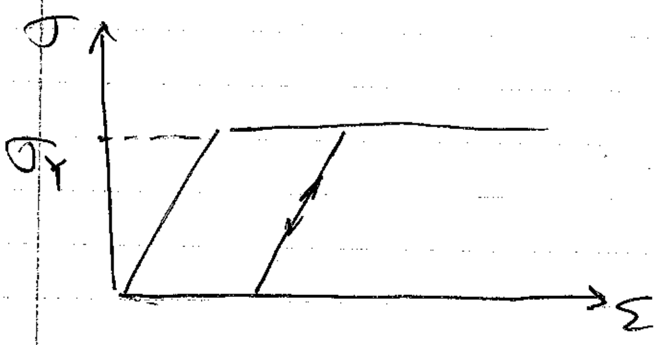


linear plasticity.



"real material"

Simplest model.



perfectly plasticity.

> Displacement. field.

$\underline{u} = \underline{x} - \underline{X} \qquad u_i(x_i)$

> Strain field.

$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

▷ Stress field, traction.

$$T_j = \sigma_{ij} n_i.$$

▷ Equilibrium condition.

$$\sigma_{ij,i} + F_j = 0$$

▷ Compatibility condition.

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$$

Strain decomp.,

$$\varepsilon_{ij} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{pl}.$$

↳ satisfy the compatibility.
(continuous body assumption).

Constitutive Equation.

~ Generalized Hooke's law.

$$\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}^{el}$$

isotropic elasticity.

$$\sigma_{ij} = \lambda \varepsilon_{kk}^{el} \delta_{ij} + 2\mu \varepsilon_{ij}^{el}.$$

where $\lambda = \frac{2\mu\nu}{1-2\nu}$

$$\sigma_{xx} = (\lambda + 2\mu)\varepsilon_{xx}^{el} + \lambda \varepsilon_{yy}^{el} + \lambda \varepsilon_{zz}^{el}$$

$$\sigma_{yy} = \lambda \varepsilon_{xx}^{el} + (\lambda + 2\mu)\varepsilon_{yy}^{el} + \lambda \varepsilon_{zz}^{el}$$

$$\sigma_{zz} = \lambda \varepsilon_{xx}^{el} + \lambda \varepsilon_{yy}^{el} + (\lambda + 2\mu)\varepsilon_{zz}^{el}.$$

$$\sigma_{xy} = 2\mu \varepsilon_{xy}^{el} \quad \sigma_{yz} = 2\mu \varepsilon_{yz}^{el}. \quad \sigma_{xz} = 2\mu \varepsilon_{xz}^{el}$$

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

$\bar{\sigma} = \frac{1}{3}\sigma_{ii}$ ⟵ Stress invariant.

i.e., hydrostatic Stress.

→ hydrostatic Strain. $\bar{\varepsilon} = \frac{1}{3}\varepsilon_{ii}$

$\boxed{\bar{\sigma} = 3K \, \bar{\varepsilon}^{el}}$ ... characterizes the vol. change.

↖ bulk modulus

deviatoric Stress

$$S_{ij} = \sigma_{ij} - \bar{\sigma}\delta_{ij} \rightsquigarrow \begin{bmatrix} \sigma_{xx}-\bar{\sigma} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\bar{\sigma} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\bar{\sigma} \end{bmatrix}$$

deviatoric Strain

$$e_{ij} = \varepsilon_{ij} - \bar{\varepsilon}\delta_{ij}$$

$$\boxed{\hookrightarrow S_{ij} = 2\mu\, e_{ij}^{el}}$$

shear modulus

Given the strain, one can decompose it

into the hydrostatic part & deviatoric part.

i.e., $\varepsilon_{ij}^{el} = \bar{\varepsilon}^{el}\delta_{ij} + e_{ij}^{el}$ ← shape change

mult bulk mod.

Volume change contr.

$$3K\bar{\varepsilon}^{el}\delta_{ij} + 2\mu e_{ij}^{el} = \sigma_{ij}$$

obtain total stress

▷ Yield condition & flow rule.

$$f(\{\sigma_{ij}\}) = 0$$

$f(\cdot)$ takes all the six stress components and $\to \mathbb{R}$.

$f<0$ in the surface,

$f>0$. outside the surface.

assumption: rate - independent . . . ①

isotropic . . ②.

coordinate transformation.

$$\sigma'_{ij} = Q_{ip} Q_{jq} \sigma_{pq}$$

$$f(\{\sigma'_{ij}\}) = 0$$

$\sigma'_{ij} \to$ invariants $\to$ phenomenology model.

Stress invariants

$[\sigma_{ij}] \to \sigma_1, \sigma_2, \sigma_3$ principal stresses.

$\downarrow$ not easy to compute

$I_1 = tr(\sigma_{ij})$

$= \sigma_1 + \sigma_2 + \sigma_3 = \sum_i \sigma_i$

$-I_2 = \frac{1}{2}(\sigma_{ii}\overset{\text{trace}}{\sigma_{jj}} - \sigma_{ij}\sigma_{ij}) = (\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}) - (\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2)$

$I_3 = det(\sigma_{ij}) = \prod_i \sigma_i$

$= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$

$f(\cdot) = f(I_1, I_2, I_3) = 0$

$\downarrow$ Bridgeman.

$f(I_2, I_3) \to f(J_2, J_3) = 0$

$\downarrow$

$f(J_2) = 0$

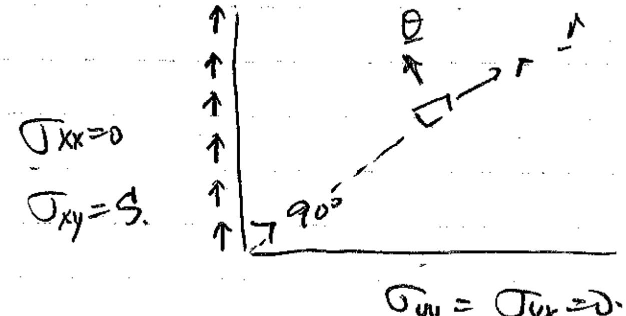
$J_1 = tr(S_{ij}) = 0$

$J_2 = \frac{1}{2} S_{ij} S_{ij} \to$ L2-norm.

$J_3 = det(S_{ij})$

Problem Session #6 5/10/2024.

Notches.



we use polar coord.

@ $\theta=0$, $\sigma_{\theta\theta}=\sigma_{r\theta}=0$.

@ $\theta=\pi/2$, $\sigma_{\theta\theta}=0$, $\sigma_{r\theta}=S$.

$\sigma_{r\theta} \propto r^0 \rightarrow \phi(r,\theta) \propto r^2$

$\phi = f(r) \cdot e^{in\theta}$

$= r^m e^{in\theta}$

$m=2$; $n=0, 2$

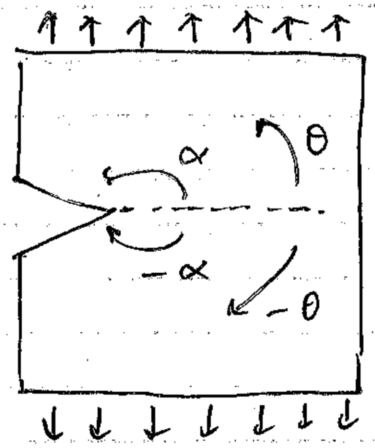
William soln : $\phi(r,\theta) = r^{n+2}\left[A_2\cos(n+2)\theta + A_2\cos(n\theta) + A_3\sin(n+1)\theta + A_4\sin(n\theta)\right]$

general soln for notch problem.

$\lambda=n-1$, $\phi(r,\theta) = r^{\lambda+1}\left[A_1\cos(\lambda+1)\theta + A_2\cos(\lambda-1)\theta + A_3\sin(\lambda+1)\theta + A_4\sin(\lambda-1)\theta\right]$

$$\sigma_{r\theta} = r^{\lambda-1}\Big[A_1 \lambda(\lambda+1)\sin(\lambda+1)\theta +$$
$$A_2\lambda(\lambda-1)\sin(\lambda-1)\theta - A_3\lambda(\lambda+1)\cos(\lambda+1)\theta$$
$$-A_4\lambda(\lambda-1)\cos(\lambda-1)\theta\Big]$$

Notch



$\sigma_{rr}, \sigma_{\theta\theta}$ same symmetry

(from σ_{xx} & σ_{yy})

B.C.s @ $\theta = \pm\alpha$, traction free.

$\sigma_{r\theta} = 0$, $\sigma_{\theta\theta} = 0$.

$\sigma_{r\theta} \rightarrow \theta = +\alpha, -\alpha$.

① $A_1(\lambda+1)\sin(\lambda+1)\alpha + A_2(\lambda-1)\sin(\lambda-1)\alpha$

$-A_3(\lambda+1)\cos(\lambda+1)\alpha - A_4(\lambda-1)\cos(\lambda-1)\alpha = 0$

② $-A_2(\lambda+1)\sin(\lambda+1)\alpha - A_2(\lambda-1)\sin(\lambda-1)\alpha$

$-A_3(\lambda+1)\cos(\lambda+1)\alpha - A_4(\lambda-1)\cos(\lambda-1)\alpha = 0$

Solve for Independent eqns for

$[A_1, A_2]$ & $[A_3, A_4]$.

$$\begin{bmatrix} M_{1\,(2\times2)} & 0 \\ 0 & M_{2\,(2\times2)} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

③ & ④ from $\sigma_{\theta\theta} = 0$.

we now want to solve:

$$M_1 \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0.$$

M_1 is singular.

$\det(M_1) = 0$

$$\begin{bmatrix} (\lambda+1)\sin(\lambda+1)\alpha & (\lambda-1)\sin(\lambda-1)\alpha \\ (\lambda+1)\cos(\lambda+1)\alpha & (\lambda+1)\cos(\lambda-1)\alpha \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

Solve this.

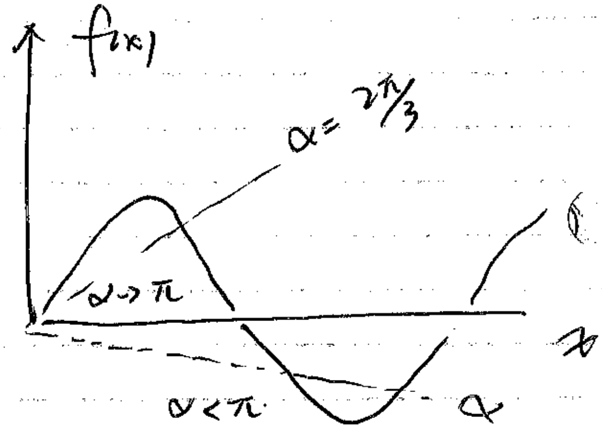
$$\det\left(\begin{bmatrix} (\lambda+1)\sin(\lambda+1)\alpha & (\lambda-1)\sin(\lambda-1)\alpha \\ (\lambda+1)\cos(\lambda+1)\alpha & (\lambda+1)\cos(\lambda-1)\alpha \end{bmatrix} \right) = 0$$

$A_1, A_2 \rightarrow \lambda \sin(2\alpha) + \sin(2\lambda\alpha) = 0$

$x = 2\lambda\alpha$

$-\frac{x}{2\alpha} \sin(2\alpha) = \sin(x)$

$A_3, A_4 \rightarrow \lambda \sin(2\alpha) - \sin(2\lambda\alpha) = 0$



if $\alpha = \frac{2\pi}{3}$,

slope of line

$= \frac{-\sin(2 \times \frac{2\pi}{3})}{4\pi/3}$

$\alpha > \pi$.

Solve for A_1, A_2

$x = 0, \pi, 2\pi, 3\pi, \dots$

$\lambda = 0, \frac{1}{2}, 1, \frac{3}{2},$

$$\begin{bmatrix} & & \\ & \ddots & \\ & & \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

non-singular

not possible

$A_1 = A(\lambda - 1)\sin(\lambda - 1)\alpha$

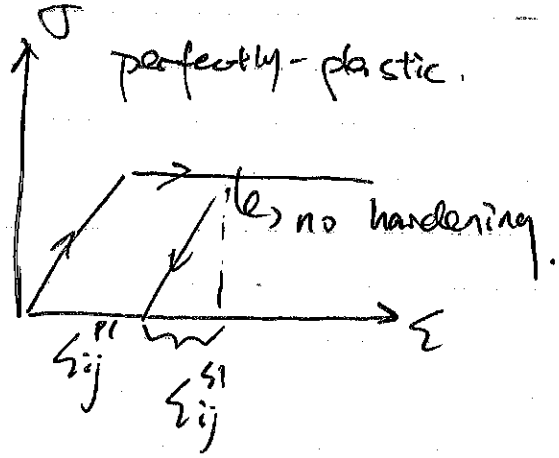
$\sigma \propto r^{\lambda - 1}$ $\{\lambda = \frac{1}{2}\}$

$A_2 = -A(\lambda + 1)\sin(\lambda + 1)\alpha$

Plasticity

Strain: $\varepsilon_{ij} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{pl}$.

Stress: $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}^{el}$



Compatibility cond'n.

→ total strain ε_{ij}.

1). hydrostatic stress. → $\bar{\sigma} = \sigma_{ii}/3$.

→ $\sigma_{xx} + \sigma_{yy} + \sigma_{zz}$.

2) Deviatoric Stress. $S_{ij} = \sigma_{ij} - \bar{\sigma}\delta_{ij}$.

hydrostatic stress + Deviatoric.

yield criteria: $f(\{\sigma_{ij}\}) = 0$.

... depends on stress invariants.

$$\begin{bmatrix} \bar{\sigma} & & \\ & \bar{\sigma} & \\ & & \bar{\sigma} \end{bmatrix} \rightarrow \text{not cause yield}$$

$\therefore f(J_2) = 0$

Stress invariants

$\longrightarrow$ Stress tensor $\rightarrow$ $\begin{cases} I_1 & \cdots \ \mathrm{tr}(\sigma_{ij}) \\ I_2 & \cdots -\frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}) \\ I_3 & \cdots \end{cases}$

$\sigma_{ij} n_i = T_j.$ eigen val. problem.

$\det(\sigma_{ij} - \lambda \underline{\underline{I}}) = 0$

$\lambda^3 - I_1\lambda^2 - I_2\lambda - I_3 = 0$

deviatoric : $\longrightarrow$ $\begin{cases} J_1 : & S_{11} + S_{22} + S_{33} = 0 \\ J_2 : & \frac{1}{2} S_{ij} S_{ij} = \frac{1}{2}(S_{11}^2 + S_{22}^2 + S_{33}^2) \\ J_3 : & S_1 S_2 S_3 \end{cases}$

$$\begin{bmatrix} S_1 & & \\ & S_2 & \\ & & S_3 \end{bmatrix}$$

Lecture 13. 5/13/2024.

plasticity : $\varepsilon_{ij} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{pl}$

generalized Hookes law: $\sigma_{ij} = \lambda \varepsilon_{kk}^{el} \delta_{ij} + 2\mu \varepsilon_{ij}^{el}$

$$\sigma_{ij} = \underbrace{\bar{\sigma}\,\delta_{ij}}_{\text{hydrostatic (spherical)}} + \underbrace{S_{ij}}_{\text{deviatoric}}$$

$\bar{\sigma} = \frac{1}{3}\sigma_{kk} = \frac{1}{3}\mathrm{tr}(\sigma_{ij})$

$\varepsilon_{ij} = \bar{\varepsilon}\,\delta_{ij} + e_{ij}$

$\bar{\varepsilon} = \frac{1}{3}\varepsilon_{kk} = \frac{1}{3}\mathrm{tr}(\varepsilon_{ij})$.

$\bar{\sigma} = 3K\bar{\varepsilon}$, $S_{ij} = 2\mu e_{ij}$

recap: elasticity part.

Yield condition. $\rightarrow f(\sigma_{ij}) = 0$

assumption : - isotropic. $\rightarrow f(\sigma_1, \sigma_2, \sigma_3) = 0$

principal stresses

(takes a lot of work to find them, avoid it...).

... Stress invariants:

$I_1 = \mathrm{tr}(\sigma_{ij}) = \sigma_1 + \sigma_2 + \sigma_3$

$I_2 = \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji})$

$I_3 = \det(\sigma_{ij}) = \sigma_1 \cdot \sigma_2 \sigma_3$

according to Bazdyenar.

Equivalently: $f(\cancel{I_1}, I_2, I_3) = 0$ (yield)

$f(S_{ij}) = 0$ ← transform I_1, I_2, I_3

↳ $f(\ J_2, J_3) = 0$ J_1, J_2, J_3

$J_1 = tr(S_{ij}) =$

$J_2 = \frac{1}{2} S_{ij} S_{ij}$ 2-norm of the deviatoric stress.

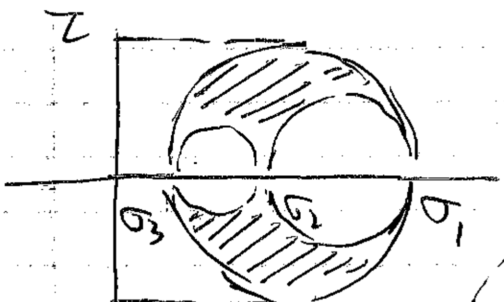
$J_3 = \cdots$

<u>Von Mises yield criteria</u>

$f(J_2) = J_2 - k^2 = 0$ ∴ $J_2 = k^2$.

∴ yield is satisfied when J_2 equals to some const.

<u>Tresca yield condition</u>

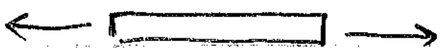


$\sigma_1 - \sigma_3 = 2k_T$. → different k from Von Mises.

↳ or the maximum difference between three principal stresses.

you can rewrite it in terms of f: $J(J_2, J_3) = 0$ (messy)

σ_Y: uniaxial tension.



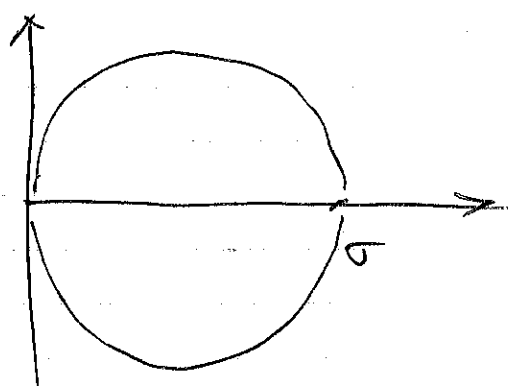
$$\sigma_{ij} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\bar{\sigma} = \frac{1}{3}\sigma \quad \rightarrow \quad S_{ij} = \begin{bmatrix} \frac{2}{3}\sigma & 0 & 0 \\ 0 & -\frac{1}{3}\sigma & 0 \\ 0 & 0 & -\frac{1}{3}\sigma \end{bmatrix}$$

$$J_2 = \frac{1}{2}\left(\frac{4}{9}\sigma^2 + \frac{1}{9}\sigma^2 + \frac{1}{9}\sigma^2\right) = \frac{1}{2}\cdot\frac{6}{9}\cdot\sigma^2 = \frac{1}{3}\sigma^2$$

$$\frac{1}{3}\sigma_Y^2 = k^2 \quad ; \quad k = \frac{\sigma_Y}{\sqrt{3}}$$

σ_Y: uniaxial tension



$$\sigma_1 - \sigma_3 = \sigma$$

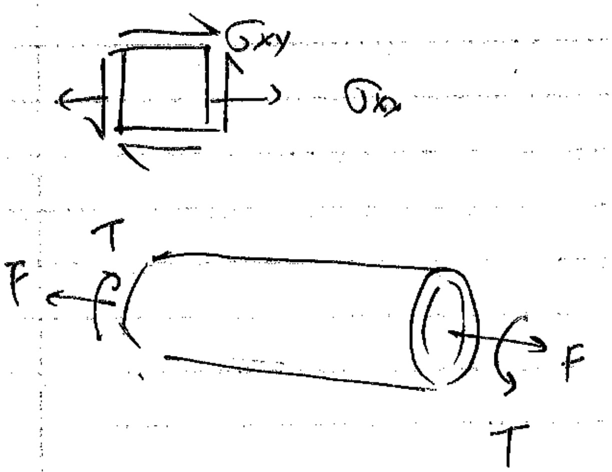
$$\sigma_Y = 2k_T$$

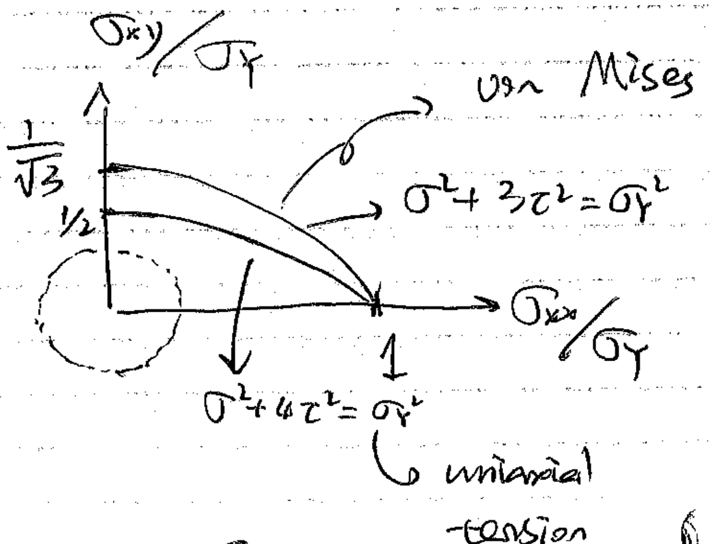
$$k_T = \frac{\sigma_Y}{2}$$

k & k_T are calibrated s.t. both vM & Tresca predicts the same yield under uniaxial tension

Taylor & Quinney (1931).

tension & shear.





"New" stress tensor $\longrightarrow$ $\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\bar{\sigma} = \frac{1}{3}\sigma_{xx}$$

$$S'_{ij} = \begin{bmatrix} \frac{2}{3}\sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & -\frac{1}{3}\sigma_{xx} & 0 \\ 0 & 0 & -\frac{1}{3}\sigma_{xx} \end{bmatrix}$$

Now, lets calc. J2 :

$$J_2 = \frac{1}{2}\left(\frac{6}{9}\sigma_{xx}^2 + 2\sigma_{xy}^2\right) = \frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2$$

Same as in pure tension

$$\frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2 = \frac{\sigma_Y^2}{3}.$$

pure shear: $\sigma_{xy}^2 = \dfrac{\sigma_Y^2}{3}.$

$\downarrow$ von Mises

$$\sigma_{xy} = \frac{\sigma_Y}{\sqrt{3}}.$$

flow rule.

Elastic perfectly-plastic material.

$J_2 = k^2$, $\dot{J}_2 = 0$. ← von Mises criterion

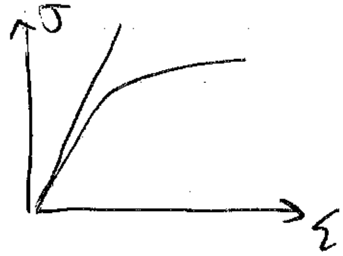
$J_2 = \frac{1}{2} S_{ij} S_{ij}$, $\dot{J}_2 = S_{ij}\dot{S}_{ij} = 0$

↑ realistic case.

$$\sigma_{ij} = \bar{\sigma}\delta_{ij} + S_{ij}.$$

$\bar{\sigma} = 3K\bar{\varepsilon}^{el}$ $\qquad S_{ij} = 2\mu \cdot e_{ij}^{el}$

$\dot{\varepsilon}_{ij}^{pl}$



↳ $\varepsilon_{ij}^{pl} = \int \dot{\varepsilon}_{ij}^{pl}(t)\,dt.$ ← the "accepted" theory

$\dot{\varepsilon}^{pl}_{ij} = \frac{\hat{\lambda}}{2\mu} S_{ij}$ ... Associated flow rule.

↑ looks like a fluid.

... sparks debate.

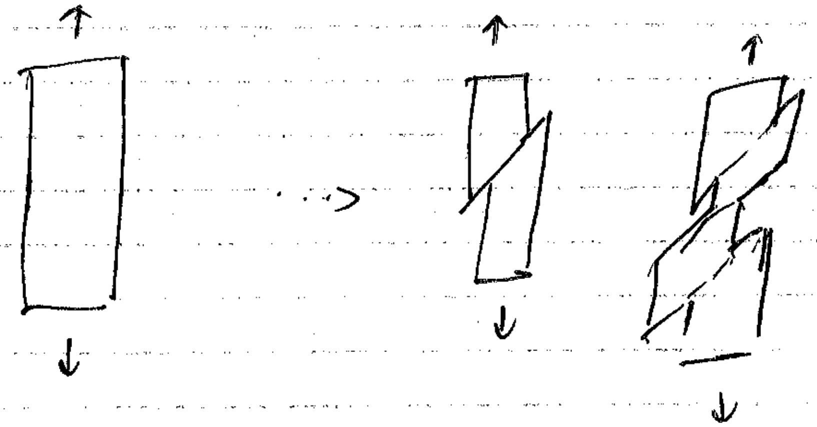
$e^{el}_{ij} = \frac{1}{2\mu} S_{ij}$

$tr\left[\dot{\varepsilon}^{pl}_{ij}\right] = 0 \quad tr\left[\varepsilon^{pl}_{ij}\right] = 0.$ ← from Bridgeman.

$$\boxed{\varepsilon^{pl}_{ij} = \cancel{\bar{\varepsilon}^{pl}\delta_{ij}} + e^{pl}_{ij}}$$

$\varepsilon^{el}_{ij} = \bar{\varepsilon}^{el}\delta_{ij} + e^{el}_{ij}.$ ← plastic strain has no volumetric part !!!

metal



(Shear)

therefore. $\dot{e}^{pl}_{ij} = \frac{\hat{\lambda}}{2\mu} S_{ij}$

Lecture 14. 5/15/2024.

Plasticity   yield criterion.

von Mises $J_2 - k^2 = 0$.

$J_2 = \frac{1}{2} S_{ij} S_{ij}$ $k = \frac{\sigma_Y}{\sqrt{3}}$

▷ flow rule → $\dot{\bar{\varepsilon}}^{pl}_{ij} = 0$ $3K \bar{\varepsilon}^{el} = \bar{\sigma}$

$$2\mu \dot{\varepsilon}^{pl}_{ij} = \tilde{\lambda} S_{ij} \qquad (2\mu e^{el}_{ij} = S_{ij})$$

$$\tilde{\lambda} = \frac{2\mu}{2k^2} \dot{W}$$

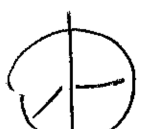
$$\dot{W} = S_{ij} \dot{e}_{ij} = S_{ij} (\dot{e}^{el}_{ij} + \dot{\varepsilon}^{pl}_{ij})$$

↳ total strain rate (deviatoric)
(because in the experiment we are impose the loading)

$$\ldots\ 2\mu \dot{W} = S_{ij}(2\mu \cdot \dot{e}^{el}_{ij} + 2\mu \cdot \dot{\varepsilon}^{pl}_{ij})$$

$$= S_{ij}(\dot{S}_{ij} + \tilde{\lambda} S_{ij}) \quad \ldots \text{(TBC)}$$

EPP: elastic perfectly-plastic if J_2 remains const.

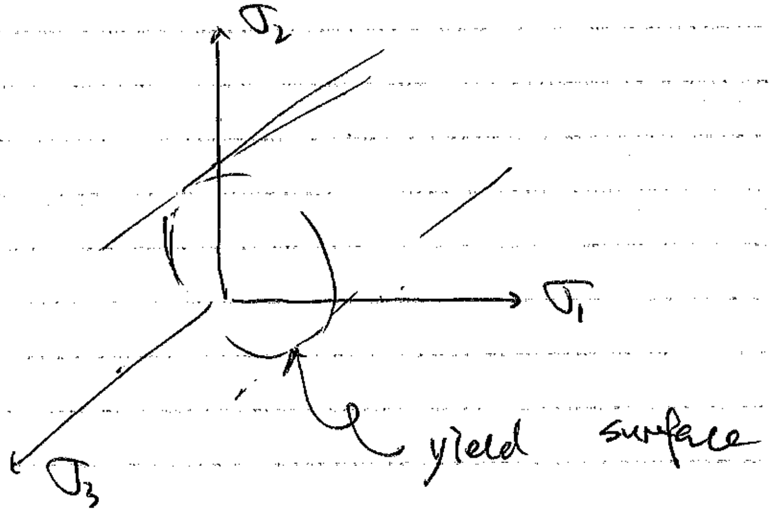
↳ for EPP.  $\dot{J}_2 = S_{ij} \dot{S}_{ij} = 0$

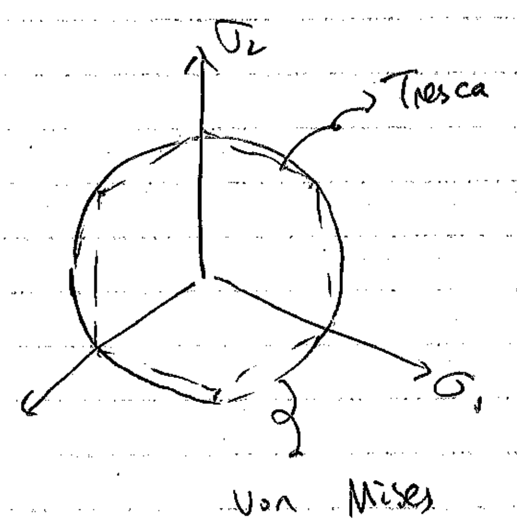
$\dot{r} \perp \Gamma$

$$\therefore \tilde{\lambda} S_{ij} S_{ij} = \tilde{\lambda}^2 2k^2$$

$$\dot{W}^{tot} = \bar{\sigma}\dot{\bar{\varepsilon}} + \dot{W}$$

if the material is isotropic, we can visualize it based on the principle stresses



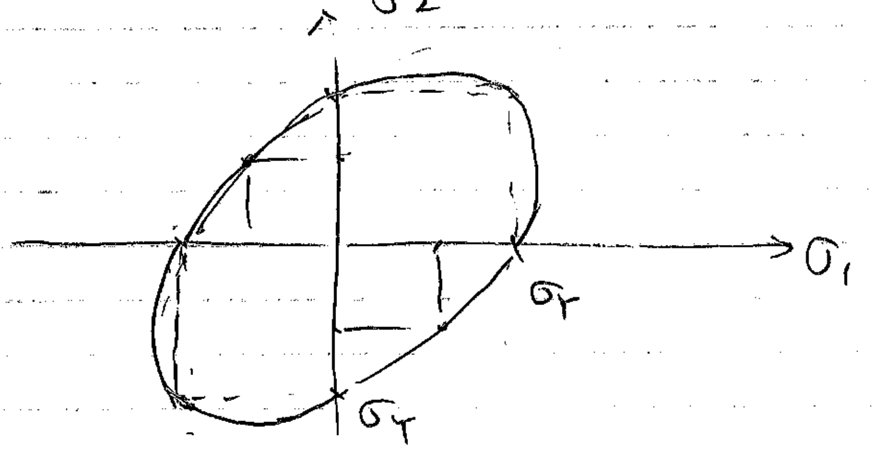


$$J_2 = \frac{1}{2}\left(S_1^2 + S_2^2 + S_3^2\right)$$

$$= \frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]$$

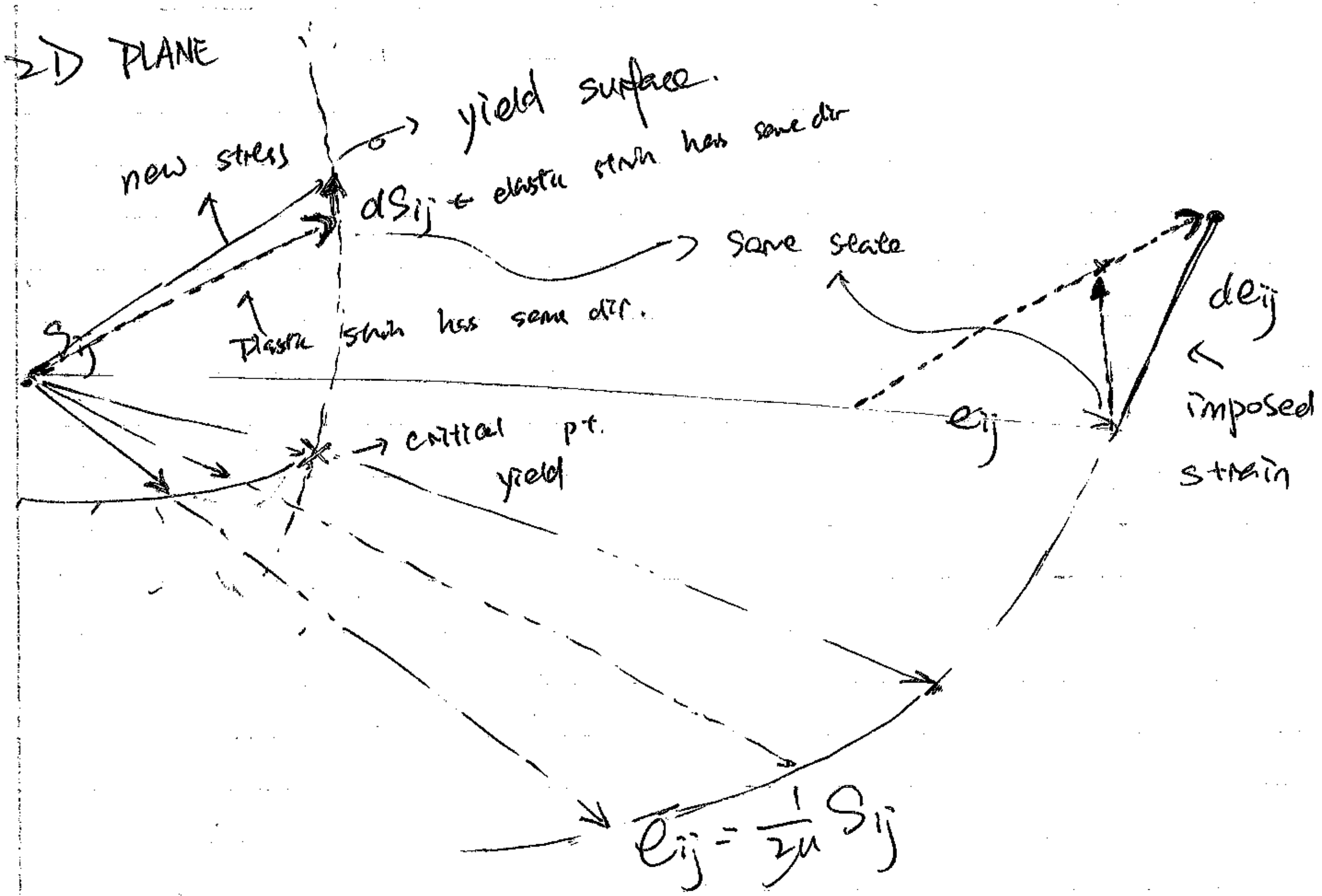
in terms of principle stresses

in plane stress.

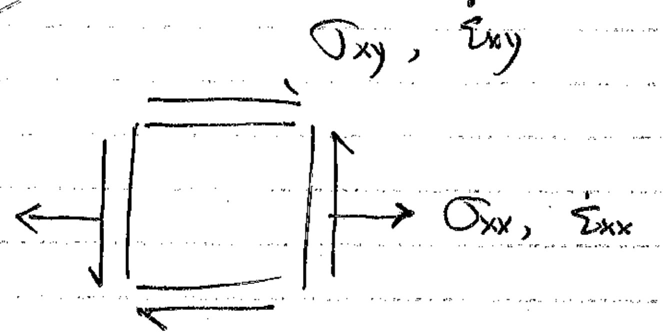


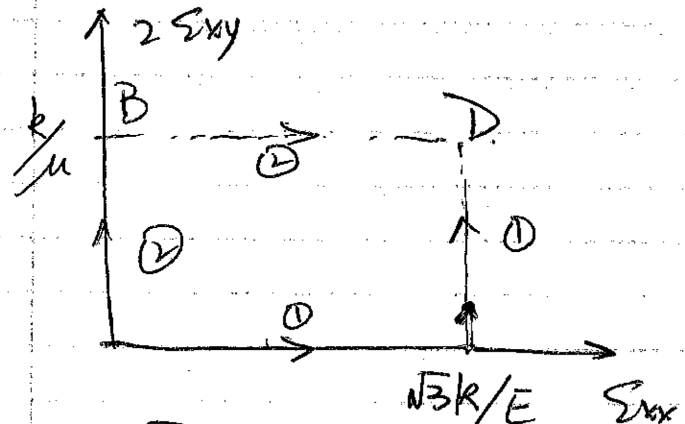
i.e." cutting a cylinder thru a perpendicular plane

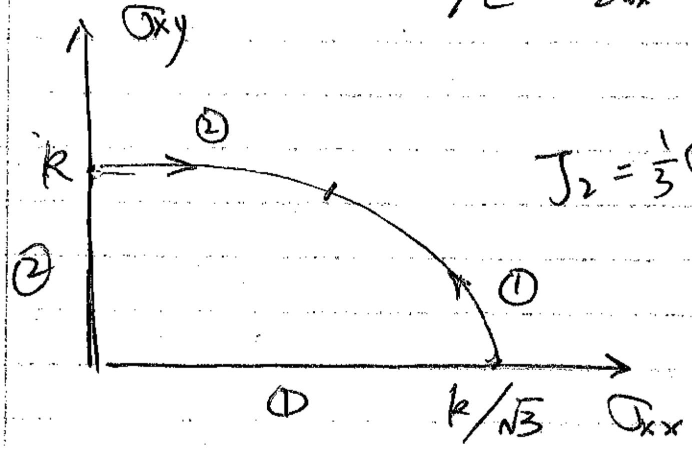
Flow Rule



Example







$$J_2 = \frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2$$

$$\dot{\varepsilon}_{xy} = \dot{\varepsilon}_{xy}^{el} + \dot{\varepsilon}_{xy}^{pl}$$

$$= \frac{\dot{\sigma}_{xy}}{2\mu} + \frac{\dot{W}}{2k^2}\sigma_{xy}$$

$$= \frac{\dot{\sigma}_{xy}}{2\mu} + \frac{\sigma_{xy}\, 2\dot{\varepsilon}_{xy}}{2k^2}\sigma_{xy}$$

incompressible $\begin{cases} \nu = 0.5 \\ E = 3\mu \end{cases}$

$$\dot{\varepsilon}_{ij}^{pl} = \frac{\dot{W}}{2k^2} S_{ij}$$

$$\dot{W} = S_{ij}\dot{e}_{ij}$$

$$= \sigma_{ij}\dot{\varepsilon}_{ij} = \sigma_{xy}\, 2\,\dot{\varepsilon}_{xy}$$

$$\rightarrow \dot{\varepsilon}_{xy} = \frac{\dot{\sigma}_{xy}}{2\mu} + \frac{\dot{\varepsilon}_{xy}}{k^2} \cdot \sigma_{xy}^2$$ final expression.

$$\left(1 - \frac{\sigma_{xy}^2}{k^2}\right)\dot{\varepsilon}_{xy} = \frac{\dot{\sigma}_{xy}}{2\mu}$$

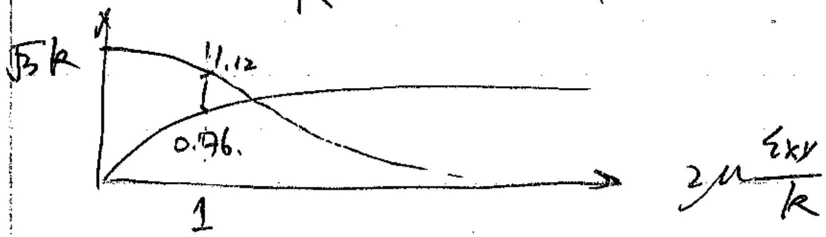
$$\frac{2\mu\dot{\varepsilon}_{xy}}{k} = \frac{\dot{\sigma}_{xy}/k}{1 - \frac{\sigma_{xy}^2}{k^2}}$$

$$2\mu\frac{\dot{\varepsilon}_{xy}}{k} = \frac{\frac{\dot{\sigma}_{xy}}{k}}{1 - \left(\frac{\sigma_{xy}}{k}\right)^2}$$

$\int dt$ on both sides.

$$2\mu\frac{\varepsilon_{xy}(t)}{k} = \text{arc-tanh}\left[\frac{\sigma_{xy}(t)}{k}\right]$$

$$\frac{\sigma_{xy}(t)}{k} = \tanh\left(2\mu \cdot \frac{\varepsilon_{xy}(t)}{k}\right)$$



$$\tanh = \frac{\sinh}{\cosh} = \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}}$$

Problem session #7 5/17/2024

<u>Problem:</u> first apply shear up to yield point $\sigma_{xy} = \sigma_Y$.

then apply tension to some strain ε_{xx}.

Assumptions: No strain hardening,

$\nu = 0.5$ (incompress)

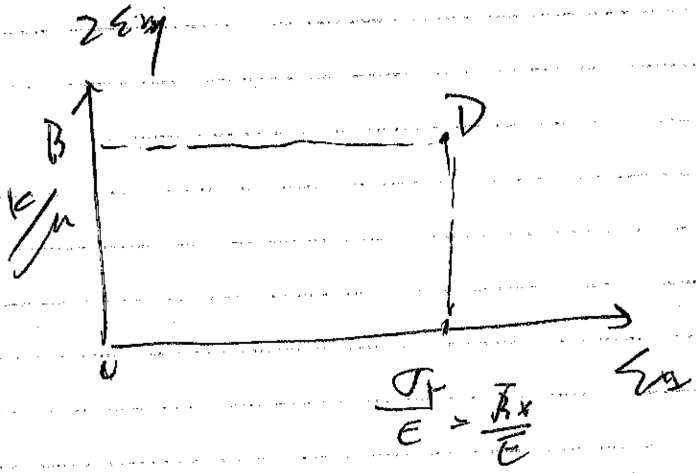
von Mises criteria. $J_2 = k^2$

VM criteria: $J_2 = k^2 = \frac{\sigma_Y^2}{3}$.

$J_2 = \frac{1}{2} s_{ij} s_{ij}$ EPP: $\dot{J}_2 = 0 \to s_{ij}\dot{s}_{ij} = 0$

plastic strain rate: $\dot{\varepsilon}_{ij}^{pl} = \frac{\dot{\lambda}}{2\mu} s_{ij} = \frac{\dot{W}}{2k^2} \cdot s_{ij}$

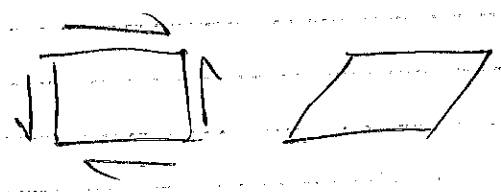
<u>Shear tension Plasticity</u>



$J_2 = \frac{\sigma_{xx}^2}{3} + \sigma_{xy}^2 = k^2$.

$\varepsilon_{xy} = \varepsilon_{xy}^{el} + \varepsilon_{xy}^{pl}$

$\nu = 0.5 \to E = 2(1+\nu)\mu$



$\downarrow$

3μ

① Along path OB. → pure elastic strain.

$$\sigma_{xy} = 2\mu\, \varepsilon_{xy} \quad \text{(goes from 0 to } k\text{)}.$$

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$$

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = \varepsilon_{yz} = 0$$

② Along path BD

$$J_2 = \frac{\sigma_{xx}^2}{3} + \sigma_{xy}^2 = k^2$$

$$\dot{\varepsilon}_{ij}^{pl} = \frac{\dot{W}}{2k^2} S_{ij}.$$

$$\dot{W} = S_{ij}\dot{e}_{ij} = S_{xx}\,\dot{e}_{xx} \quad \Big\rangle \begin{array}{l} e_{xy} = \text{const.} \\ \dot{e}_{xy} = 0 \end{array}$$

shape change
rate of work

$$\dot{W} = S_{xx}\,\dot{e}_{xx} = \frac{2}{3}\sigma_{xx}\, e_{xx}$$

$$\rightarrow \dot{\varepsilon}_{xx}^{pl} = \frac{\dot{W}}{2k^2} \cdot S_{xx} = \frac{\dot{W}}{3k^2}\,\sigma_{xx}$$

$$\dot{\varepsilon}_{xx} = \dot{\varepsilon}_{xx}^{el} + \dot{\varepsilon}_{xx}^{pl}$$

$$\cancel{\dot{\varepsilon}} + e_{xx}^{el}$$

$$\left(\frac{\dot{\sigma}_{xx}}{E} \right)$$

$$= \frac{\dot{\sigma}_{xx}}{E} + \frac{\dot{\lambda}}{3k^2} \sigma_{xx}$$

$$= \frac{\dot{\sigma}_{xx}}{E} + \frac{\dot{\varepsilon}_{xx}}{3k^2} (\sigma_{xx})^2$$

$$\dot{\varepsilon}_{xx} \left(1 - \frac{\sigma_{xx}^2}{3k^2} \right) = \frac{\dot{\sigma}_{xx}}{E}$$

$$\frac{E \dot{\varepsilon}_{xx}}{\sqrt{3} k} = \frac{\frac{\dot{\sigma}_{xx}}{\sqrt{3} k}}{1 - \left(\frac{\sigma_{xx}}{\sqrt{3} k} \right)^2}$$

some algebra

$$\operatorname{atan}'(x) = \frac{1}{1 - x^2}$$

Integrate both sides.

$$\frac{E \varepsilon_{xx}(t)}{\sqrt{3} k} = \operatorname{arctanh}\left(\frac{\sigma_{xx}(t)}{\sqrt{3} k} \right)$$

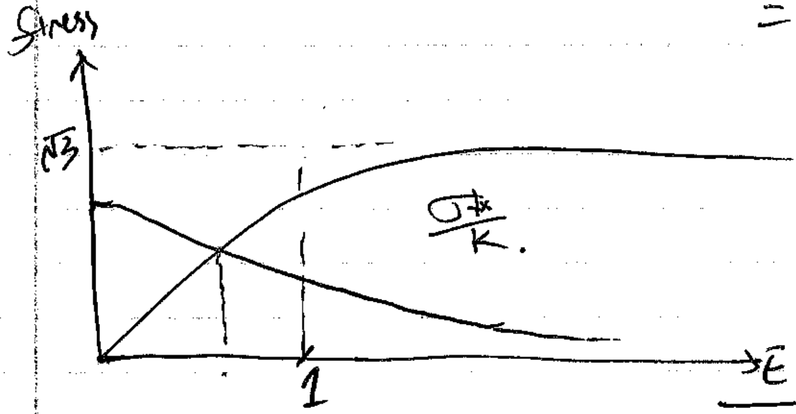
$$\frac{\sigma_{xx}(t)}{\sqrt{3} k} = \tanh\left(\frac{E \varepsilon_{xx}(t)}{\sqrt{3} k} \right)$$

Using $J_2 = k^2$

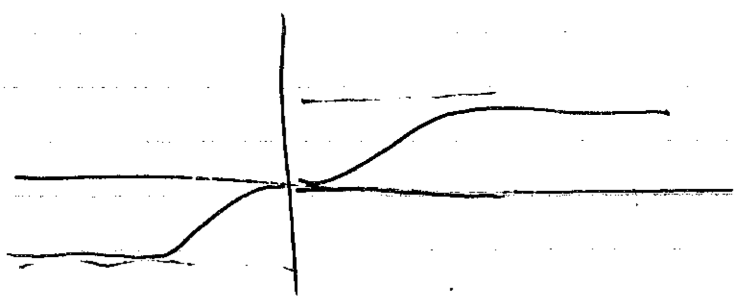
$$\frac{\sigma_{xx}^2}{3} + \sigma_{xy}^2 = k^2 \Rightarrow \frac{\sigma_{xy}}{K} = \sqrt{1 - \left(\frac{\sigma_{xx}}{\sqrt{3}k}\right)^2}$$

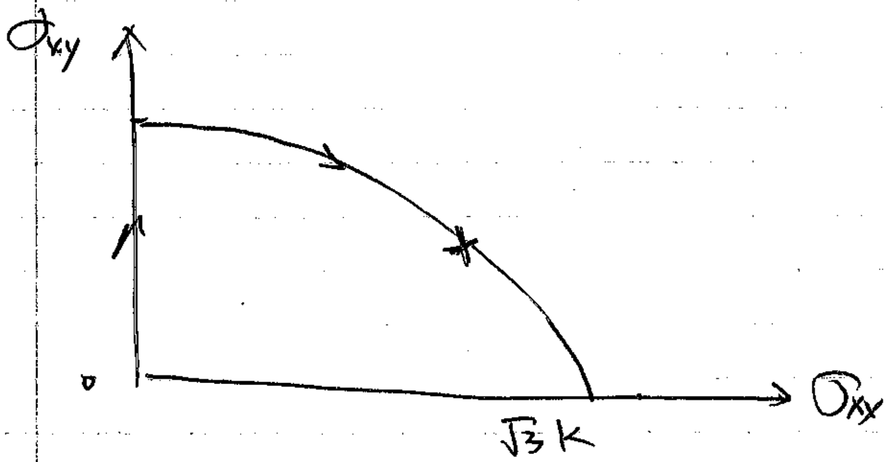
$\hookrightarrow \tanh^2$

$$= \frac{1}{\cosh\left(\frac{E\varepsilon_{xx}(t)}{\sqrt{3}k}\right)}$$



$$\frac{E\varepsilon_{xx}(t)}{\sqrt{3}k} \rightarrow \frac{E\sigma_Y}{E\sqrt{3}k}$$





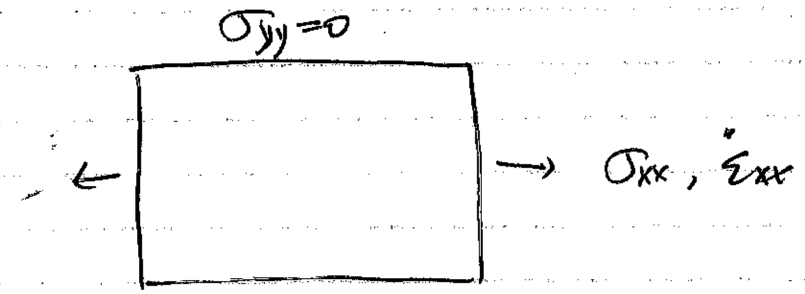
If $\nu < 0.5$ (not incompressible).

$\downarrow$

no analytical sol'n

$\downarrow$

numerical methods.



plane strain pro.

$$J_2 = k^2 = \frac{\sigma_Y^2}{3} \qquad \sigma_{zz} = \nu \sigma_{xx}$$

$$= \frac{1}{2}\left(S_{xx}^2 + S_{yy}^2 + S_{zz}^2 \right)$$

$$\sigma_{xx} = \frac{\sigma_Y}{\sqrt{\nu^2 - \nu + 1}} > \sigma_Y$$

Finite time steps.

$\sigma_{xx}(t)$, $\sigma_{zz}(t)$, $\varepsilon_{xx}(t)$, $\varepsilon_{zz}(t)$.

$\varepsilon_{xx}(t+\Delta t)$, $\varepsilon_{zz}(t+\Delta t)$.

Unknowns: $\sigma_{xx}(t+\Delta t)$. $\sigma_{zz}(t+\Delta t)$

$(\bar{\sigma}/2\mu)$

$$\Delta\bar{\sigma} = \bar{\sigma}(t+\Delta t) - \bar{\sigma}(t)$$

$$\Delta S_{xx} = S_{xx}(t+\Delta t) - S_{xx}(t).$$

$$\Delta S_{yy} = S_{yy}(t+\Delta t) - S_{yy}(t).$$

Total strains. elastic

$$\Delta \varepsilon_{xx}^{el} = \Delta\bar{\varepsilon} + \Delta e_{xx}^{el}$$

$$= \frac{\Delta\bar{\sigma}}{3K} + \frac{\Delta S_{xx}}{2\mu}$$

$$\Delta \varepsilon_{zz}^{el} = \cdots$$

plastic strain

$$\Delta \varepsilon_{xx}^{pl(t)} = \frac{\hat{\lambda}\Delta t}{2\mu}\cdot\left(\frac{S_{xx}(t) + S_{xx}(t+\Delta t)}{2}\right)$$

$$\dot{\varepsilon}_{xx}^{pl} = \frac{\hat{\lambda}}{2\mu} S_{xx}$$

$\downarrow$

$$\frac{\Delta \varepsilon_{xx}^{pl}}{\Delta t} \qquad \Delta \varepsilon_{zz}^{pl} = \cdots$$

Final eqns.

$$\varepsilon_{xx}(t) + \Delta \varepsilon_{xx}^{el} + \Delta \varepsilon_{xx}^{pl} - \varepsilon_{xx}(t+\Delta t) = 0$$

$$\varepsilon_{zz}(t) + \Delta \varepsilon_{zz}^{el} + \Delta \varepsilon_{zz}^{pl} - \varepsilon_{zz}(t+\Delta t) = 0$$

$$\rightarrow \frac{1}{2}\left[S_{xx}(t+\Delta t)^2 + S_{yy}(t+\Delta t)^2 + S_{zz}(t+\Delta t)^2 \right] = k^2$$

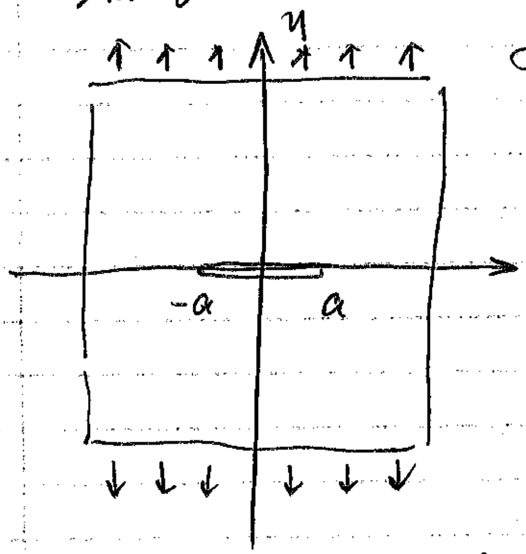
$$J_2 = k^2$$

5/20/24 Lecture 15

LEFM $\rightarrow$ EPFM

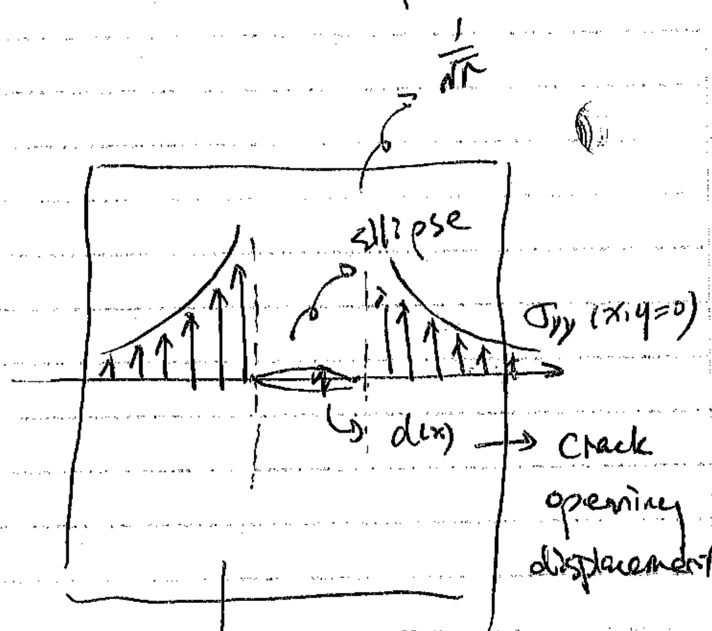
Slit-like crack

$\sigma_{yy}^A = S$.



undeformed config.

"recall previous lecture".



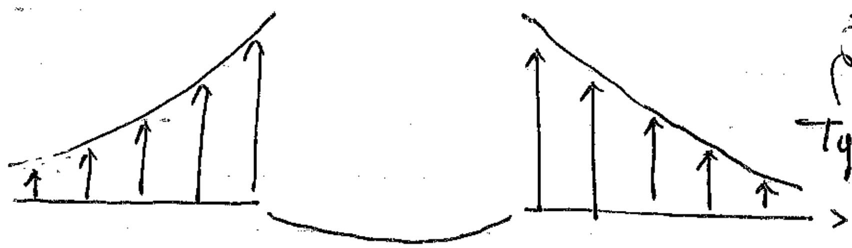
Def'n: stress intensity factor.

$$\sim \frac{K_I}{\sqrt{2\pi r}} \rightarrow r = x - a$$

We want to know K_I

$\rightarrow$ for crack growth

View the crack problem as the half-space



"pulling up"

$$T_y(x) = -P_y(x)$$

"pushing down"

surface displacement.

half-space.

$$\tilde{u}_y(x) = \int_{-\infty}^{+\infty} T_y(x') \left(-\frac{\kappa+1}{4\pi\mu}\right) \log|x-x'|\, dx'$$

from surface Green's function.

Invert it to solve for T_y

$$T_y(x) = \underbrace{\left[\int \frac{u_y'(x')}{\cdots}\right]}_{=0} + \frac{Ax+B}{(x+a)^{1/2}\,(x-a)^{1/2}}$$

even func. of x

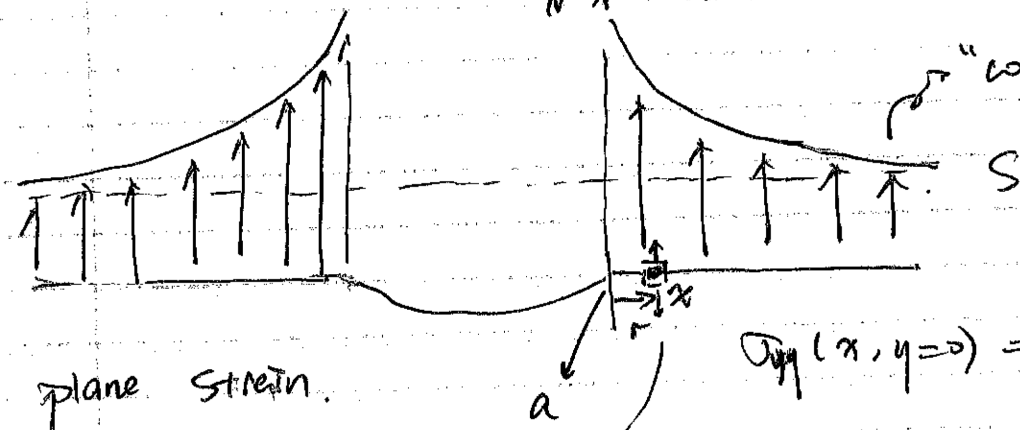
$$T_y(x) = \frac{Ax+B}{(x+a)^{1/2}\,(x-a)^{1/2}} = \frac{A+\cancel{B/x}^{\,0}}{\sqrt{1-(a/x)^2}}$$

$$T_y(x) = \frac{A}{\sqrt{1-(a/x)^2}} = \frac{S}{\sqrt{1-(a/x)^2}}$$

"elegant sol'n"

$$\sigma_{yy}(x, y=0) = \frac{S \cdot |x|}{\sqrt{x^2 - a^2}}$$

to make sure the loading is still an even function



$\sigma_{yy}(x, y=0) = T_y(x)$.

plane strain.

look at point "x"

$x = a + r$

$\sigma_{yy}(x, y=0)$

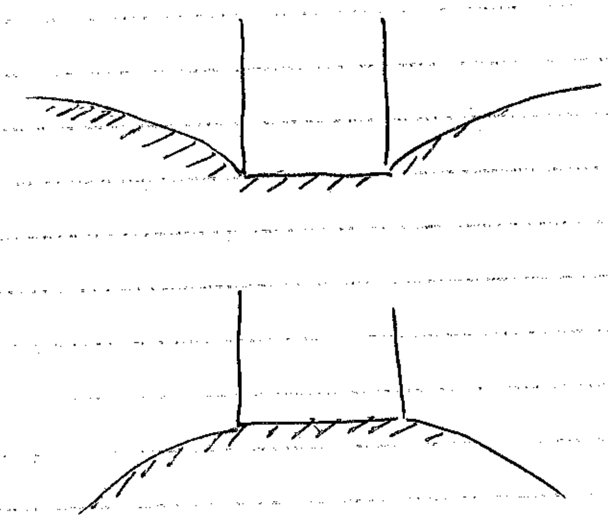
$= \sigma_{\theta\theta}(r, \theta=0)$

$$= \frac{S(a+r)}{\sqrt{(a+r)^2 - a^2}} \Bigg\} \lim_{r \to 0}$$

$$= \frac{Sa}{\sqrt{2ar}}$$

$$= S\sqrt{\frac{a}{2}} \cdot \frac{1}{\sqrt{r}} = \frac{K_I}{\sqrt{2\pi r}}$$

analogous to the flat punch



~ Stress Intensity factor for mode-I fracture

$$\boxed{\therefore K_I = S\sqrt{a\pi}.}$$ ← IMPORTANT

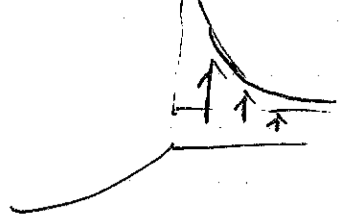
unit: $[Pa \cdot m^{1/2}]$

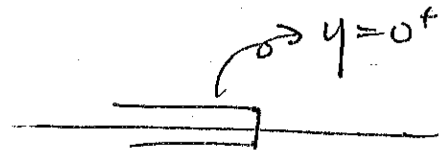
Q: why [Pa$\cdot\sqrt{m}$].

... K divide by $\sqrt{r}$ to get stress.

$$\tilde{u}_y(x) = -\frac{\kappa+1}{4\mu} S a \sqrt{1-\left(\frac{x}{a}\right)^2}$$

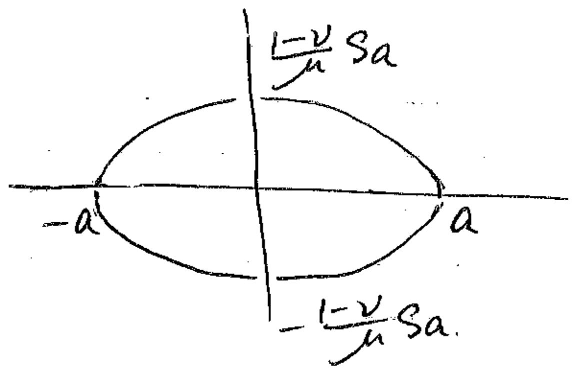
$|x| < a,\ y = 0^-$





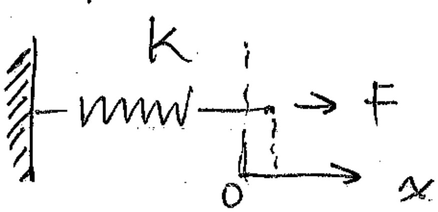
$$d(x) = -2\,\tilde{u}_y(x)$$

$$\boxed{= \frac{2(1-\nu)}{\mu} S a \sqrt{1-\left(\frac{x}{a}\right)^2}}$$ — # IMPORTANT RESULT



Enthalpy

Example



enthalpy: $H = E - \Delta W_{LM}$

$\downarrow$

$\Delta W_{LM} = Fx$

$F = kx$

$E = \frac{1}{2} k x^2$

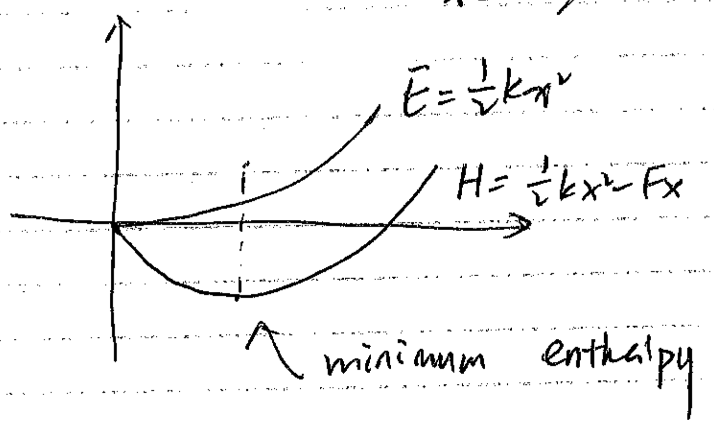
principle: under load mechanism, system go to the state where H is minimized

"When equilibrium is reached"
↓
H is minimized.

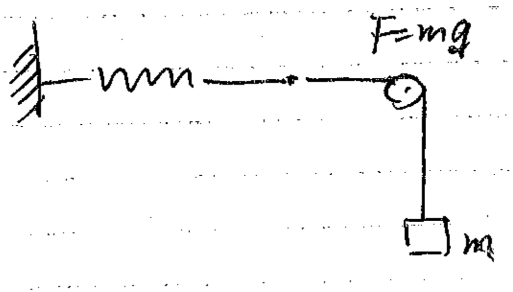
formulate optimization problem.

$$\min_x \left(\tfrac{1}{2}kx^2 - Fx\right).$$

$$x = F/k.$$



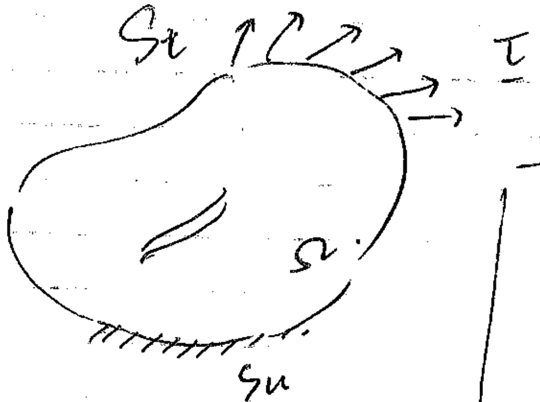
" F shift down to spring elastic energy"



$$E^{tot} = \tfrac{1}{2}kx^2 - mgx.$$

Goal: minimize $H = E - \Delta W_{in}$

via some traction B.C.s



$$\boxed{\begin{aligned} H &= E - \Delta W_{im} \\ &= \int_\Omega \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij}\, dV \\ &\quad - \int_{S_t} T_j u_j\, dS \end{aligned}}$$

hypothesis: the system is minimizing H.

Q: what is the enthalpy of the crack?

→ one may derive the enthalpy of the whole system.

∴ "enthalpy of crack"

	system w/o crack	system w/ crack ($2a$)
energy	$E_0 = \frac{1}{2}\sigma_{ij}^A \varepsilon_{ij}^A V$	E_{2a} → $\Delta E = E_{2a} - E_0$
enthalpy	H_0	H_{2a} → $\Delta H = H_{2a} - H_0$

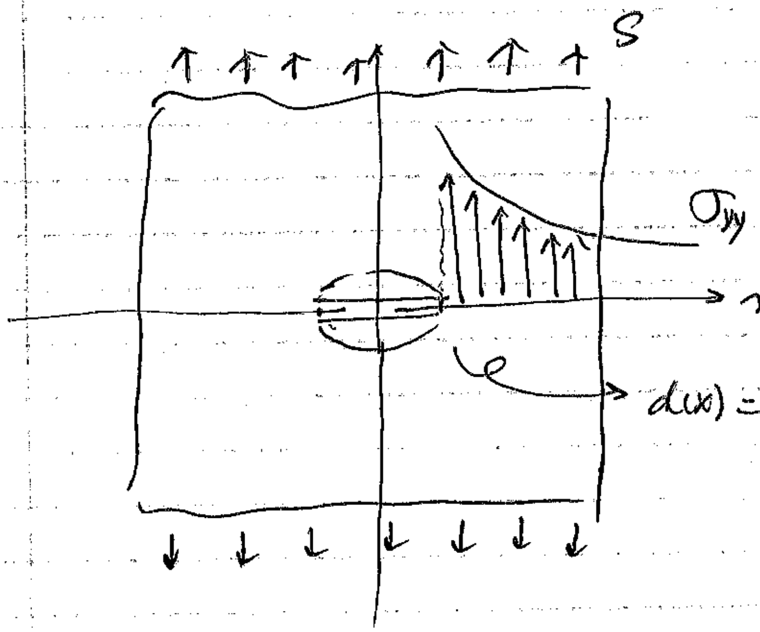
$$\Delta E = \frac{1-\nu}{2\mu} S^2 \pi a^2$$

Conclusion:

$$\boxed{\Delta H = -\frac{1-\nu}{2\mu} S^2 \pi^2 a^2}$$ ← plane strain

Lecture 16.   5/22/2024.

Slit-like crack.

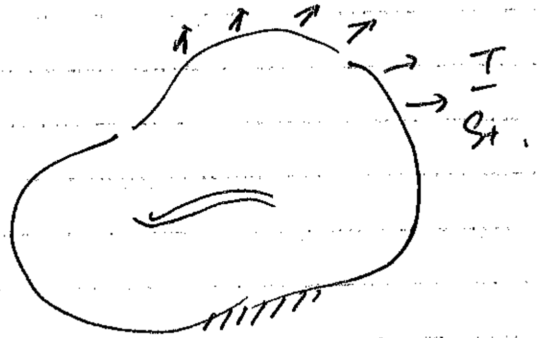


$$\sigma_{yy}(x, y=0) = \frac{S|x|}{\sqrt{x^2-a^2}}.$$

$$d(x) = \frac{2(1-\nu)}{\mu} S \cdot a \sqrt{1-\left(\frac{x}{a}\right)^2}$$

Q: what is the enthalpy of the crack?

$$H = E - \Delta W_{in}.$$



$$H = \int \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \, dV.$$

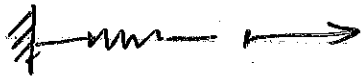
the system is trying to minimize H.

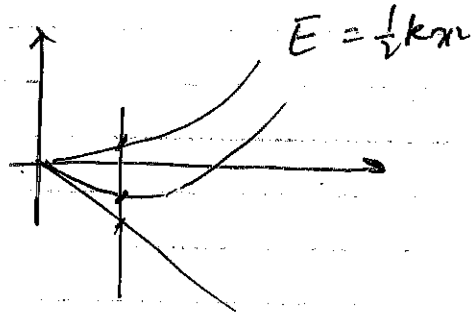
$$- \int_{S_t} T_i u_i \, dS.$$

... body with no pre-existing internal stress.

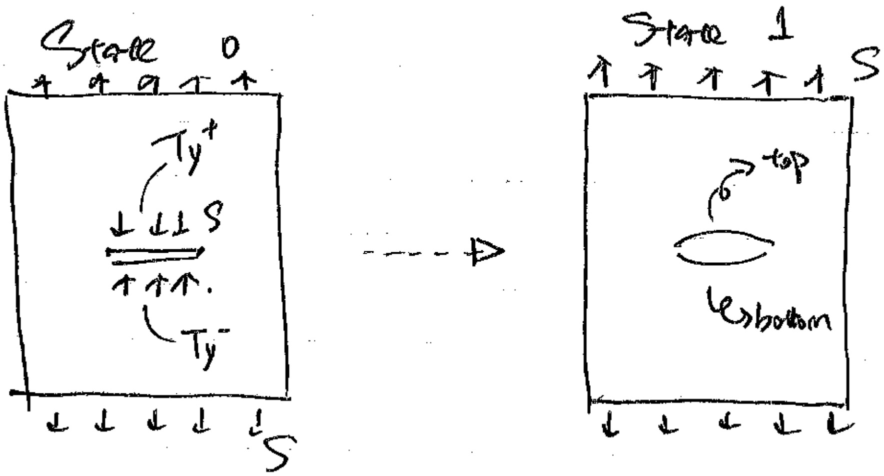
then $H = -E$

toy example





... thought experiment.



E_0, $H_0 = -E_0$ $\qquad$ E_1, $H_1 = -E_1$.

$$\Delta E = E_1 - E_0$$

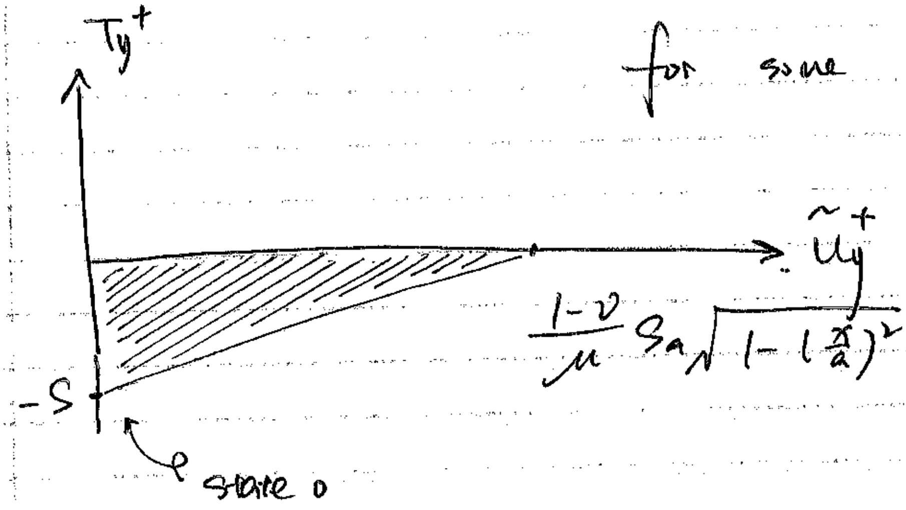
$$E_0 = \frac{1}{2}\sigma_{yy}^A \cdot \varepsilon_{yy}^A V.$$

$$\Delta H = H_1 - H_0 = -\Delta E$$

$$H_0 = -\frac{1}{2}\sigma_{yy}^A \cdot \varepsilon_{yy}^A V.$$

Calculate the work done along the path from the "closed crack" to the "opened crack".

for some x: $-a < x < a$



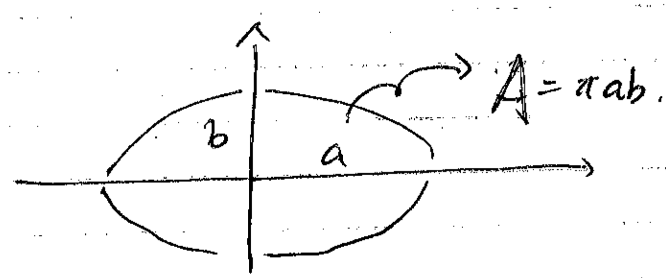
$$\Delta W^+ = \int_{-a}^{a} \frac{1}{2} S \frac{1-\nu}{\mu} S a \sqrt{1-\left(\frac{x}{a}\right)^2}\, dx.$$

$$\Delta H = 2\Delta W^+ = \int_{-a}^{a} \frac{1-\nu}{\mu} S^2 a \sqrt{1-\left(\frac{x}{a}\right)^2}\, dx$$

$$\Delta H = -\frac{1-\nu}{2\mu} S^2 \pi a^2$$

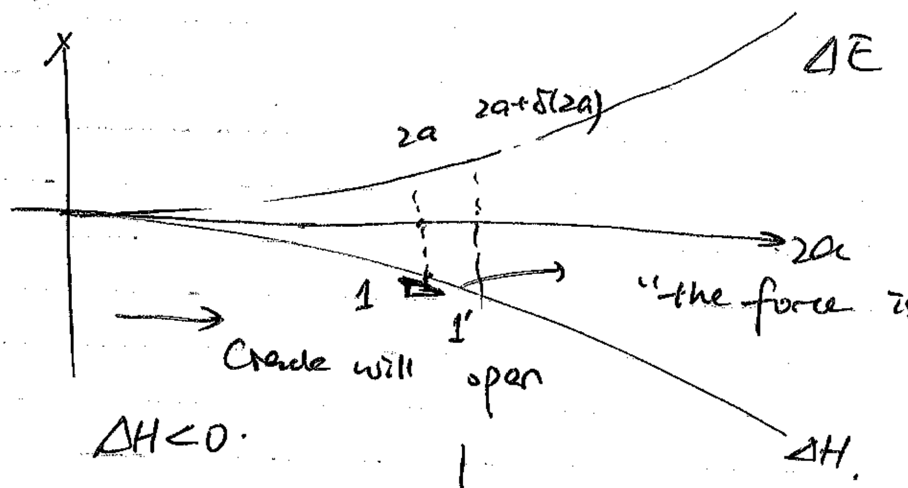
physically: the enthalpy of crack is

$-\frac{1}{2} S$ multiplies the area of the crack opening.



$\Delta H < 0$.

$$\Delta H = -\frac{1}{2} S A.$$



"the force is the derivative of the energy"...

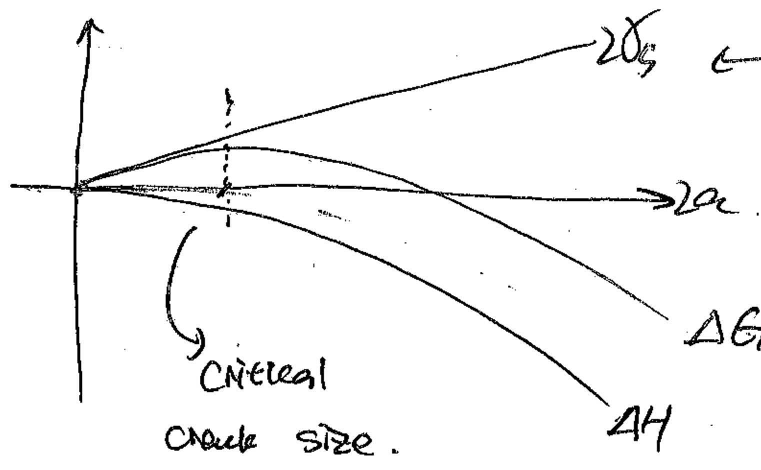
↳ goes to a lower enthalpy state.

Driving force for crack extension,

$$f_{el} = -\frac{\partial(\Delta H)}{\partial(2a)} = \frac{\pi(1-\nu)}{2\mu} S^2 a$$

$$K_I = S\sqrt{\pi a}, \quad \underbrace{f_{el} = \frac{1-\nu}{2\mu} K_I^2}_{\text{explicit relationship.}}$$

Griffith Criteria. (1921).



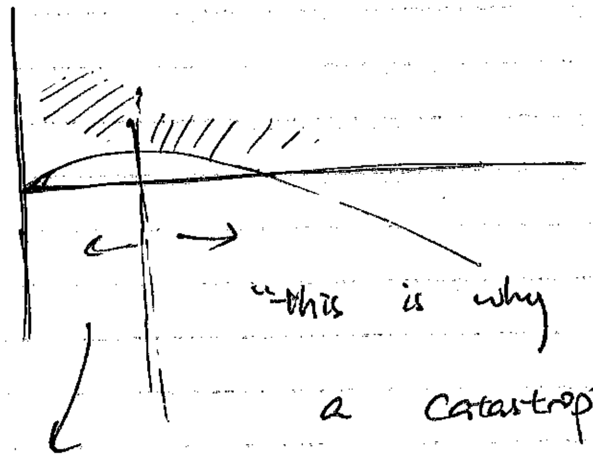
← 2 surfaces.

"pseudo-entropy"

$$\Delta G = \Delta H + \overbrace{4\gamma_s a}$$

$$\boxed{2a_c = \frac{8\mu}{\pi(1-\nu)}\,\frac{\gamma_s}{S^2}}$$

↳ critical crack size for fracture.



"this is why crack is usually a catastrophic process".

"stable crack size"

$$S_c = \sqrt{\frac{8\mu\,\gamma_s}{\pi(1-\nu)\,(2a)}}$$

↳ plane strain.

$$f_{el} = 2\gamma_s.$$

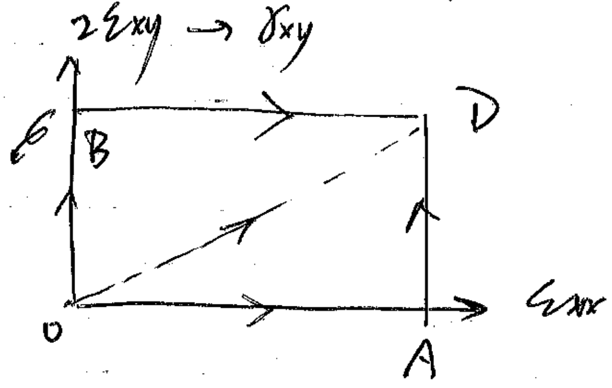
↑ surface energy

crack-driving force.

Problem Session 8

Plasticity code for numerical soln

$$\frac{\sigma_y}{\sqrt{3}\mu} = \frac{K}{\mu}$$



$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

Differential $\to$ algebraic

$\downarrow$

Numerical methods

At any time t. ... known: $\sigma_{xx}(t)$, $\sigma_{xy}(t)$

$\varepsilon_{xx}(t)$, $\varepsilon_{xy}(t)$.

$\downarrow$ yield point

$\varepsilon_{xx}(t+\Delta t)$.

Up to yield $\to$ Hooke's law is valid. $\varepsilon_{xy}(t+\Delta t)$.

$\downarrow$

Unknowns: $\sigma_{xx}(t+\Delta t)$, $\sigma_{xy}(t+\Delta t)$.

$\frac{\gamma}{2\mu}$.

Equations.

① change in stress. finite time steps

$\hookrightarrow \Delta t$, $\to \Delta\sigma_{xx}$, $\Delta\sigma_{xy}$, etc. ...

$$\Delta\bar{\sigma} = \bar{\sigma}(t+\Delta t) - \bar{\sigma}(t).$$

$$\hookrightarrow \begin{cases} \bar{\sigma}(t+\Delta t) = \dfrac{\sigma_{xx}(t+\Delta t)}{3}. \\ \bar{\sigma}(t) = \dfrac{\sigma_{xx}(t)}{3}. \end{cases}$$

$$S_{xx}(t+\Delta t) = \sigma_{xx}(t+\Delta t) - \bar{\sigma}(t+\Delta t)$$

$$S_{xy}(t+\Delta t) = \sigma_{xy}(t+\Delta t)$$

② → change in elastic strain.

$$\Delta \varepsilon_{xx}^{el} = \Delta \bar{\varepsilon}^{el} + \Delta e_{xx}^{el}$$

$$\Delta \varepsilon_{xx}^{el} = \frac{\Delta S_{xx}}{2\mu}$$

$$\Delta \bar{\varepsilon} = \frac{\Delta \bar{\sigma}}{3K}$$

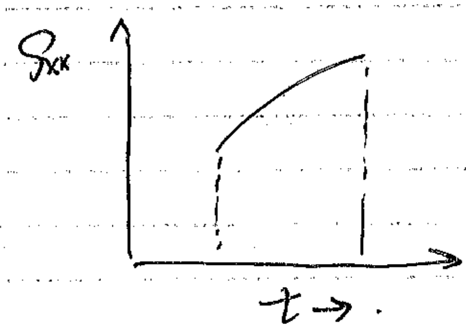
$$\Delta \varepsilon_{xy}^{el} = \frac{\Delta S_{xy}}{2\mu}$$

③ Change in plastic strain.

$$\dot{\varepsilon}_{xx}^{pl} = \frac{\tilde{\lambda}}{2\mu} S_{xx} \rightarrow \text{analytical}$$

$$\downarrow$$

$$\frac{\Delta \dot{\varepsilon}_{xx}^{pl}}{\Delta t}$$



From $t \rightarrow t+\Delta t$.

$$\int_t^{t+\Delta t} \dot{\varepsilon}_{xx}^{pl}\, dt = \frac{\tilde{\lambda}}{2\mu} \int_t^{t+\Delta t} S_{xx}(t)\, dt \rightsquigarrow \Delta \varepsilon_{xx}^{pl}$$

$$= \frac{\tilde{\lambda}}{2\mu}\left(\frac{S_{xx}(t) + S_{xx}(t+\Delta t)}{2}\right)$$

assume $\tilde{\lambda}$ is const. from $t \rightarrow t+\Delta t$.

Similarly for

$$\Delta \varepsilon_{xy}^{el} = \frac{\tilde{\lambda}}{2\mu}\left(\frac{S_{xy}(t) + S_{xy}(t+\Delta t)}{2}\right)$$

Find 3 eqns

1). $\varepsilon_{xx}(t) + \Delta\varepsilon_{xx}^{el} + \Delta\varepsilon_{xx}^{pl} = \varepsilon_{xx}(t+\Delta t)$

2). $\varepsilon_{xy}(t) + \Delta\varepsilon_{xy}^{el} + \Delta\varepsilon_{xy}^{pl} = \varepsilon_{xy}(t+\Delta t)$

3) $J_2 = k^2$

$$\frac{1}{2}\left(S_{xx}^2 + S_{yy}^2 + S_{zz}^2\right) + S_{xy}^2 = k^2$$

(or) $\dfrac{\sigma_{xx}^2}{3} + \sigma_{xy}^2 = k^2$ and solve: $\dfrac{\tilde{\lambda}}{2\mu}$

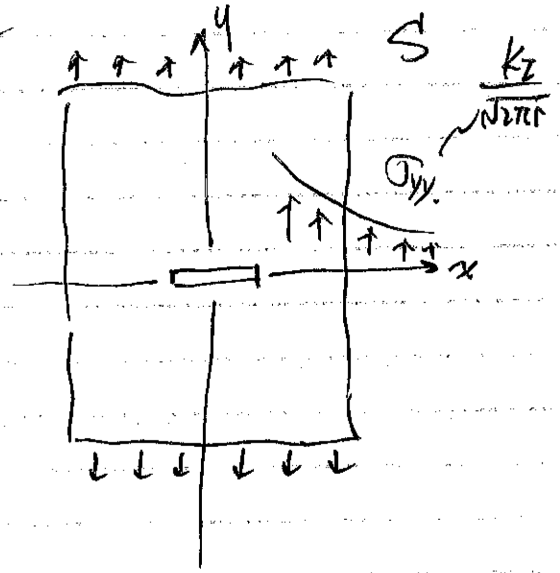
$(t+\Delta t)$ $(t+\Delta t)$.

MATLAB fsolve (fun, [trial], hyper param ...).

current stress store, σ_{xx}, σ_{xy}, $\tilde{\lambda}$

5/29/2024. Lecture 17.

Recap

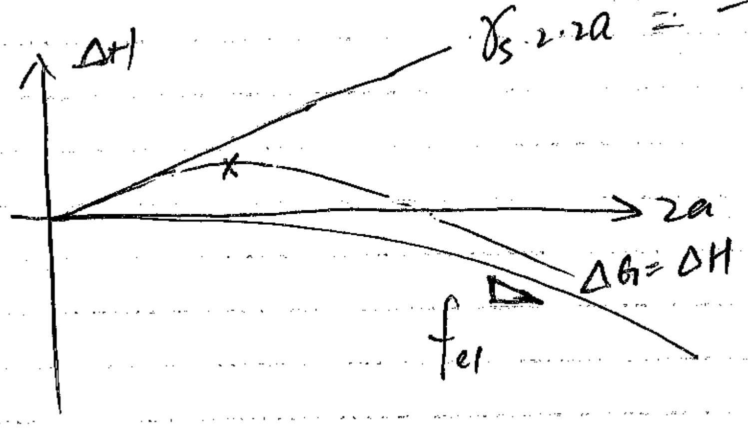


$\Delta H = H_1 - H_0$

$= -\frac{1-\nu}{2\mu} S^2 \pi a^2$

(plane strain)

$\gamma_s \cdot 2 \cdot 2a = -\frac{K+1}{8\mu} S^2 \pi a^2$



$f_{el} = -\frac{\partial \Delta H}{\partial (2a)} = \frac{\pi(1-\nu)}{2\mu} S^2 a$ (plane strain)

$\Delta G = -\frac{1-\nu}{2\mu} S^2 \pi a^2 + 4\gamma_s a$

$f_{tot} = \frac{\partial \Delta G}{\partial (2a)} = \frac{\pi(1-\nu)}{2\mu} S^2 a - 2\gamma_s$

∴ Griffith criteria. $\frac{\pi(1-\nu)}{2\mu} S^2 a \geq 2\gamma_s$ ... (*)

energy release rate $\mathcal{G}$

LHS of (*) $\nearrow$

i.e., $\mathcal{G} = \frac{\pi(1-\nu)}{2\mu} (\sigma_{yy}^A)^2 a$ ... (**)

replace S'

RHS: critical energy release rate

$\mathcal{G}_c = 2\gamma_s$ ... (***).

Eqn. (**) can be rewritten as:

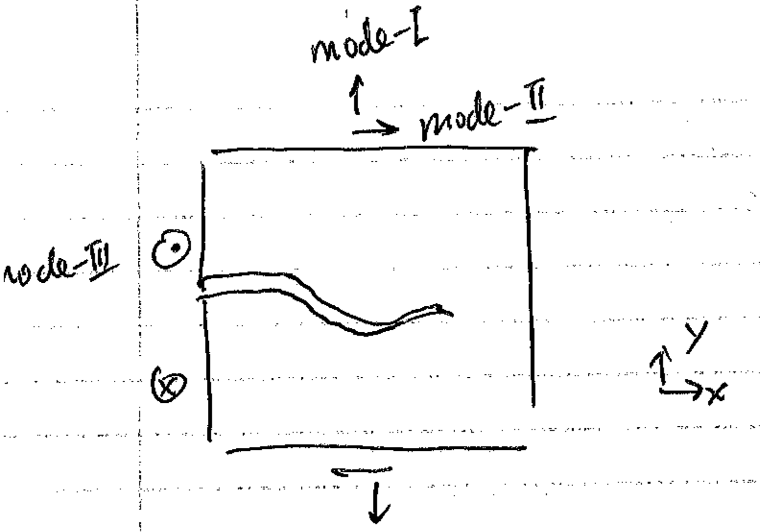
$\mathcal{G} = \frac{\pi(1-\nu)(1+\nu)}{E} (\sigma_{yy}^A)^2 a$. (plane strain)

define $E' = \frac{E}{1-\nu^2} \longrightarrow \frac{\pi}{E'}$

$K_I = \sigma_{yy}^A \sqrt{\pi a}$ stress intensity factor

... $\mathcal{G} = \frac{K_I^2}{E'}$

$\hookrightarrow$ for mode-I loading

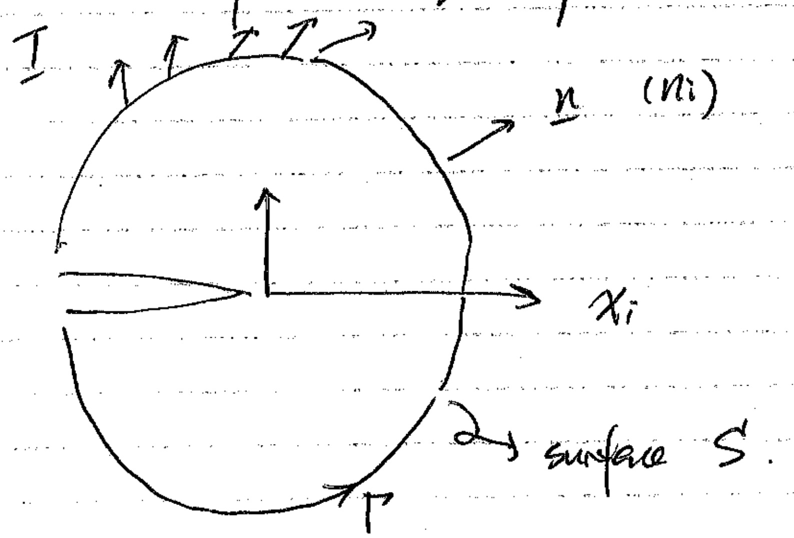


$$G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

fracture criteria: $G \geq G_c$.

J- Integral

Q: what is the generalized force for any kinds of singularity?

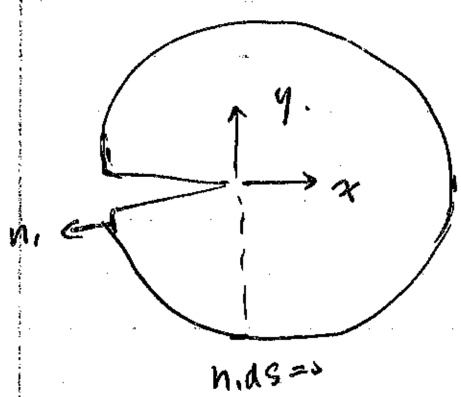


$$J_i = \int_S (W n_i - T_j u_{j,i})\, dS$$

general force — surface — energy density $\frac{1}{2}\sigma_{ij}\varepsilon_{ij}$

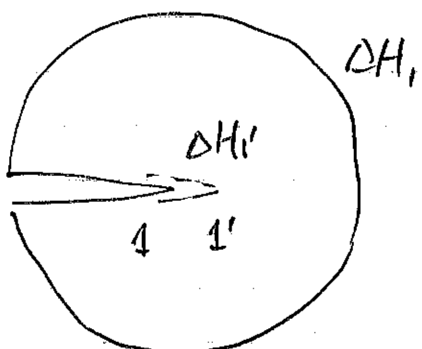
$$J_1 = \int_{\Gamma} w\, dy - T \frac{\partial u}{\partial x} ds$$

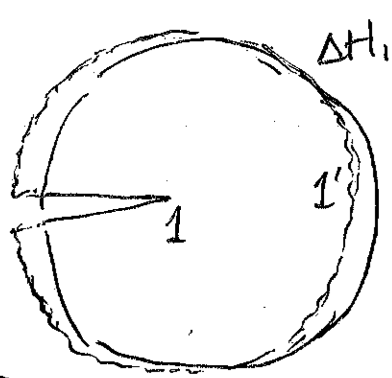
$$G = J$$



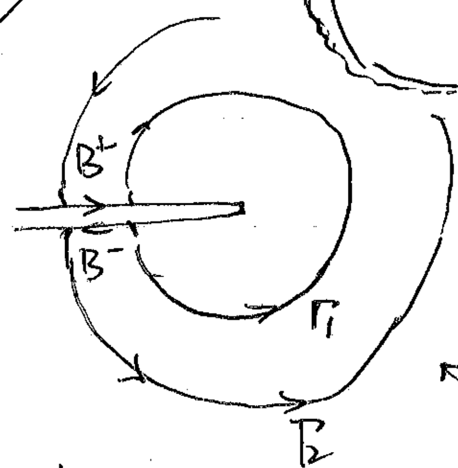
doesn't have to be a perfect circle

$n ds = dy$





Example 1



contour integral along the crack surface

take any contour you want.

(Very nice!!!)

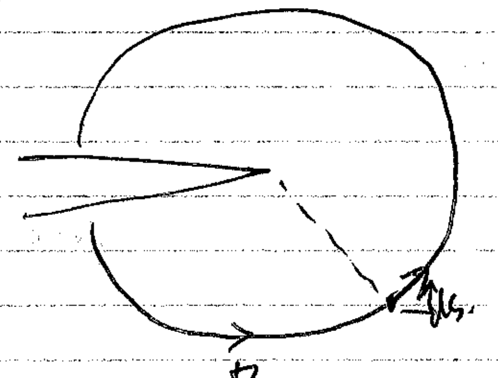
5/31/2014. Lecture 19.

Some review on fracture mechanics.

$\mathcal{G} \geq \mathcal{G}_c$

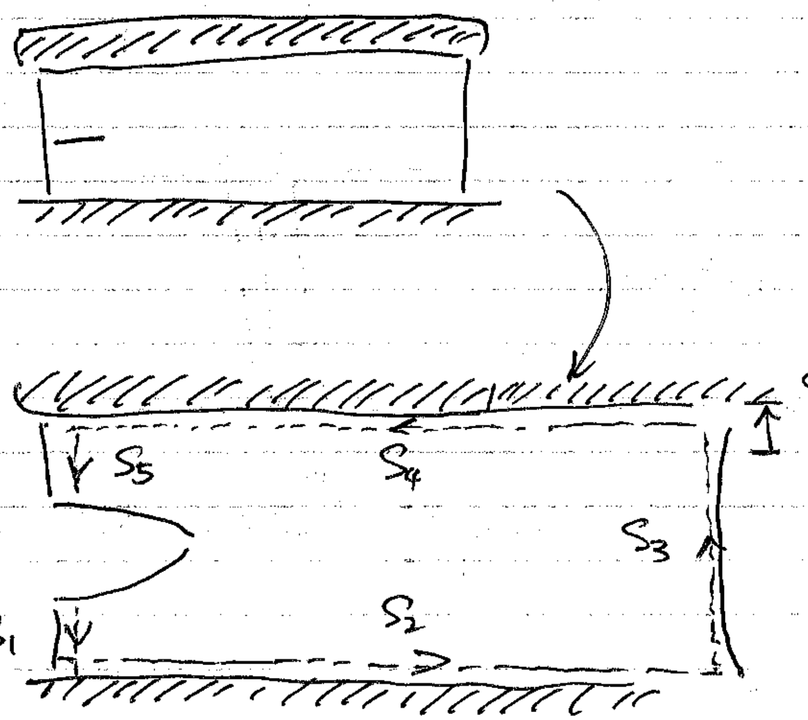
J- integral

2D



$$J = \int_\Gamma w\,dy - \underline{T} \cdot \frac{\partial \underline{u}}{\partial s}\, ds.$$

Example 2



$U_x = 0$

$U_y = \text{const.}$

$$J(S_2) = \int w\,dy - \underline{T}\frac{\partial \underline{u}}{\partial x}\, ds. = 0 \quad \text{almost zero}$$

$$J(S_4) = 0, \quad J(S_1) = \int w\, \cancel{dy}^{\,0} - \underline{T}\frac{\partial \underline{u}}{\partial x}\, ds, \quad J(S_5) = 0$$

the only J survived.

$$J(S_3) = \int w\,dy - \cancel{T\frac{\partial u}{\partial x}ds}^{\,0} = wh$$

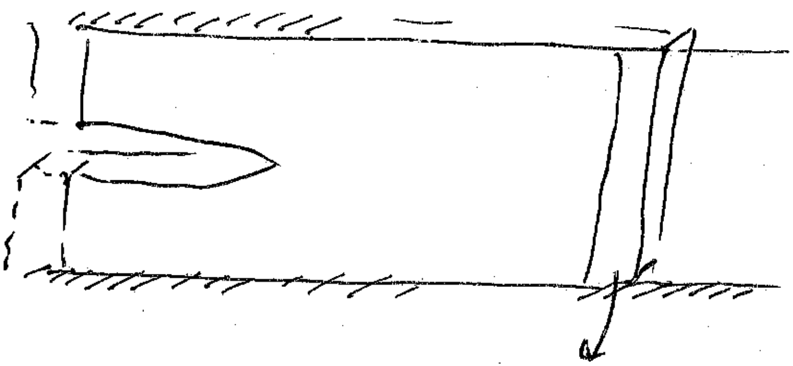
strain energy.

$$\mathcal{G} = J = wh$$

$$\mathcal{G} \geq \mathcal{G}_c$$

not the material properties

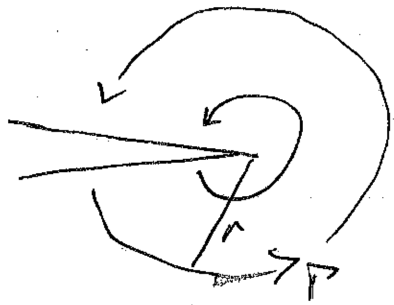
we are just solving for the force (LHS).



physics intuition behind $\mathcal{G} = wh$.

if it's 3D. it should consider the whole plane

Example 3

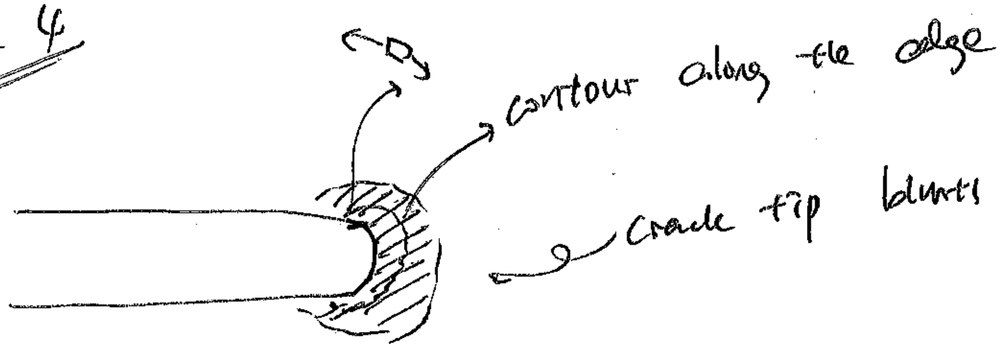


$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right)$$

$$\downarrow \qquad \text{(shrink the contour)}$$

$$J = \frac{K_I^2}{E}.$$

Example 4



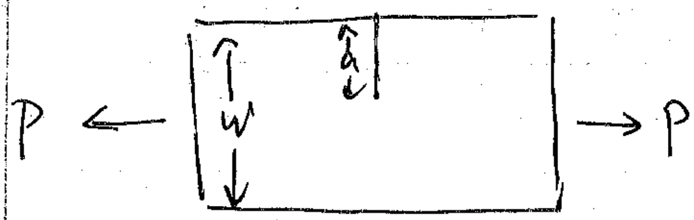
$$J = \int_\Gamma w\,dy - T\frac{\partial u}{\partial x}\,ds$$

$$= \int_\Gamma w\,dy \qquad \left(w = \frac{\sigma^2}{2E}\right)$$

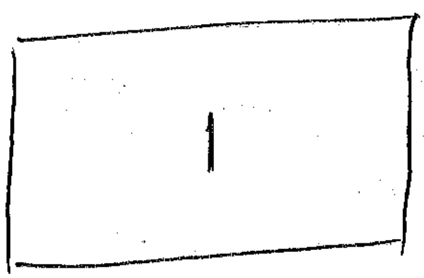
LEFM.

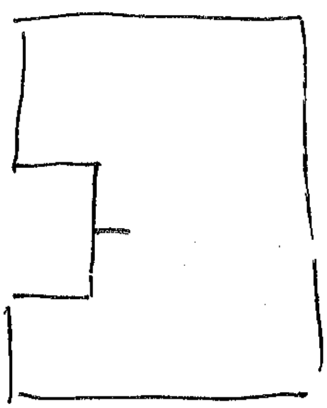
Recap. $G = \frac{K_I^2}{E'}$, $K_I = \sqrt{GE'}$, $K_{Ic} = \sqrt{G_c E'}$

$\rightarrow \; K_I \geq K_{Ic}$

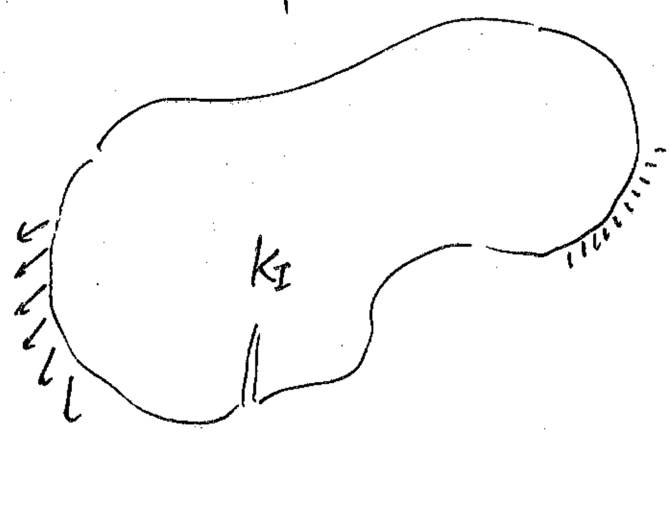


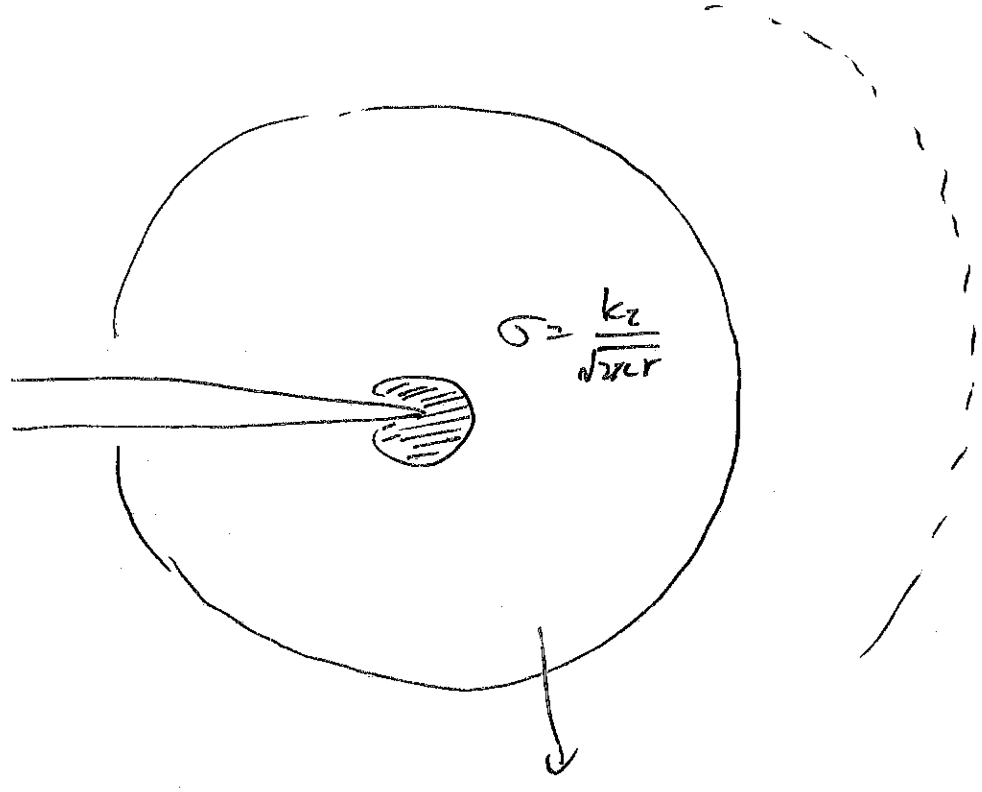
$$K_I = \frac{P}{B\sqrt{W}}$$





reality

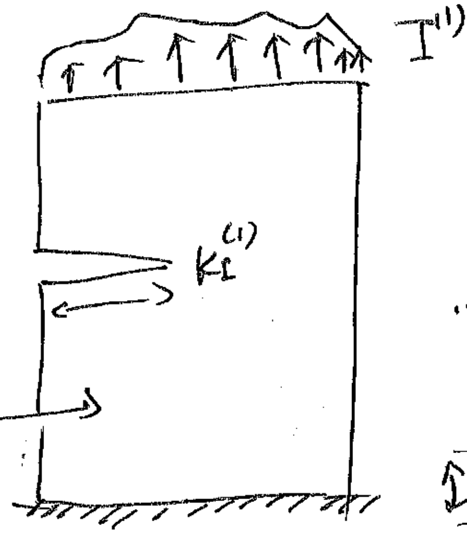




K- field
(zone)

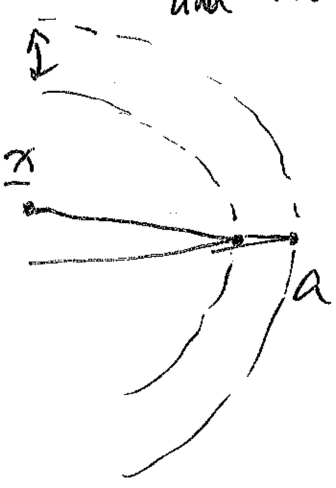
Rice (1972).

LEFM - K_I.



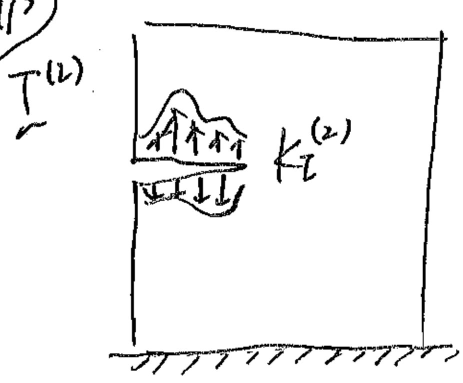
assume I have the solution.

"coupling between motion of crack and loading"

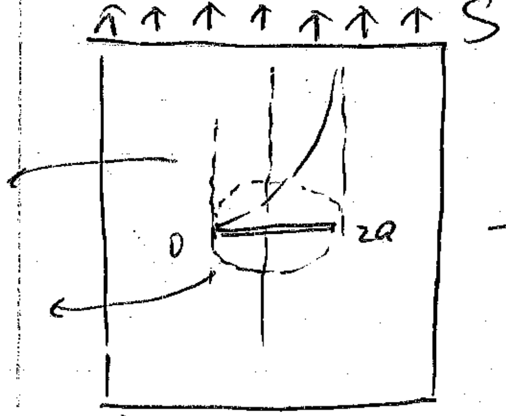


loading (1)

$$K_I^{(2)} = \frac{E'}{2K_I^{(1)}} \int \underline{T}_i^{(2)} \frac{\partial u_i^{(1)}}{\partial a} d\Gamma$$



loading (1)

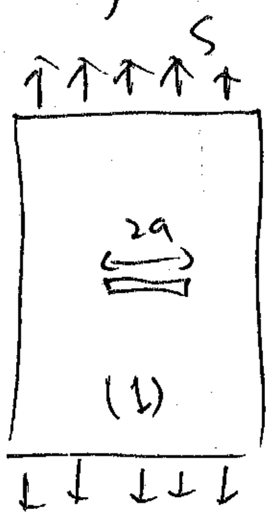


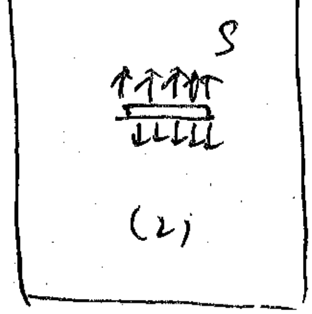
$$\frac{\partial u_i^{(1)}}{\partial a} = \sqrt{\frac{x}{2a-x}}.$$

$$K_I^{(2)} = \frac{E'}{2K_I^{(1)}} \int S \sqrt{\frac{x}{2a-x}}\, dP$$

$$= S\sqrt{\pi a}$$ ∴ results are the same.

loading (1)





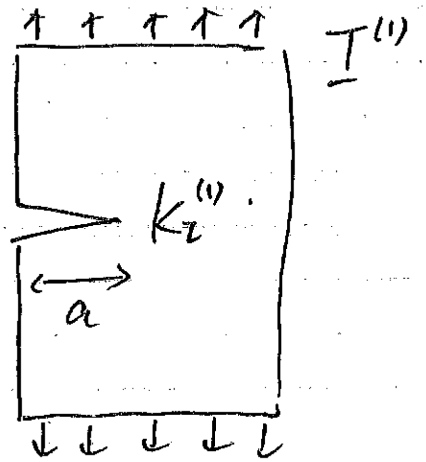
$K_I = S\sqrt{\pi a}$

$K_I = S\sqrt{\pi a}$

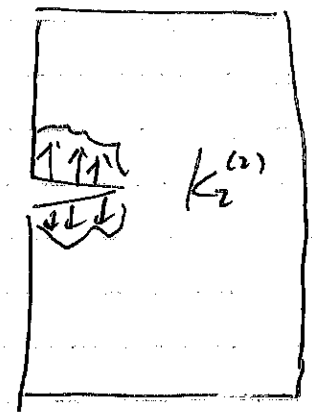
Lecture 19 6/3/2024

EPFM.

Recap LEFM. $K_I \geq K_{IC}$.



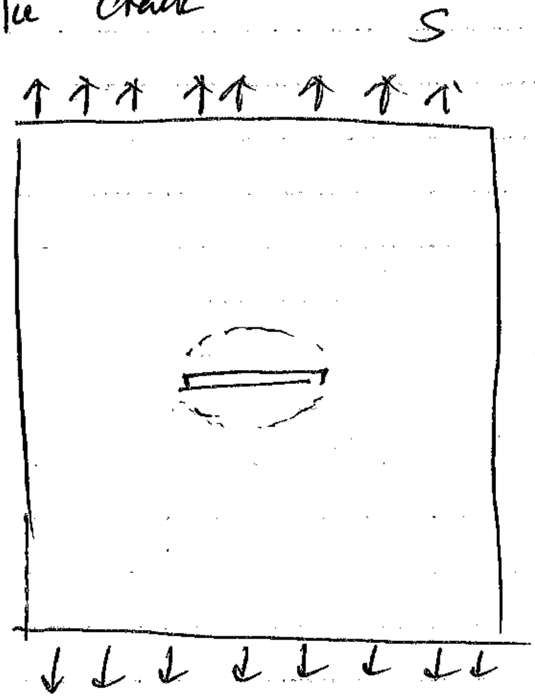
loading (1)

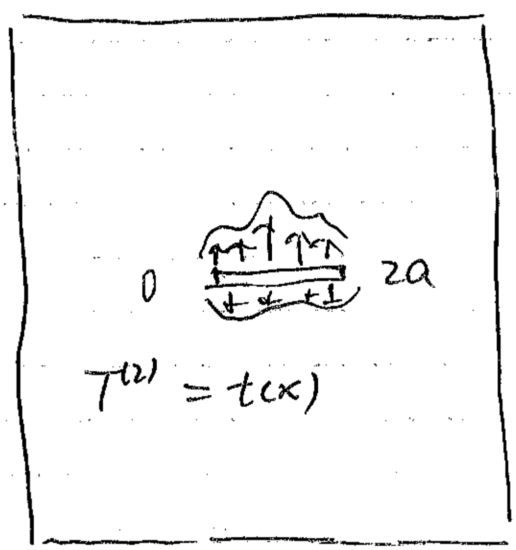


loading (2)

$$K_I^{(1)} = \frac{E'}{2K_I^{(1)}} \int_\Gamma T_i^{(2)} \frac{\partial u_i^{(1)}}{\partial a} d\Gamma$$

Slit-like crack





$$K_I^{(2)} = \frac{1}{\sqrt{\pi a}} \int_0^{2a} t(x) \sqrt{\frac{x}{2a-x}}\, dx$$

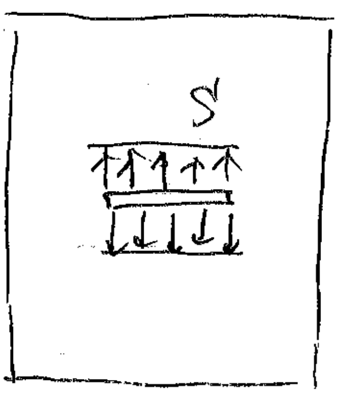
(for $x = 2a$)

if we shift the coordinate:

$$K_I^{(2)} = \frac{1}{\sqrt{\pi a}} \int_{-a}^{a} t(x) \sqrt{\frac{a+x}{a-x}}\, dx \quad \rightarrow \text{elasticity theory.}$$

crack: $-a \le x \le a$ for $x = a$

Example 1

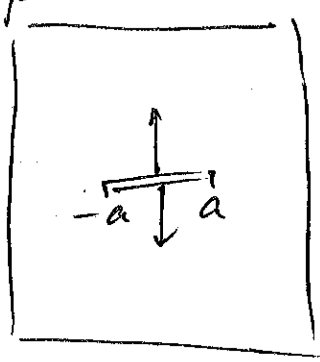


$$K_I^{(2)} = \frac{1}{\sqrt{\pi a}} \cdot \int_{-a}^{a} S \sqrt{\frac{a+x}{a-x}}.\, dx$$

$$= \frac{1}{\sqrt{\pi a}} . S . \pi a$$

$$= S\sqrt{\pi a}$$

Example 2



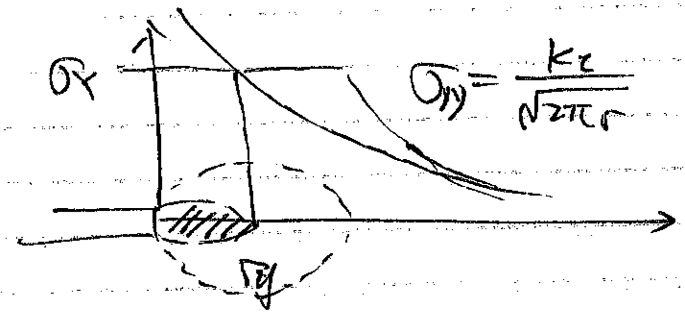
$$K_I^{(2)} = \frac{E'}{\sqrt{\pi a}} \int_{-a}^{a} F\, \delta(x) \sqrt{\frac{a+x}{a-x}}\, dx$$

$$= \frac{E'F}{\sqrt{\pi a}}$$

EPFM

Irwin's approach

w/o plasticity



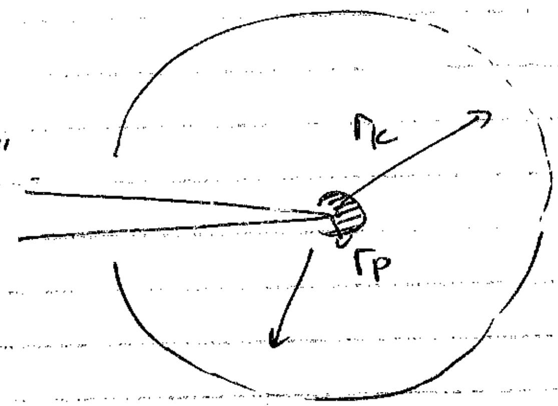
$$\sigma_Y = \frac{K_I}{\sqrt{2\pi r_y}}$$

$$r_y = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_Y}\right)^2$$

Irwin did some "debug"

$$r_p = 2 r_y = \frac{1}{\pi}\left(\frac{K_I}{\sigma_Y}\right)^2$$

empirical



K-field

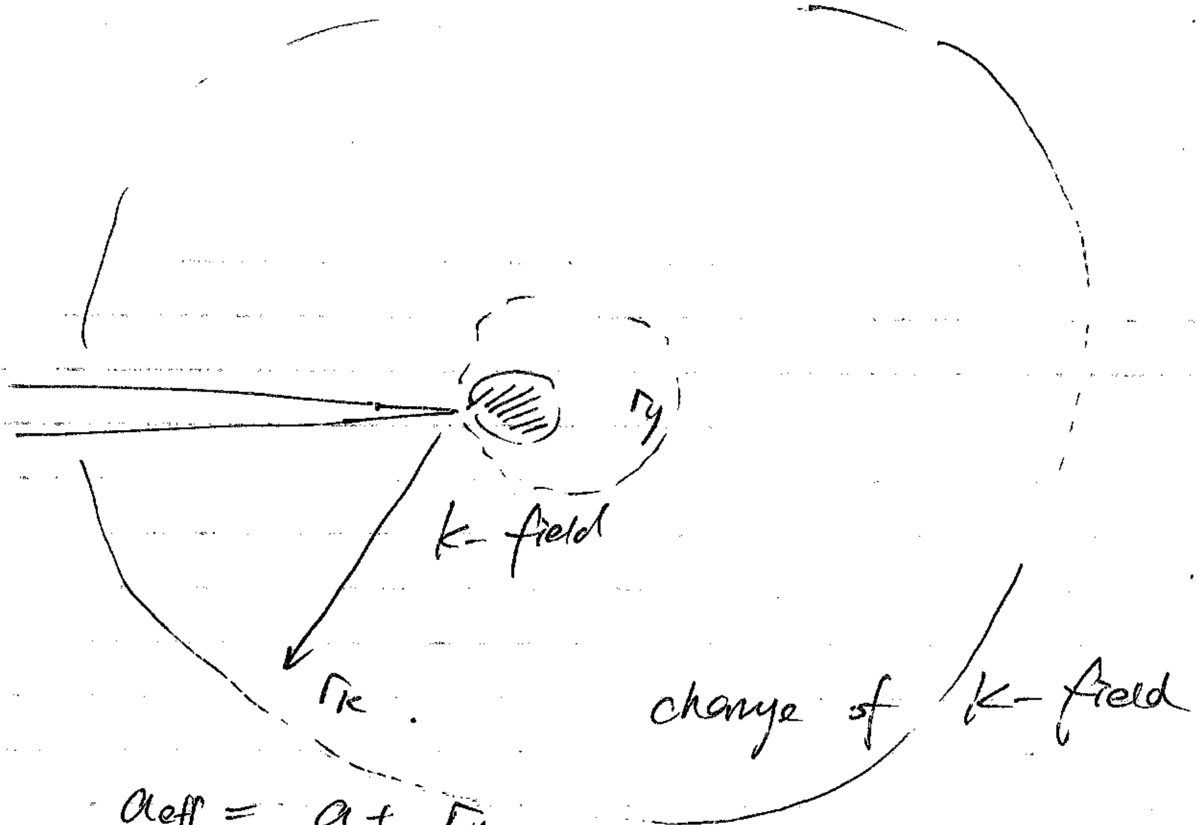
$\sigma_{ij} \sim \frac{K_I}{\sqrt{2\pi r}} f(\theta)$

$K_I \geq K_{IC}$

how does plasticity change this criteria?

σ_Y.

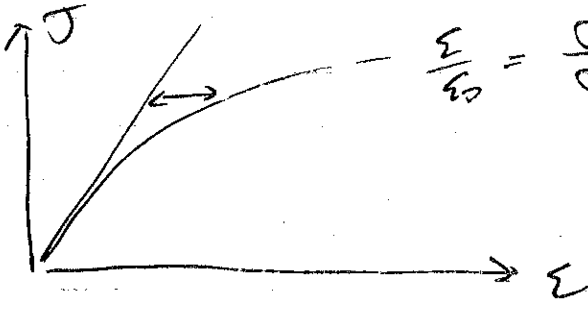
yield increases K_{IC} (mat. param.)



$$a_{eff} = a + r_y$$

$$K_{Ieff} = \frac{P}{B\sqrt{W}} f\left(\frac{a_{eff}}{W}\right)$$

HRR Solution 1968



$$\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha\left(\frac{\sigma}{\sigma_0}\right)^n$$

closer to the crack tip,

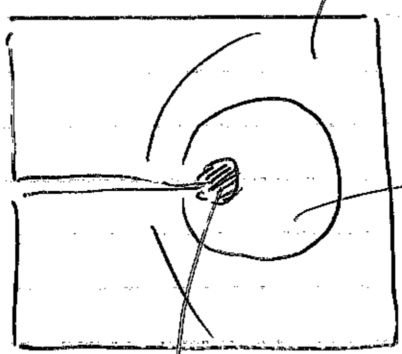
$$\sigma_{ij} = k_1 \left(\frac{J}{r}\right)^{\frac{1}{n+1}}$$

$$\sigma \cdot \varepsilon \propto \frac{1}{r}$$

$$\varepsilon_{ij} = k_2 \left(\frac{J}{r}\right)^{\frac{n}{n+1}}$$

... HRR singularity

linear elastic region.
find a boundary : K - dominated

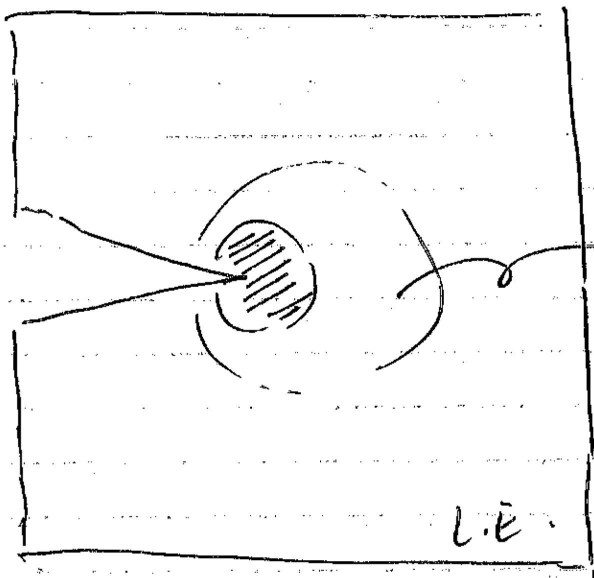


J- dominated region.

↳ LEFM works
J-integral works.

large strain

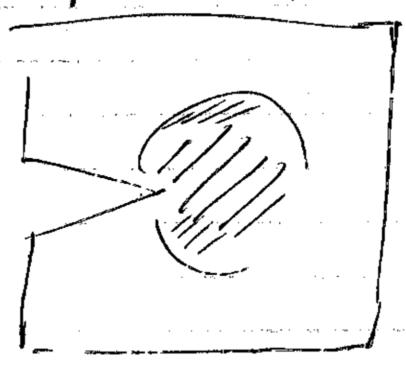
increasing the load.



zone expand
"K-dominated zone vanishes...

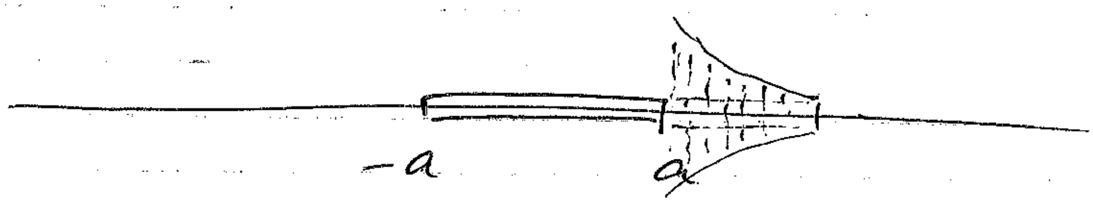
↳ LEFM not works
J- integral works

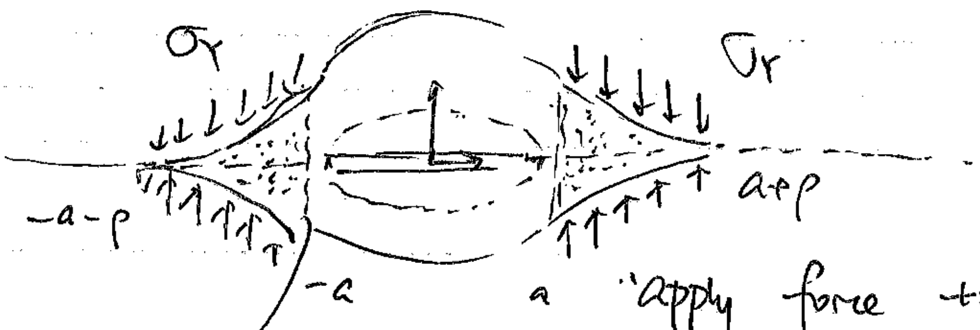
... keep loading



LEFM & J- integral
all dont work.

Strip-yield model





"apply force to close the crack"

plastic flow.

blunts the crack tip.

$$K_I^{tot} = \sigma\sqrt{\pi(a+\rho)} - 2\sigma_Y\sqrt{\frac{a+\rho}{\pi}}\arccos\left(\frac{a}{a+\rho}\right) = 0$$

Solve for the correct ρ.

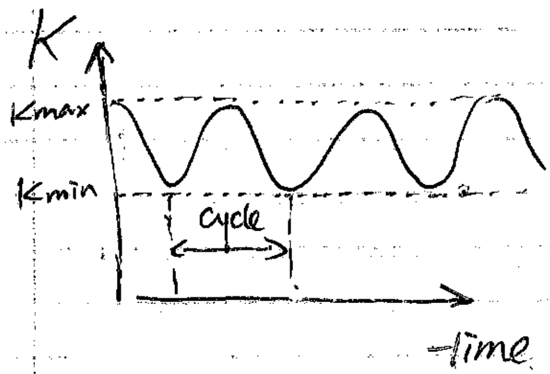
$$\rho = \frac{\pi}{8}\left(\frac{K_I^{old}}{\sigma_Y}\right)^2$$

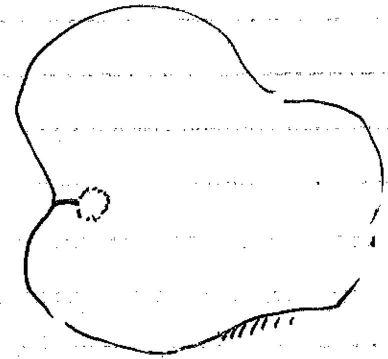
0.393

Lecture 20 6/5/2014.

Fatigue — example, airplane, vehicles, etc. ...

Paris law.



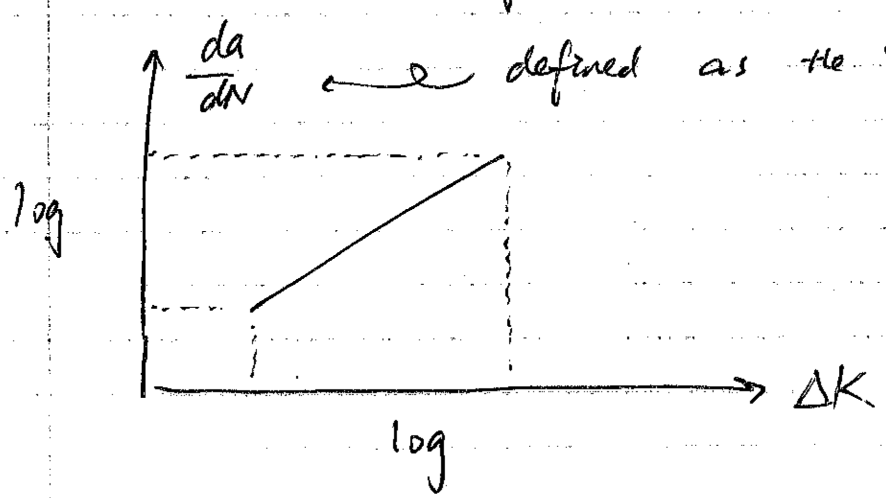


↳ not a time-dependent problem in this approximation.

$\Delta K = K_{max} - K_{min}$.

$$R = \frac{K_{min}}{K_{max}}$$

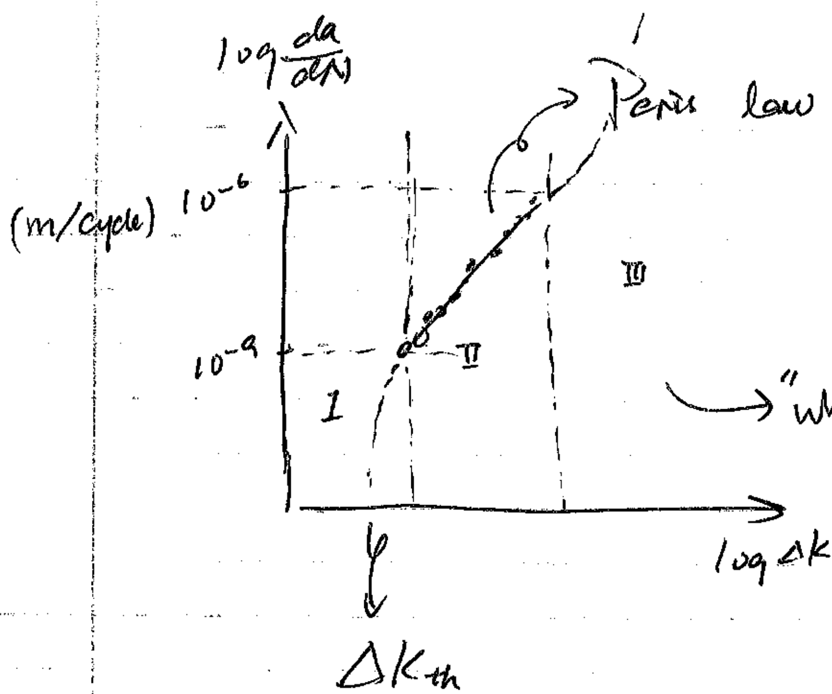
if you just focus on ΔK.



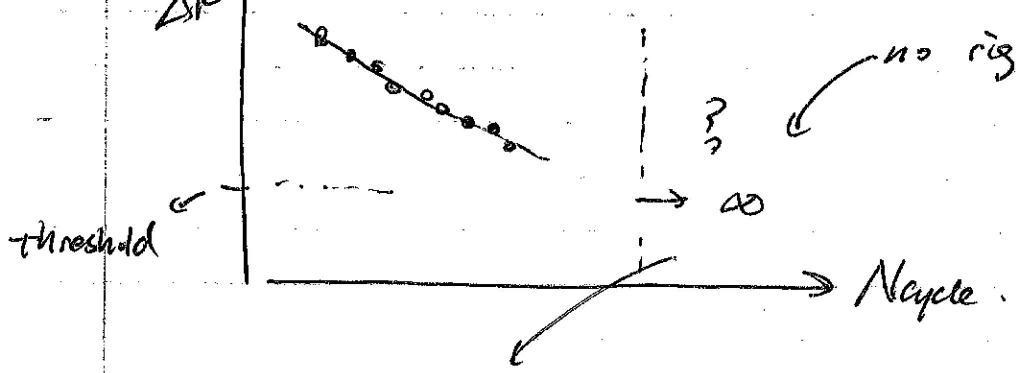
$$\frac{da}{dN} = f(\Delta K, R)$$

$$\cong C(\Delta K)^m.$$

$2 \le m \le 4 \quad m \sim 3$



"when k is big fracture happens".



no rigorous theory to verify, just from observation...

We donte know if thees a limit

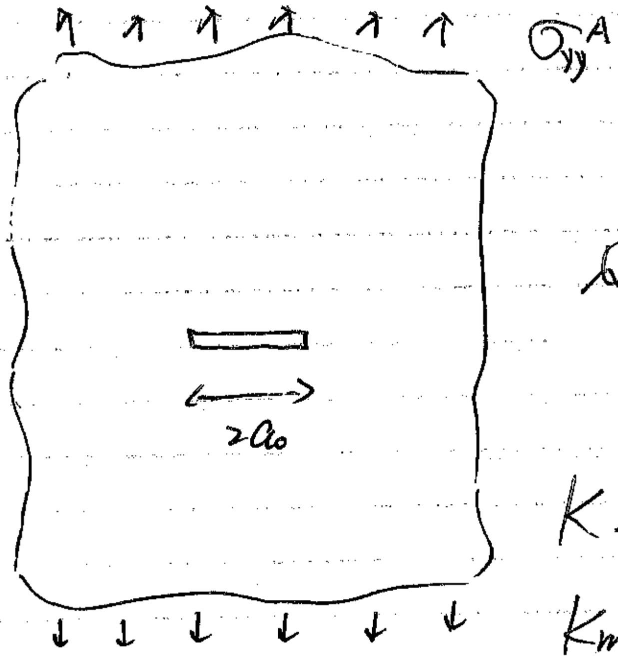
$$\frac{da}{dN} = C\,(\Delta k)^m \cdot \frac{\left(1 - \frac{\Delta K_{th}}{\Delta K}\right)^p}{\left(1 - \frac{K_{max}}{K_c}\right)^{q}}$$

a modified theory for fatigue

material constants:

$C,\ m,\ \Delta K_{th},\ K_c,\ p,\ q.$

Example

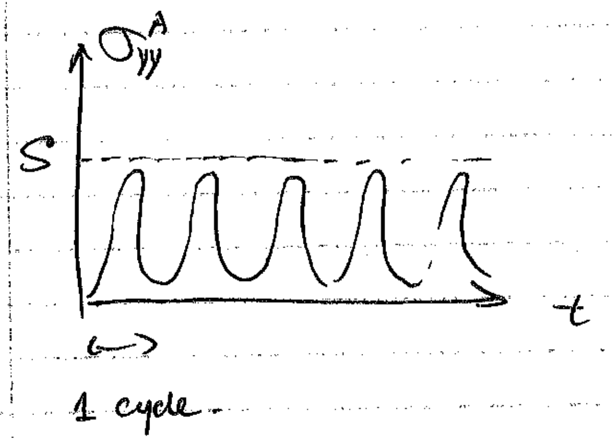


Q: How many cycles until fracture?

$$K = \sigma_{yy}^A \sqrt{\pi a}.$$

$$K_{max} = S\sqrt{\pi a} = \Delta K.$$

$$K_{min} = 0$$

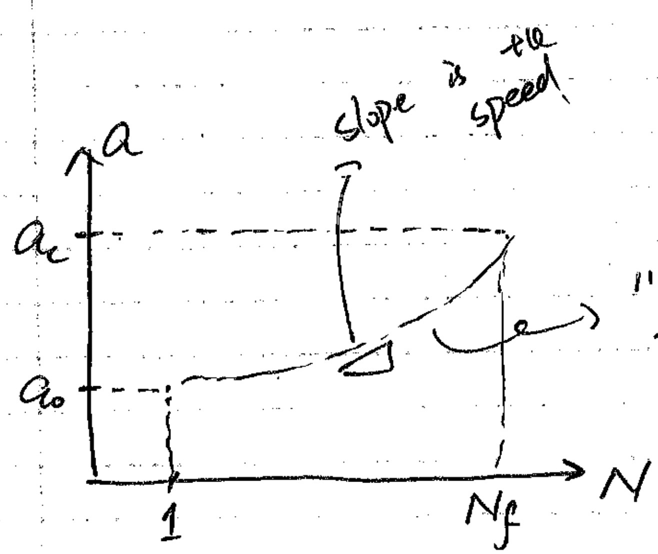


assume $\frac{da}{dN} = C(\Delta K)^m$

we need to find critical size a_c

when $K = K_{Ic}$

$$\sigma_{yy}^A \sqrt{\pi a_c} = K_{Ic} \Rightarrow a_c = \frac{1}{\pi}\left(\frac{K_{Ic}}{S}\right)^2$$



"just a guess"

$$\Delta K = S\sqrt{\pi a}$$

$$\frac{da}{dN} = C(\Delta K)^m = C S^m (\pi a)^{\frac{m}{2}}$$

$$\frac{dN}{da} = \frac{1}{C S^m \pi^{m/2}} a^{-\frac{m}{2}}$$

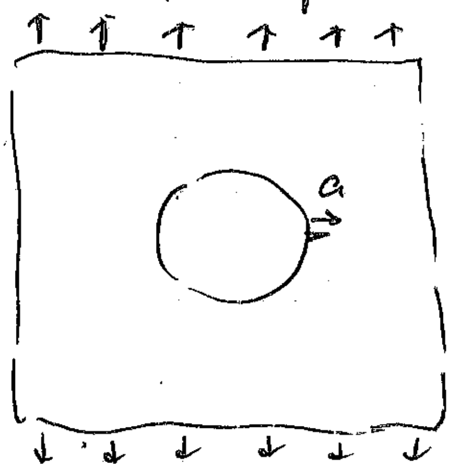
$$N = \int_{a_0}^{a_c} \frac{1}{C S^m \pi^{m/2}} a^{-\frac{m}{2}} \, da$$

$$N_f = \frac{1}{(-\frac{m}{2}+1) C S^m \pi^{m/2}} \left(a_c^{-\frac{m}{2}+1} - a_0^{-\frac{m}{2}+1} \right)$$

$$N_f = \frac{a_0^{-\frac{m}{2}+1} - a_c^{-\frac{m}{2}+1}}{(\frac{m}{2}-1)\ C \cdot S^m \cdot \pi^{m/2}}$$

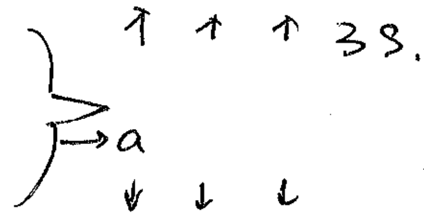
Example 2

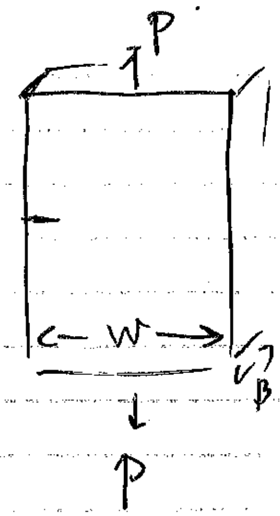
infinite plate



pure hole, stress concen. factor 3.

locally





$$K_I = \frac{P}{B\sqrt{W}} \left[\frac{\sqrt{2\tan\left(\frac{\pi a}{2W}\right)}}{\cos\left(\frac{\pi a}{2W}\right)} \left(0.752 + 2.02\frac{a}{W}\right) \right]$$

from table

$$\xrightarrow{W\to\infty} \frac{P}{B\sqrt{W}}\sqrt{\frac{\pi a}{W}}\left(0.752 + 0.37\right).$$

$$= \frac{P}{BW}$$ ← just the stress

$$\cdot\sqrt{\pi a} \cdot 1.122$$

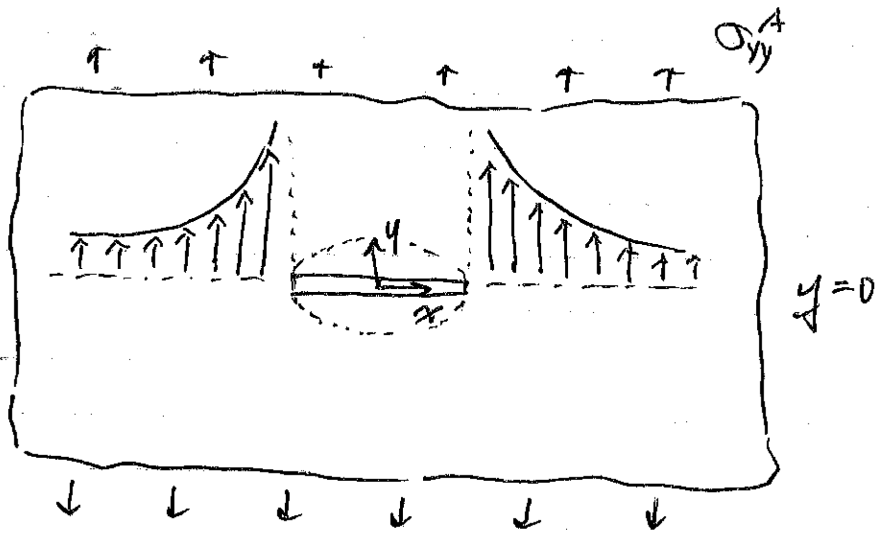
$$= 1.122\, \sigma_y^{app} \sqrt{\pi a}$$

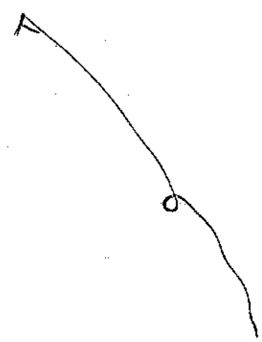
35

from the hole solution.

Problem Session 10 $6/7/2014$

LEFM → Slit-like crack.





apply stress "S"

$$g(x) = \int_{-\infty}^{\infty} \frac{p(x')}{x-x'}\,dx' , \quad p(x) = \frac{S\,|x|}{\sqrt{x^2-a^2}}$$

$\frac{du(x)}{dx}$

let $x = a+r$, $\lim_{r\to 0} \sigma_{yy}(x) = \frac{K_I}{\sqrt{2\pi r}}$ @ x, $y=0$.

$$d(x) = \frac{2(1-\nu)}{\mu}\sqrt{1-\left(\frac{x}{a}\right)^2} \qquad -a \le x \le a.$$

↳ enthalpy: $H = E - W_{in} = -E$ linear elastic medium.

should decrease during loading.

—mmm→ F $\quad W = Fx$

& no pre-existing

this kind of system: $W_{in} = 2E$ ← Stress.

internal energy

State 0 $\downarrow$ no crack

E_0, H_0.

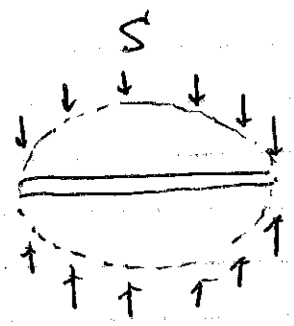
$\rightarrow H_0 = -E_0$

State 1. $\downarrow$ With crack & opened.

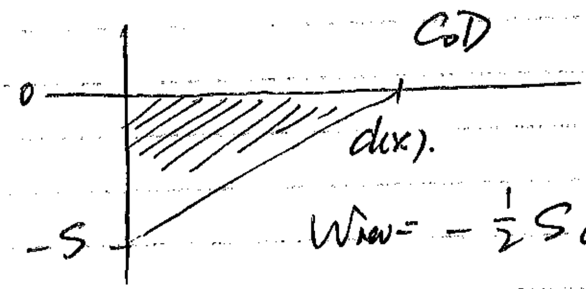
E_1, H_1

$\Delta H = -\Delta E$.

$\underset{\text{neg.}}{\downarrow} \rightarrow \Delta E > 0$



apply $S \rightarrow$ State 0



$-S \rightarrow 0$

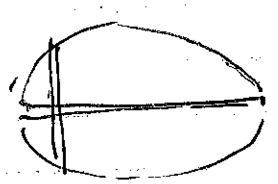
$W_{in} = -\frac{1}{2} S\, d(x)\, 2a$

if stress is changing: $\int_{-a}^{a} \left[\frac{1}{2}\, dF\, d(x) \right]$

$\downarrow$

$S\,dx$

We obtain: $\Delta H = -\frac{(1-\nu)}{2\mu} S^2 \pi a^2$



slit-like crack

<u>Griffith Criteria</u>

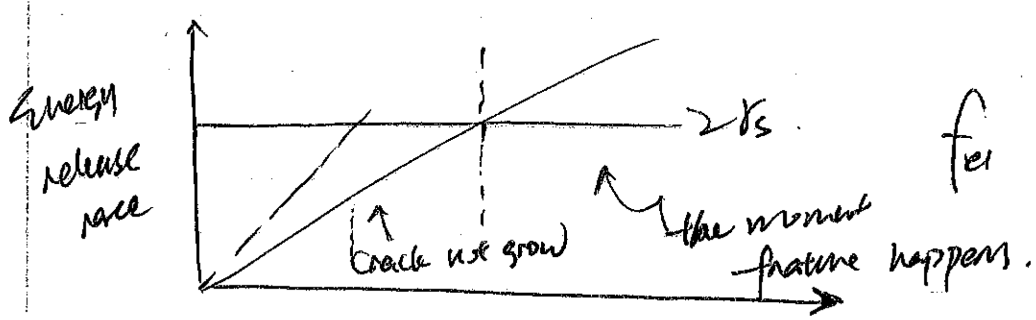
Slit-like

$$f_{el} = -\frac{\partial(\Delta H)}{\partial(\text{crack length})} = -\frac{\partial(\Delta H)}{\partial(2a)}$$

$$f_{tot} = -\frac{\partial(\overbrace{\Delta H + 2\cdot\gamma_s\cdot 2a}^{\Delta G})}{\partial(2a)} \longrightarrow \text{two surfaces.}$$

Griffith

$$= -\frac{\partial(\Delta H)}{\partial(2a)} - 2\gamma_s$$ material property



$$\Delta H = -\frac{(1-\nu)}{2\mu}S^2\pi a^2 \rightarrow f_{el} = \frac{\pi(1-\nu)}{2\mu}S^2 a$$

$$a^2 \hookrightarrow \frac{(2a)^2}{4}$$

$$f_{el} = 2\gamma_s. \quad S_c = \sqrt{\frac{8\mu\gamma_s}{\pi(1-\nu)(2a)}} \leftarrow \text{crack length fixed}$$

$$\text{or} \quad 2a_c = \frac{8\mu\gamma_s}{\pi(1-\nu)S^2} \leftarrow \text{stress fixed}$$

mode-I loading

Energy release rate $\mathcal{G} = -\dfrac{\partial(\Delta H)}{\partial(\text{crack length})}$

$= \dfrac{\pi(1-\nu)}{2\mu} S^2 a$ slit-like

generalized expression $= \dfrac{K_I^2}{E'}$ $\begin{cases} K_I = S\sqrt{\pi a} \\ E' = \dfrac{E}{1-\nu^2} \text{ (plane strain)} \end{cases}$

True loadings

$\sigma_{yy}^{(1)}$ $\sigma_{yy}^{(2)}$ — can use principles of superposition due to linear elasticity.

$K_I^{(1)}$ $K_I^{(2)}$

$$\mathcal{G} = \frac{(K_I^{(1)} + K_I^{(2)})^2}{E'}$$

Multiple modes

$$\mathcal{G} = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

J-Integral

$\mathcal{G} \geq \mathcal{G}_c \rightarrow$ material property

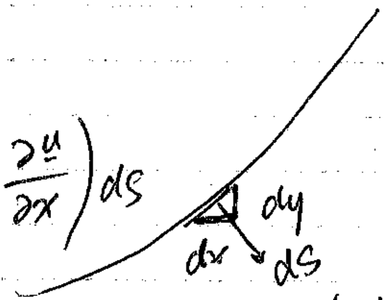
$\dfrac{K_I^2}{E'} \geq \dfrac{K_{Ic}^2}{E'} \Rightarrow K_I \geq \underbrace{K_{Ic}}_{\text{fracture toughness}}$ Crack growth condition

$$J_i = \int_S (w n_i - T_j u_{j,i})\, dS.$$

$\downarrow$

x, y, z

In 1D, $J_x = \int_\Gamma \left(w\, dy - T \frac{\partial u}{\partial x} \right) dS$

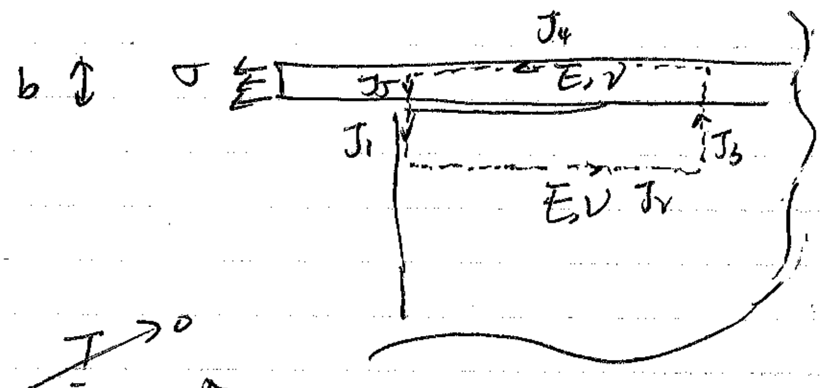


path $\rightarrow dx\, \underline{i} + dy\, \underline{j}$

$dS \rightarrow dy\, \underline{i} - dx\, \underline{j}$

$$J_x = \int_\Gamma w\,dy - \left(T_x \frac{\partial u_x}{\partial x} + T_y \frac{\partial u_y}{\partial x} \right) dS.$$

Example



$$J_x = \sum_{i=1}^{5} J_i$$

①: $\cancel{w} \rightarrow 0$ $\quad \cancel{\underline{T}} \rightarrow 0$

J_1 by surface

②: $\cancel{dy} \rightarrow 0$ $\quad \cancel{\frac{\partial \underline{u}}{\partial x}} \rightarrow 0$

Q: Z_dx?

③: $\cancel{W} \rightarrow 0$ $\quad \cancel{\underline{T}} \rightarrow 0$ (infinitely faraway)

④: $\cancel{W\,dy} \rightarrow 0$ $\quad \cancel{\underline{T}}$ traction free.

⑤. $\int w\,dy = \int_b^0 \frac{1}{2} \sigma_{ij} \varepsilon_{ij}\, dy$

$$dS = dx\,\underline{i} + dy\,\underline{j}$$

$$W = \frac{1}{2}\sigma\varepsilon$$

$$W = \frac{\sigma^2}{2E} \qquad \varepsilon_{xx} = \frac{\sigma_{xx}}{E}$$

$$\int_0^b w\,dy = \frac{\sigma^2 b}{2E}$$

$$\int_0^b T_x \frac{\partial u_x}{\partial x}\,dy \qquad T_x = \sigma_{xx} = \sigma$$

$$\frac{\partial u_x}{\partial x} = \varepsilon_{xx}$$

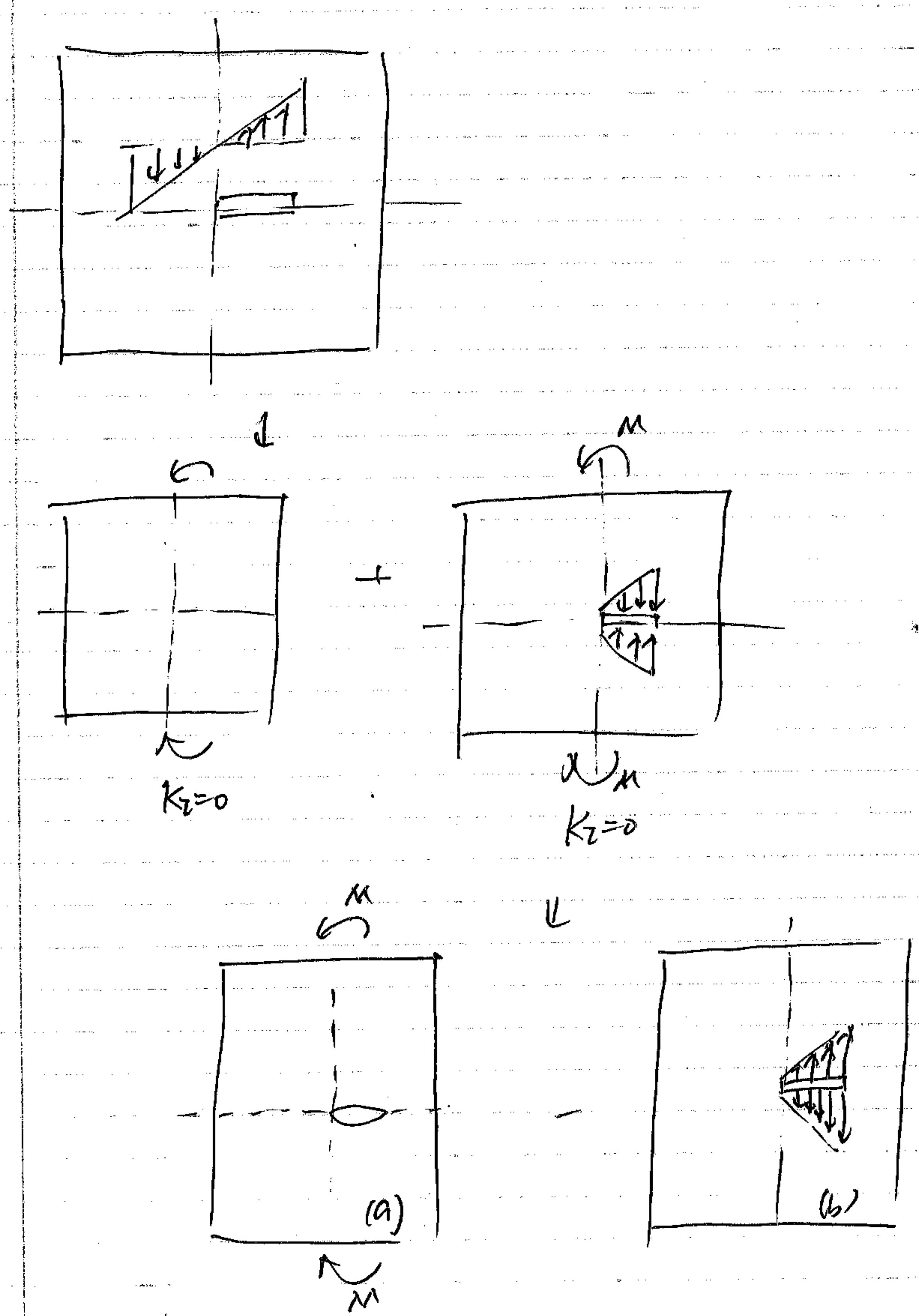
$$= -\int_0^b \sigma_{xx}\varepsilon_{xx}\,dy$$

$$= -\int_0^b \frac{\sigma^2}{E}\,dy = \frac{-\sigma^2 b}{E}$$

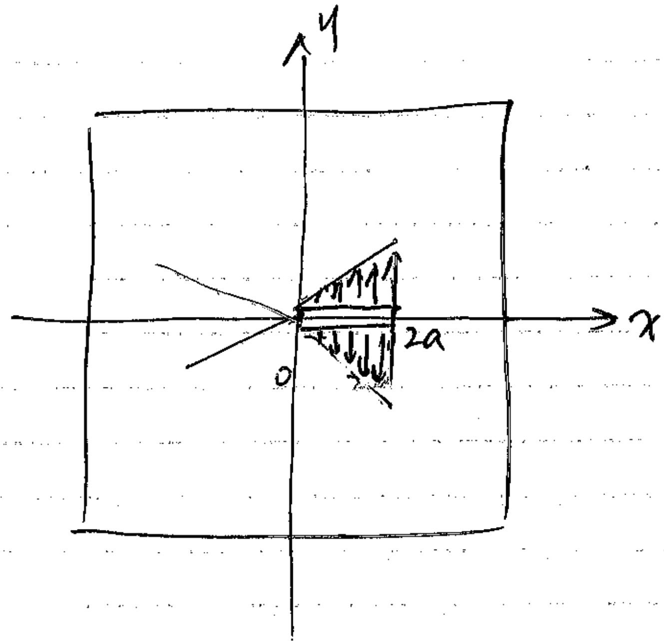
$$J_x = \frac{\sigma^2 b}{2E} - \left(-\frac{\sigma^2 b}{E}\right) = \frac{3\sigma^2 b}{2E} = \frac{K_I^2}{E}$$

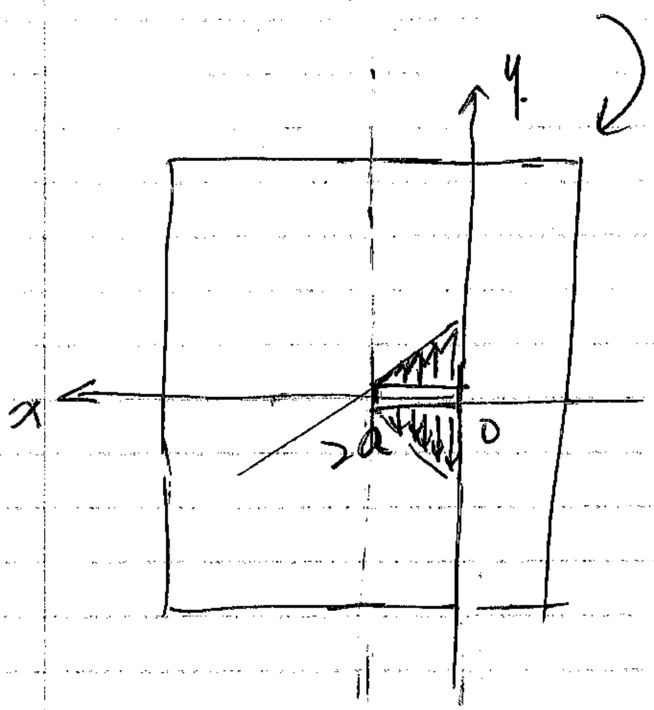
plane stress

for plane strain, we replace E with E'.



$$K_I^{(a)} = K_I^{(b)}$$





looking from the back side.

"Shift coordinate to obtain K_I at
the left side → integral goes from $0 \rightarrow 2a$".

Fracture mechanics review

Contact problem

from the surface Green function we know the equation for contact:

$$\frac{du_0}{dx} = \frac{K+1}{4\pi\mu} \int_{-c}^{c} \frac{P_y(x')}{x-x'} dx'$$

→ integral equation soln:

$$P_y(x) = -\frac{1}{\pi^2\sqrt{c^2-x^2}} \int_{-c}^{c} \frac{\sqrt{c^2-x'^2} \cdot \frac{4\pi\mu}{K+1} \frac{du_0(x')}{dx'}}{x-x'} dx'$$

$$+ \frac{F}{\pi\sqrt{c^2-x^2}}.$$

From the flat punch contact we know

$$P_y \sim \frac{F}{\pi}(2cr)^{-1/2}$$

$$\rightarrow \sigma_{yy} \propto \frac{1}{\sqrt{r}}$$

Wedge and Notch

Trial soln for wedge:

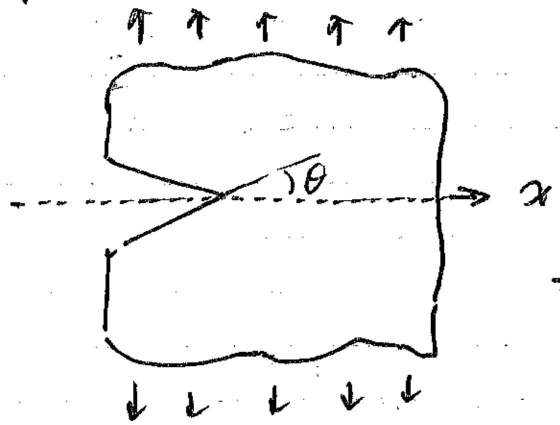
$$\phi = r^2(A_1\cos 2\theta + A_2 + A_3 \sin 2\theta + A_4\theta)$$

$$\sigma_{rr} = -2A_1\cos 2\theta + 2A_2 - 2A_3\sin 2\theta + 2A_4\theta.$$

$$\sigma_{r\theta} = 2A_1 \sin 2\theta + 0 - 2A_3\cos 2\theta - A_4$$

$$\sigma_{\theta\theta} = 2A_1\cos 2\theta + 2A_2 + 2A_3\sin 2\theta + 2A_4\theta.$$

formulate the notch problem.



William's soln. $(n = \lambda - 1)$

$$\phi = r^{\lambda+1}\{A_1 \cos(\lambda+1)\theta + A_2 \cos(\lambda-1)\theta + A_3 \sin(\lambda+1)\theta + A_4 \sin(\lambda-1)\theta\}.$$

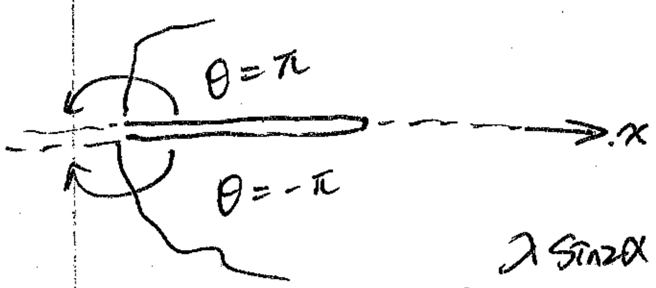
$\cdots \rightarrow \sigma_{rr}, \sigma_{r\theta}, \sigma_{\theta\theta}$.

Find the symmetric & anti-symmetric part based on the nature of the loading.

$\det(M_1) = 0$ ← symmetric

$\det(M_2) = 0$ ← anti-symmetric.

$\cdots \rightarrow$ when the notch turns into a crack



$\alpha = 2\pi$

$\updownarrow$

$\sin 2\alpha = 0.$

$\lambda \sin 2\alpha \pm \sin 2\lambda\alpha = 0 \quad \cdots \rightarrow \quad \sin 2\pi\lambda = 0$

$\lambda = 0, \frac{1}{2}, 1, \frac{3}{2}$

$\lambda = 0$: Strain energy is infinite.

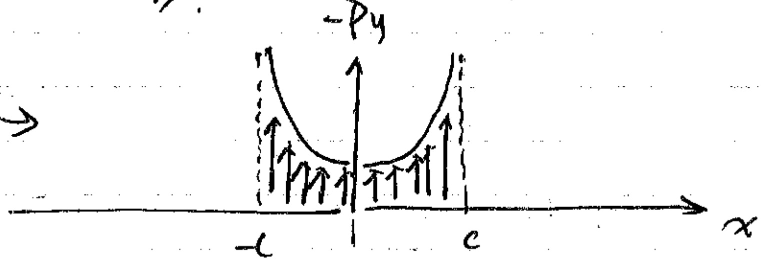
$\lambda = 1, \frac{3}{2}$: stress fields nonsingular.

$\lambda = \frac{1}{2}$: $\sigma \sim \frac{1}{\sqrt{r}}$ ... crack tip singularity.

Equivalence between crack & flat-punch problems

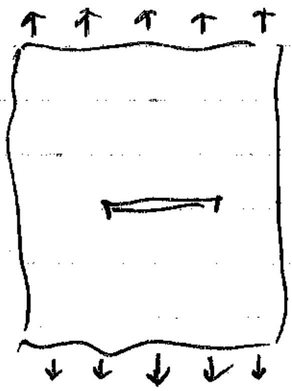
$$-P_y(x) = \frac{F}{\pi\sqrt{c^2-x^2}} = -\sigma_{yy}.$$

$$-P_y(x) = \frac{F}{\pi\sqrt{c^2-x^2}} = \sigma_{yy}.$$

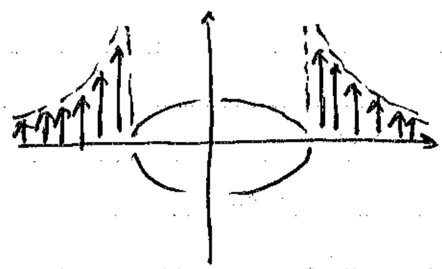


(... skip plasticity).

Slit-like Crack



Equivalence between contact & cracks.



recall soln to the singular integral equation:

$$P_y(x) = -\frac{1}{\pi^2}\frac{(x-a)^{1/2}}{(x+a)^{1/2}}\left[\int_{-\infty}^{-a} + \int_{a}^{+\infty}\right]\frac{(x'+a)^{1/2}}{(x'-a)^{1/2}}\frac{q(x')}{x-x'}\,dx' + \frac{Ax+B}{(x+a)^{1/2}(x-a)^{1/2}}$$

$$P_y(x) = \frac{A + B/x}{\sqrt{1-(a/x)^2}}$$

$$\ldots\ \sigma_{yy}(x, y=0) = \frac{S\cdot|x|}{\sqrt{x^2-a^2}}$$

to find the stress singularity at crack tip,

let $x = a + r$. (taking $r \to 0^+$)

$$\sigma_{yy} \sim \frac{Sa}{\sqrt{2ar}} = S\sqrt{\frac{a}{2}}\frac{1}{\sqrt{r}}$$

recall wedge & notch: $\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right)$

$\to \sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}$ ... stress intensity factor: $K_I = S\sqrt{\pi a}$

for slit-like crack:

$$\tilde{u}_y(x) = -\frac{1-\nu}{\mu} Sa \cdot \sqrt{1-(x/a)^2}$$

$$\to d(x) = -2\tilde{u}_y(x) = \frac{2(1-\nu)}{\mu} Sa \cdot \sqrt{1-(x/a)^2}$$

▷ enthalpy of the crack.

$H = E - \Delta W_{lm}$.

linear elastic medium sub. traction force $\underset{\sim}{T}$ on S_t.

the enthalpy writes:

$$H = \underbrace{\int_\Omega \frac{1}{2}\sigma_{ij}\varepsilon_{ij}\, dV}_{\text{elastic strain energy}} - \int_{S_t} T_j u_j\, dS.$$

For slit-like crack,

works like magic !!!

$$\begin{cases} E = \frac{1}{2}\sigma_{yy}^A \varepsilon_{yy} V \\ \Delta W_{lm} = (\sigma_{yy}^A A)\cdot(\varepsilon_{yy} L) = \sigma_{yy}^A \varepsilon_{yy}\cdot V. \\ H = \frac{1}{2}\sigma_{yy}^A \varepsilon_{yy} V - \sigma_{yy}^A \varepsilon_{yy} V = -\frac{1}{2}\sigma_{yy}^A \varepsilon_{yy} V = -E \end{cases}$$

- enthalpy: $\Delta H = \Delta W^{+} + \Delta W^{-}$

$$= \frac{1}{2} S \int_{-a}^{a} 2\hat{u}_y(x)\,dx$$

$$= -\frac{1}{2} S \int_{-a}^{a} d(x)\,dx$$

crack-opening displacement:

$$d(x) = \frac{2(1-\nu)}{\mu} S a \sqrt{1-\left(\frac{x}{a}\right)^2}$$

enthalpy change proportional to applied stress S:

$$\Delta H = -\frac{1-\nu}{2\mu} S^2 \pi a^2 \qquad \text{(plane strain)}$$

- driving force for crack propagation.

$$f_{el} = -\frac{\partial \Delta H}{\partial (2a)} = \frac{\pi(1-\nu)}{2\mu} S^2 a = \frac{1-\nu}{2\mu} K_I^2$$

Griffith criteria. free energy:

$$\Delta G = \Delta H + \gamma_s \cdot 2 \cdot 2a \quad \leftarrow \text{slit-like.}$$

plug in: $\Delta G = -\frac{1-\nu}{2\mu} S^2 \pi a^2 + 4\gamma_s a$

$$f_{tot} = \frac{\pi(1-\nu)}{2\mu} S^2 a - 2\gamma_s$$

Solving for $f_{tot} = 0$, one can solve for the critical crack size & critical stress:

$$a_c = \frac{4\mu}{\pi(1-\nu)} \frac{\gamma_s}{S^2} \quad \& \quad S_c = \sqrt{\frac{4\mu\gamma_s}{\pi(1-\nu)a}}$$

(plane strain)

See eqn. (50) for general expression $d(x)$, ΔH, f_{el}, f_{tot}.

Energy release rate

mode-I, mode-II, mode-III.

Solutions for $\begin{cases} \sigma_{xx}^{(I)} \\ \sigma_{yy}^{(I)} \\ \sigma_{xy}^{(I)} \end{cases}$, $\begin{cases} \sigma_{xx}^{(II)} \\ \sigma_{yy}^{(II)} \\ \sigma_{xy}^{(II)} \end{cases}$, $\begin{cases} \sigma_{xz}^{(III)} \\ \sigma_{yz}^{(III)} \end{cases}$

energy release rate $\mathcal{G}$ (crack extension force)
elastic contribution.

$$\mathcal{G} = -\frac{\partial (\Delta H)}{\partial (2a)} = \frac{\pi(1-\nu)}{2\mu} (\sigma_{yy}^{A})^2 a \qquad E' \equiv \frac{E}{1-\nu^2}$$

$$\mathcal{G} = \frac{\pi}{E'} (\sigma_{yy}^{A})^2 a$$

recall $K_I = \sigma_{yy}^{A} \sqrt{\pi a} \rightarrow \mathcal{G} = \frac{K_I^2}{E'}$ ... mode-I

general crack case: $\mathcal{G} = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$

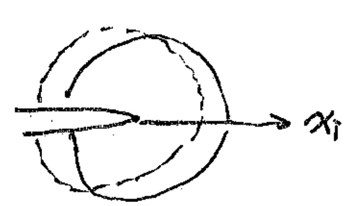
one may also derive the energy release rate based on the variation of H w.r.t. a.

J-Integral (2D)) $J = \int_{\Gamma} w\, dy - \underline{T} \cdot \frac{\partial \underline{u}}{\partial x} dS$

$J = \mathcal{G}$

$$= \int_{V} \frac{\partial w}{\partial x_i} dV - \int_{S} T_j \frac{\partial u_j}{\partial x_i} dS$$

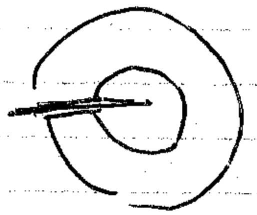
→ the work done : $-J_i \delta x_i = \int_{V'} w\, dV - \int_{V} w\, dV$



$$+ \int_{S'} T_j u_j dS - \int_{S} T_j u_j dS$$

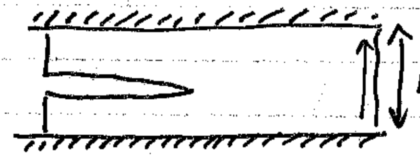
$$= H' - H$$

Example 1



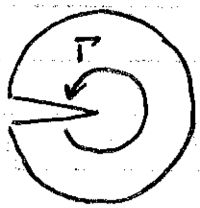
← no singularity, $J(\Gamma) = 0$.

Example 2



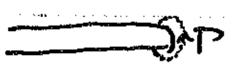
$J = wh$ boundary work.

Example 3



$$J = \int_\Gamma w\,dy - \underline{T} \cdot \frac{\partial u}{\partial x} ds = \frac{1-\nu}{2\mu} K_I^2 = \frac{K_I^2}{E'}$$

Example 4

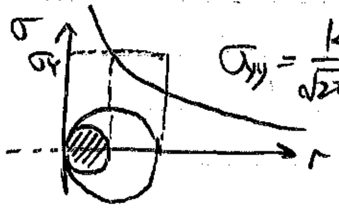


$$J = \int_\Gamma w\,dy - \cancel{\underline{T} \cdot \frac{\partial u}{\partial x} ds}$$

"under tension"

Elastic-Plastic Fracture Mechanics

fracture criteria: $J = J_c$



$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$

$$\sigma_Y = \frac{K_I}{\sqrt{2\pi r_y}} \rightarrow r_y = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_Y}\right)^2 \qquad r_p = 2 r_y$$

plastic yielding changes K_I.

$$K_I = \frac{P}{B\sqrt{W}} f\left(\frac{a_{eff}}{W}\right) \leftarrow \text{estimate: } a_{eff} = a + r_y$$

HRR Solution: $$\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha\left(\frac{\sigma}{\sigma_0}\right)^n$$ ← Strain-hardening exp.

HRR singularity: $\sigma \cdot \varepsilon \propto \frac{1}{r}$.

▷ Strip yield model

$$K_{I_{+a}} = \frac{P}{\sqrt{\pi a}}\sqrt{\frac{a+x}{a-x}}, \quad K_{I_{-a}} = \frac{P}{\sqrt{\pi a}}\sqrt{\frac{a-x}{a+x}}$$

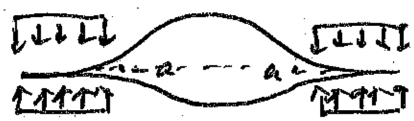
total stress intensity factor @ $a+\rho = 0$

↑ cancel singularity

$$K_{tot} = \sigma_{yy}^{a}\sqrt{\pi(a+\rho)} + K_{closure} = 0$$

$$\sigma_{yy}^{A}\sqrt{\pi(a+\rho)} - 2\sigma_Y\sqrt{\frac{a+\rho}{\pi}}\arccos\left(\frac{a}{a+\rho}\right) = 0$$

... some algebra.



$$\rho \approx 0.393\left(\frac{K_I^{old}}{\sigma_Y}\right)^2$$

$$r_p = \frac{1}{\pi}\left(\frac{K_I}{\sigma_Y}\right)^2 = 0.318\left(\frac{K_I}{\sigma_Y}\right)^2$$ ... Irwin's approach.

▷ Crack tip opening displacement.

$$\delta = \frac{8\sigma_Y a}{\pi E'}\cdot \ln\left(\frac{a+\rho}{\rho}\right)$$

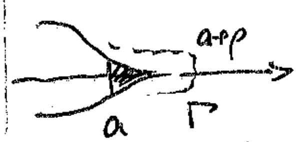
$$\delta = \frac{8\sigma_Y a}{\pi E'}\ln\left[\sec\left(\frac{\pi}{2}\cdot\frac{\sigma_{yy}^{A}}{\sigma_Y}\right)\right]$$

at small σ_{yy}^{A}: $\delta \approx \frac{K_I^{old\,2}}{E'\sigma_Y}$ as $\sigma_{yy}^{A} \to \sigma_Y$, $\delta \to \infty$

▷ energy release rate of strip yield.

$$\mathcal{G} = J = \int_{\Gamma} \cancel{w\,dy} - T\frac{\partial u}{\partial x}ds$$

$$= -\sigma_Y u_y(x=a, y=0^-) + \sigma_Y u_y(x=a, y=0^+)$$



$$= \sigma_Y \cdot \delta$$

.... $\mathcal{G} = J = \sigma_Y \cdot \delta$ . ... fracture: $\sigma_Y \cdot \delta_c$

↑ fracture CTOD

Plasticity review

- Displacement: $\underline{u} = \underline{x} - \underline{X}$
- Strain: $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$
- Stress: $T_j = \sigma_{ij} n_i$ → same with elasticity
- Equilibrium: $\sigma_{ij,i} + F_j = 0$

For strain: $\varepsilon_{ij} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{pl}$

Elastic constitutive relationship: $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}^{el}$

important

hydrostatic stress: $\bar{\sigma} \equiv \frac{1}{3}\sigma_{ii}$

$\hookrightarrow \bar{\sigma} = 3K\bar{\varepsilon}$ ... $K = \frac{E}{3(1-2\nu)}$

deviatoric stress & strain relationship. $S_{ij} = 2\mu e_{ij}$

- Yield condition. $f(\{\sigma_{ij}\}) = 0$

original stress invariants: I_1, I_2, I_3 (too complicated!)

Solving eigenequation for deviatoric stress.

$\det(\underline{\underline{S}} - \hat{\sigma}\underline{\underline{I}}) = \left| \quad \right| = 0 \rightarrow \hat{\sigma}^3 - J_1\hat{\sigma}^2 + J_2\hat{\sigma} + J_3 = 0$

after some algebra, we have $J_2 = \frac{1}{2} S_{ij} S_{ij}$

a measure of deviatoric stress.
can be think of as the norm.

the new stress invariants:

$$J_1 = 0, \quad J_2 = \frac{1}{2} S_{ij} S_{ij}, \quad J_3 = \det(S_{ij})$$

$$\rightarrow f(\bar{\sigma}, J_2, J_3) = 0$$

some further simplification:

$$\ldots \quad f(J_2) = J_2 - k^2 = 0 \text{ ; if EPP is assumed:}$$

$$\rightarrow \dot{J}_2 = 0 \quad \rightarrow \quad \dot{J}_2 = S_{ij} \dot{S}_{ij} = 0$$

a constraint on the stress rate.

☆ Q: How to determine ε_{ij}^{pl}?

$$\varepsilon_{ij}^{pl} = \int_0^t \dot{\varepsilon}_{ij}^{pl}(t)\, dt \quad \Rightarrow \quad 2\mu \dot{\varepsilon}_{ij}^{pl} = \hat{\lambda} S_{ij} \quad \text{(flow rule)}$$

$$\ldots \text{ recall } \quad 2\mu e_{ij}^{el} = S_{ij} \quad \& \quad 2\mu \dot{e}_{ij}^{el} = \dot{S}_{ij}$$

$$2\mu \dot{e}_{ij} = 2\mu(\dot{e}_{ij}^{el} + \dot{\varepsilon}_{ij}^{pl}) = \dot{S}_{ij} + \lambda S_{ij}$$

total deviatoric strain rate

define work rate: $\dot{W} = S_{ij} \dot{e}_{ij} = S_{ij}(\dot{e}_{ij}^{el} + \dot{\varepsilon}_{ij}^{pl})$

With EPP assumption: $2\mu \dot{w} = 2\tilde{\lambda} k^2$

$\rightarrow \tilde{\lambda} = \frac{2\mu}{2k^2} \dot{w} \quad \dot{\varepsilon}_{ij}^{pl} = \frac{\dot{w}}{2k^2} \dot{w}$

<u>Overall summary</u>

$$\dot{\varepsilon}_{ij} \begin{cases} \dot{\bar{\varepsilon}} = \frac{1}{3} \dot{\varepsilon}_{ii} \rightarrow \dot{\bar{\sigma}} = 3K \dot{\bar{\varepsilon}} \\ \dot{e}_{ij} = \dot{\varepsilon}_{ij} - \dot{\bar{\varepsilon}} \delta_{ij} \end{cases}$$

$$\dot{\sigma}_{ij} = \dot{S}_{ij} + \dot{\bar{\sigma}} \delta_{ij}$$

$$\sigma_{ij} \begin{cases} \bar{\sigma} = \frac{1}{3} \sigma_{ii} \\ S_{ij} = \sigma_{ij} - \bar{\sigma} \delta_{ij} \end{cases}$$

$$\dot{w} = S_{ij} \dot{e}_{ij} \rightarrow \dot{S}_{ij} = 2\mu \left(\dot{e}_{ij} - \frac{\dot{w}}{2k^2} S_{ij} \right)$$